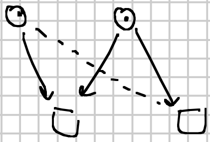
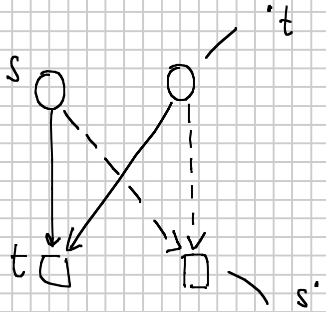
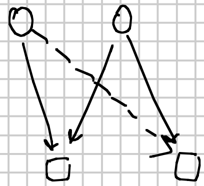
Definition 4.1 *Free-choice nets, free-choice systems*

A net $N = (S, T, F)$ is free-choice if $(s, t) \in F$ implies ${}^\bullet t \times s^\bullet \subseteq F$ for every place s and every transition t.

A system (N, M_0) is free-choice if its underlying net N is free-choice.



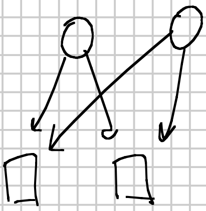
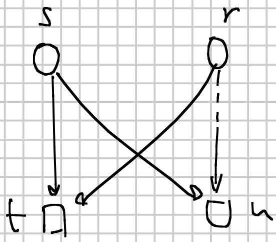
Proposition 4.2 *Characterizations of free-choice nets*

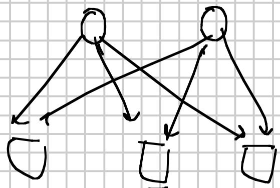
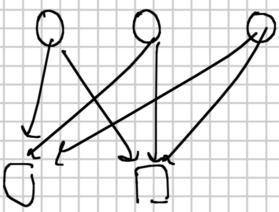
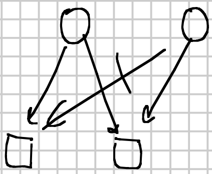
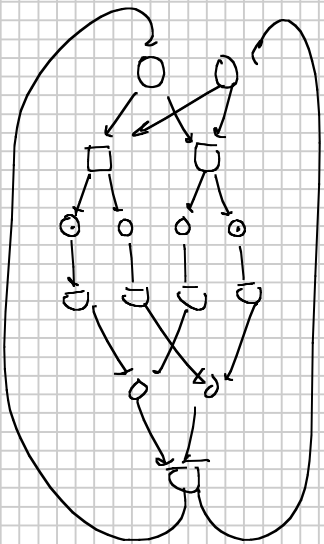
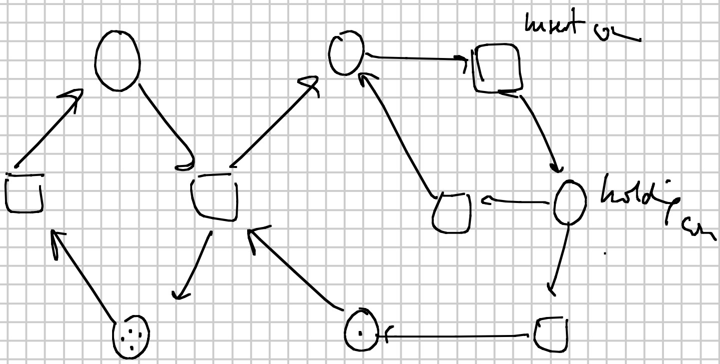
(1) A net (S, T, F) is free-choice iff for every two places s and r and every two transitions t and u

$$\{(s,t), (r,t), (s,u)\} \subseteq F \implies (r,u) \in F.$$

(2) A net is free-choice iff for every two places s and r either $s^\bullet \cap r^\bullet = \emptyset$ or $s^\bullet = r^\bullet$.

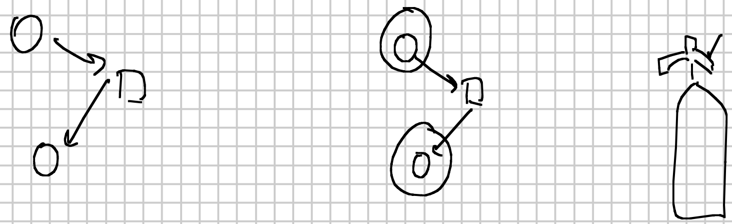
(3) A net is free-choice iff for every two transitions t and u either ${}^\bullet t \cap {}^\bullet u = \emptyset$ or ${}^\bullet t = {}^\bullet u$.

Definition 4.7 *Stable sets, stable predicates*

A set $\mathcal{M}$ of markings of a net is stable if $M \in \mathcal{M}$ implies $[M\rangle \subseteq \mathcal{M}$. The membership predicate of a stable set is called a stable predicate.



$R^\bullet \subseteq {}^\bullet R$

Definition 4.8 *Siphons, proper siphons*

A set R of places of a net is a siphon if ${}^\bullet R \subseteq R^\bullet$. A siphon is called proper if it is not the empty set.

Proposition 4.9 *Unmarked siphons remain unmarked*

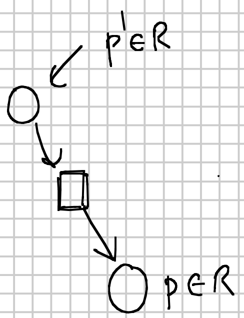
If R is a siphon then the set of markings M satisfying $M(R) = 0$ is stable.

Proof:
Follows easily from the definition. □

Proposition 4.11 *Deadlocked systems have an unmarked proper siphon*

Let (N, M_0) be a deadlocked system, i.e., M_0 is a dead marking of N. Then the set R of places of N unmarked at M_0 is a proper siphon.

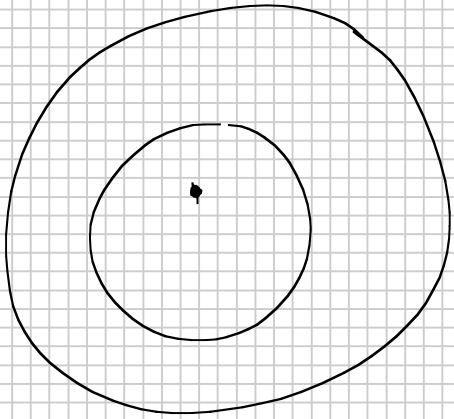
M deadlock



${}^\bullet p \subseteq R^\bullet$

${}^\bullet R \subseteq R^\bullet$

R



Definition 4.12 *Traps, proper traps*

A set R of places of a net is a trap if $R^\bullet \subseteq {}^\bullet R$. A trap is called proper if it is not the empty set.

Proposition 4.13 *Marked traps remain marked*

If R is a trap then the set of markings M satisfying $M(R) > 0$ is stable.

Proposition 4.14 *A sufficient condition for deadlock-freedom*

If every proper siphon of a system includes an initially marked trap, then the system is deadlock-free.

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Commoner's Theorem

A free-choice system is live if and only if every proper siphon includes an initially marked trap.

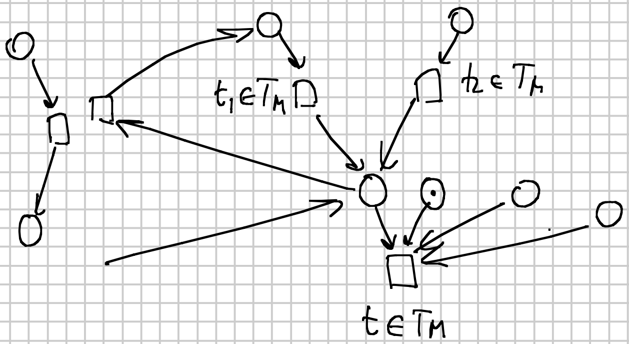
Lemma 4.20

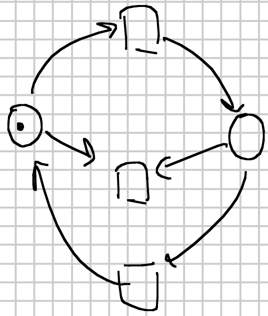
Every non-live free-choice system has a proper siphon R and a reachable marking M such that R is unmarked at M.

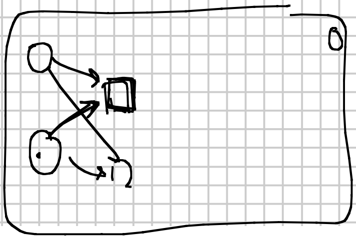
Proof Let (N, M_0) be free choice and non-live

$\Rightarrow$ there is a reachable marking M and a transition t such that t can never fire again from M.

Let T_M be the set of transitions that can never occur from M.





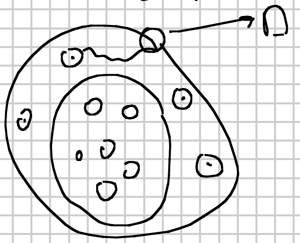


Theorem 4.21 *'If' direction of Commoner's Theorem*

If every proper siphon of a free-choice system includes an initially marked trap, then the system is live.

Proof By the lemma above, if the system is not live, then we can "empty" a syphon (proper syphon). But then this syphon cannot contain an initially marked trap.

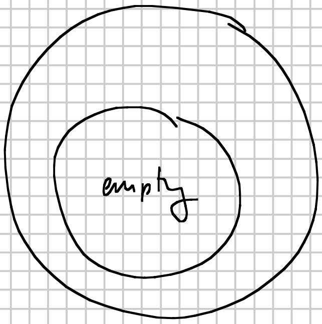
Idea for the "only if" direction:



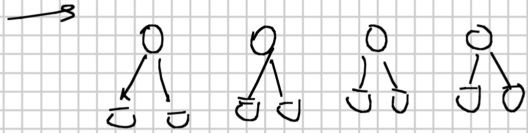
Theorem 4.28 *Complexity of the non-liveness problem of free-choice systems*

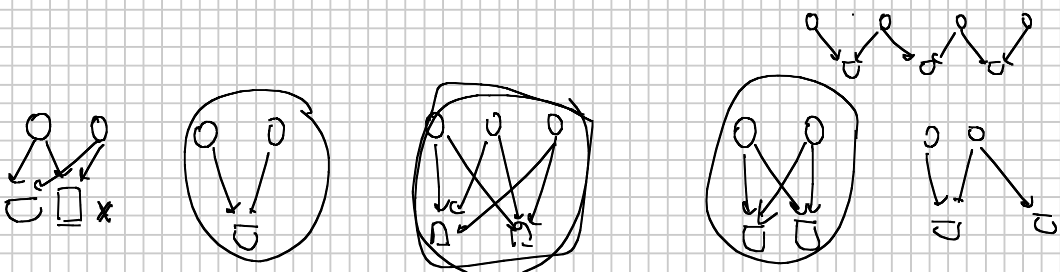
The following problem is NP-complete:

Given a free-choice system, to decide if it is not live.



Reduction from SAT





Definition 4.4 *Clusters*

Let x be a node of a net. The cluster of x, denoted by $[x]$, is the minimal set of nodes such that

- $x \in [x]$,
- if a place s belongs to x then $s^\bullet$ is included in $[x]$, and
- if a transition t belongs to $[x]$ then ${}^\bullet t$ is included in $[x]$.

Proposition 4.5 *A property of clusters*

Let N be a net. The set $\{[x] \mid x \text{ is a node of } N\}$ is a partition of the nodes of N.

Theorem 6.14 *The Rank Theorem*

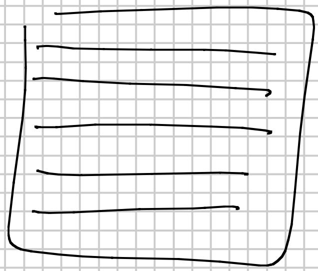
Let N be a free-choice net. Let $\mathbf{N}$ be the incidence matrix of N and C_N the set of clusters of N. The net N is well-formed iff

(a) it is connected, and has at least one place and one transition,

(b) it has a positive S-invariant,

(c) it has a positive T-invariant, and

(d) $\text{Rank}(\mathbf{N}) = |C_N| - 1$. □

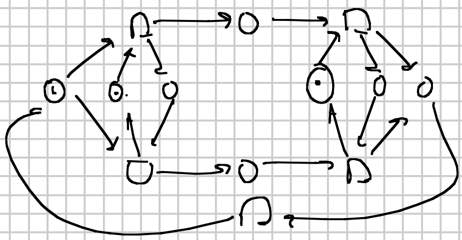


Exercise 1

SAT $\longrightarrow$ Non liveness in fc net.

Exercise

Find a net that is live and bounded, but after adding tokens it is no longer bounded



The reachability theorem

Definition 8.1 *Home marking*

Let (N, M_0) be a system. A marking M of the net N is a home marking of (N, M_0) if it is reachable from every marking of $[M_0\rangle$.

We say that (N, M_0) has a home marking if some reachable marking is a home marking.

Theorem 8.11 *Home Marking Theorem*

A reachable marking of a live and bounded free-choice system (N, M_0) is a home marking iff it marks every proper trap of N.

Theorem 9.6 *Reachability Theorem*

Let (N, M_0) be a live and bounded free-choice system, which is moreover cyclic (i.e., M_0 is a home marking). A marking M is reachable from M_0 iff M and M_0 agree on all S-invariants and M marks every proper trap of N.

Theorem 9.17 *Shortest Sequence Theorem*

Let (N, M_0) be a live and b-bounded free-choice system with n transitions, and let M be a reachable marking. There exists an occurrence sequence $M_0 \xrightarrow{\sigma} M$ such that the length of σ is at most

$$b \cdot \frac{n \cdot (n+1) \cdot (n+2)}{6}$$