Reachability Theorem for T-systems.

Theorem Let (N, M_0) be a live T-system. A marking M is reachable iff $M_0 \sim M$ iff $\exists\, X \in \mathbb{Q}^{|T|} : M = M_0 + \underline{N} \cdot X$

Proof $(\Rightarrow)$ Holds in general.

$(\Leftarrow)$ Assume $\exists X \in \mathbb{Q}^{|T|} : M = M_0 + \underline{N} . X$

(a). We prove: $\exists Y \in \mathbb{Q}_{\geq 0}^{|T|} : M = M_0 + \underline{N} \cdot Y$

Let $J = \underbrace{(1, \ldots, 1)}_{|T|}$ and choose $\lambda > 0$ such that

$Y := X + \lambda J > 0$

We have $M_0 + \underline{N}\, Y$

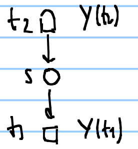
$= M_0 + \underline{N}\,(X + \lambda J)$

$= M_0 + \underline{N} X + \lambda \underbrace{\underline{N} . J}_{0.} \rightarrow J$ is T-invariant

$= M_0 + \underline{N}\, X$

(b) $\exists\, Z \in \mathbb{N}^{|T|} : M = M_0 + \underline{N} \cdot Z$

Let s be an arbitrary place



$M(s) = M_0(s) + Y(t_2) - Y(t_1)$

Since $M(s), M_0(s) \in \mathbb{N}^{|S|}$ we have

$Y(t_2) - Y(t_1) \in \mathbb{Z}$ and so

$\lceil Y(t_2) \rceil - \lceil Y(t_1) \rceil = Y(t_2) - Y(t_1)$

Let $Z := \lceil Y \rceil$. We have $M = M_0 + \underline{N} Y = M_0 + \underline{N} . Z$

(c) $M_0 \xrightarrow{*} M$

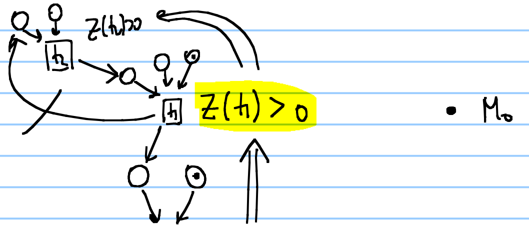
By induction on $|Z| = \sum_{t \in T} Z(t)$

Basis $|Z| = 0$ ✓

Step $|Z| > 0$.

Claim M_0 enables some transition t s.t. $Z(t) > 0$

Proof of the claim We use the "backwards construct" starting at a transition t_0 such that $Z(t_0) > 0$



The construction cannot "run into a circuit" because the T-system is live and so all circuits are marked at M_0. So the construction terminates with at least one transition t enabled at M_0. □

Let $M_0 \xrightarrow{t} M_1$ and $Z_1 = Z - \overset{\overset{(0 . 0 \overset{t}{1} 0 - 0)}{\shortparallel}}{e_t}$

then $M_0 + \underline{N}\, Z = M = M_1 + \underline{N} \cdot Z_1$

By IH $(|Z_1| = |Z| - 1)$ we have

$M_1 \xrightarrow{*} M$ and so $M_0 \xrightarrow{t} M_1 \xrightarrow{*} M$