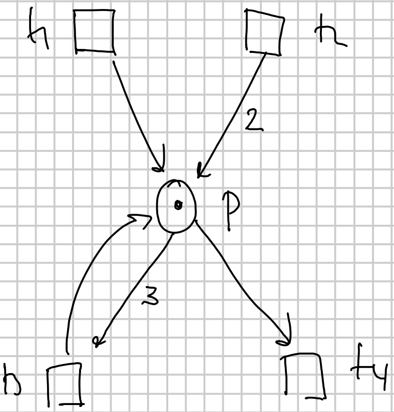
The marking equation

Let σ be a firing sequence

$\sigma = t_1\, t_2\, t_3\, t_1\, t_2\, t_3\, t_3\, t_2 \rightsquigarrow$ firing count vector

$X = (2, 3, 3, 0) \quad (\vec{\sigma})$



$X' = (2, 3, 2, 2, \underline{6, 1, 0})$

$$M(p) = 1 + 2 \cdot 1 + 3 \cdot 2 - 2 \cdot 2 - 2 \cdot 1 + 6 \cdot 0 + 1 \cdot 0 + 0 \cdot 0 = 3$$

$$M(p) = M_0(p) + X(t_1)\big(W(t_1, p) - W(p, t_1)\big) + X(t_2)\big(W(t_2, p) - W(p, t_2)\big) + \dots + X(t_7)\big(W(t_7, p) - W(p, t_7)\big)$$

"token conservation law"

$$M(p) = M_0(p) + \sum_{i=1}^{n} X(t_i) \underbrace{\big(W(t_i, p) - W(p, t_i)\big)}_{C(t_i, p)}$$

$$= M_0(p) + (C \cdot X)(p)$$

$$M = M_0 + C \cdot X \quad \leftarrow$$ marking equation

The marking equation for acyclic nets

Theorem Let $N = (P, T, F, W, M_0)$ be an acyclic net and let M be a marking of N. M is reachable **iff** there exists $X \in \mathbb{N}^{|T|}$ such that

$$M = M_0 + C \cdot X$$

where C is the incidence matrix of N.

Proof $\Rightarrow$ If $M_0 \xrightarrow{\sigma} M$ then take X as the firing count vector of σ

$\Leftarrow$ By induction on $K = \sum_{t \in T} X(t)$

Base $K = 0$. then $X = (0, 0, \ldots, 0)$ ✓

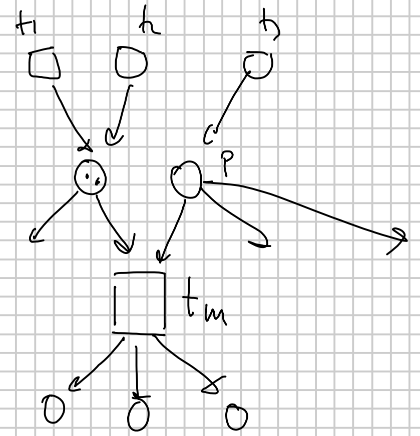
Step $K > 0$. Say $t_1 \leq t_2$ if there is a (possibly empty) path from t_1 to t_2.

Since N is acyclic, $\leq$ is a partial order

Let $\|X\|$ be the set of transitions t such that $X(t) > 0$

Let t_m be a minimal element of $\|X\|$ w.r.t $\leq$

We prove that M_0 enables t.



$X(t_1) = X(t_2) = X(t_3) = 0$

$X(t_m) > 0$

If M_0 does not enable t_m, then there is $p \in {}^\bullet t_m$ such that $M_0(p) = 0$. Since $X(t) = 0$ for every $t \in {}^\bullet p$ but $X(t_m) > 0$ we have $M(p) = M_0(p) + (C \cdot X)(p) < 0$, contradicting that M is a marking

So M_0 enables t_m. Let $M_0 \xrightarrow{t_m} M_1$

Consider the firing count vector X'

$$X'(t) = \begin{cases} X(t)-1 & \text{if } t = t_m \\ X(t) & \text{otherwise} \end{cases}$$

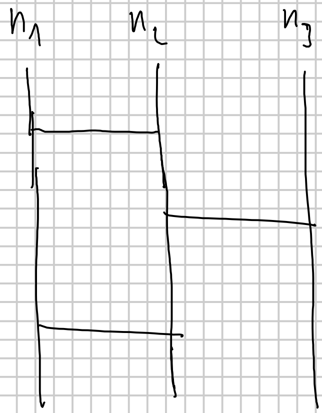
We have $M = M_1 + C \cdot X'$

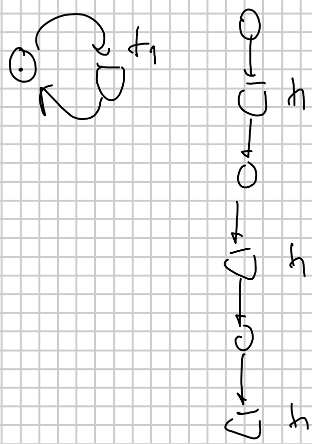
$$\begin{aligned} M = M_0 + C \cdot X &= M_0 + C \cdot (X' + e_{t_m}) \\ &= \underbrace{M_0 + C \cdot e_{t_m}}_{M_1} + C \cdot X' \\ &= M_1 + C \cdot X' \end{aligned}$$

Since $\sum_{t \in T} X'(t) < \sum_{t \in T} X(t)$ by induction hypothesis

there exists σ' with $M_1 \xrightarrow{\sigma'} M$

But then $M_0 \xrightarrow{t_m} M_1 \xrightarrow{\sigma'} M$ and so M is reachable $\boxtimes$





Theorem The reachability problem for acyclic Petri nets is NP-complete.

Proof Reachability is in NP

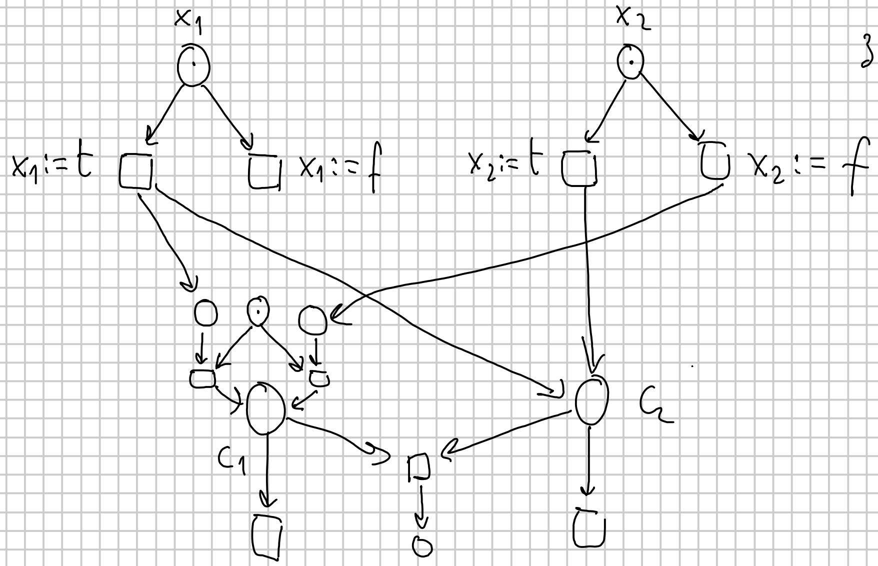
Checking if $M = M_0 + C \cdot X$ has some solution $X \in \mathbb{N}^{|T|}$ is in NP

Reachability is NP-hard.

By reduction from CNF-SAT

$$\phi = \overbrace{(x_1 \vee \neg x_2)}^{C_1} \wedge \overbrace{(x_1 \vee x_2)}^{C_2} \quad \longleftarrow$$

3 CNF-SAT



ϕ satisfiable $\iff$ Marking with exactly one token in $C_1, \ldots, C_n$ and 0 tokens elsewhere is reachable

Exercise Show NP-hardness of

Given: a 1-safe acyclic net N, a place p of N

Decide: is there a reachable marking M such that $M(p) = 1$?