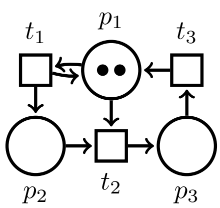
Petri nets — Revision Exercise Sheet

Exercise R.1

(From SS 2016, Exercise sheet 2)

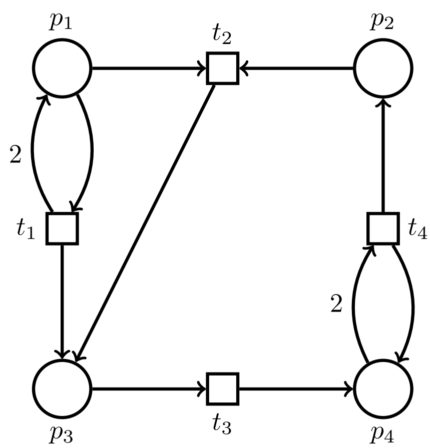
Apply the backwards reachability algorithm to the Petri net below to decide if the marking $M = (0, 0, 2)$ can be covered. Record all intermediate sets of markings with their finite representation of minimal elements.



Exercise R.2

(From SS 2018, Exercise sheet 5)

Consider the following Petri net (with weights) $\mathcal{N}$:



(a) Identify the traps of only one place. Identify the proper siphons of this net.
Hint: there are 4 siphons.

(b) Use siphons/traps to prove or disprove that $\mathcal{N}$ is live from $M_0 = \{p_2, 3 \cdot p_4\}$,

(c) Can the marking equation be used to prove or disprove that $\{p_2, p_4\} \xrightarrow{*} \{p_1, 3 \cdot p_2, p_3\}$? Is so, why? If not, can traps or siphons help?

Exercise R.3

(From SS 2016, Exercise sheet 3)

For a Petri net (N, M_0) and a transition t of N, we define liveness levels in the following way:

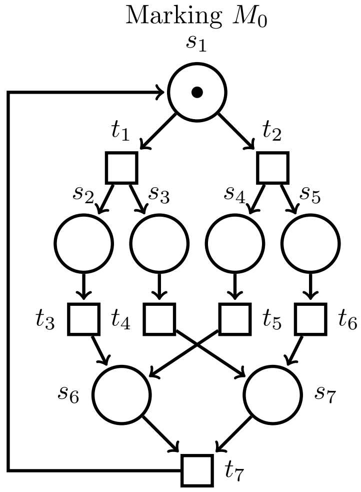
- t is L_0-live (or dead) if t occurs in no firing sequence σ of N enabled at M_0.
- t is L_1-live if t occurs in some firing sequence σ of N enabled at M_0.
- t is L_2-live if for any $k \in \mathbb{N}$, t occurs at least k times in some firing sequence σ of N enabled at M_0.
- t is L_3-live if t occurs infinitely often in some infinite firing sequence σ of N enabled at M_0.
- t is L_4-live if for any reachable marking M from M_0, t occurs in some firing sequence σ of N enabled at M, i.e. t can always fire again. *Note*: If this holds for all transitions, this coincides with our standard definition of liveness for Petri nets.

For each $i \in \{0, 1, 2, 3\}$, exhibit a Petri net (N, M_0) and a transition t of N such that t is L_i-live, but not L_{i+1}-live.

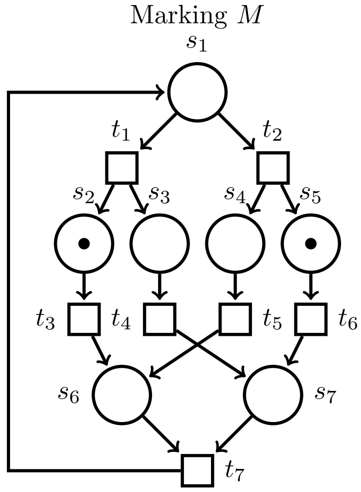
Exercise R.4
(From SS 2019, Exercise sheet 5)

1. Give a procedure that, given a net $\mathcal{N}$, constructs a boolean formula φ satisfying the following properties:
   - The formula contains variables r_s for each place $s \in S$,
   - if φ is satisfiable, then $\mathcal{N}$ has a trap,
   - and if φ is not satisfiable, then $\mathcal{N}$ has no trap.
   - Additionally, if A is a model of φ, then the set given by $R = \{s \mid A(r_s)\}$ is a trap of $\mathcal{N}$.
2. Apply your procedure to the Petri net on the left below and give the resulting constraints.
3. Adapt your procedure such that, given two marking M_0 and M, it adds additional constraints to ensure that any trap R obtained as a solution by the constraints is marked at M_0 and unmarked at M. The constraints should be satisfiable iff a trap marked at M_0 and unmarked at M exists.
4. Construct the constraints for the Petri net below with the markings M_0 and M.
5. Use your constraints and the trap property to show that M is not reachable from M_0 in the net below.

Marking M_0



Marking M

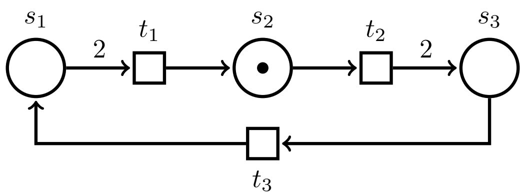


Exercise R.5

(From SS 2016, Exercise sheet 2)
Reduce the reachability problem for Petri nets with weighted arcs to the reachability problem for Petri nets without weighted arcs.

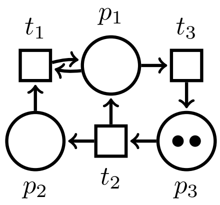
For that, describe an algorithm that, given a Petri net with weighted arcs $N = (S, T, W, M_0)$ and a marking M, constructs a Petri net $N' = (S', T', F', M_0')$ and a marking M' such that M is reachable in N if and only if M' is reachable in N'. The algorithm should run in polynomial time (you may assume unary encoding for the weights in the input, although it is also possible with a binary encoding).

Apply the algorithm to the Petri net below with the target marking $M = (2, 0, 0)$ and give the resulting Petri net N' and marking M'.



Solution R.1

For the backwards reachability algorithm, it can be helpful to construct the reverse net, with all arcs inverted, to compute the predecessors of markings. Then start with the target marking M and add tokens as necessary for firing transitions.



We start with $m_0 = \{(0,0,2)\}$, which is the set of minimal elements for $\{M\}$. Recall that the backwards reachability algorithm iteratively updates m_i to

$$m_{i+1} = min(m_i \cup \bigcup_{t \in T} pre(m_i \wedge R[t], t)).$$

We have $R[t_1] = (1,1,0)$, $R[t_2] = (0,0,1)$ and $R[t_3] = (1,0,0)$.

$$\begin{aligned}
pre((0,0,2) \wedge R[t_1], t_1) &= pre((1,1,2), t_1) = (1,0,2)\\
pre((0,0,2) \wedge R[t_2], t_2) &= (1,1,1)\\
pre((0,0,2) \wedge R[t_3], t_3) &= pre((1,0,2), t_3) = (0,0,3)
\end{aligned}$$

After adding the new markings to m_0 and eliminating non-minimal markings, our new set is $m_1 = \{(0,0,2),(1,1,1)\}$. For the new marking $(1,1,1)$, we compute the predecessors:

$$\begin{aligned}
pre((1,1,1), t_1) &= (1,0,1)\\
pre((1,1,1), t_2) &= (2,2,0)\\
pre((1,1,1), t_3) &= (0,1,2)
\end{aligned}$$

We add the new markings, take the minimal elements and obtain $m_2 = \{(0,0,2),(1,0,1),(2,2,0)\}$. For $(1,0,1)$ and $(2,2,0)$, we compute the predecessors:

$$\begin{aligned}
pre((1,0,1) \wedge R[t_1], t_1) &= pre((1,1,1), t_1) = (1,0,1)\\
pre((1,0,1), t_2) &= (2,1,0)\\
pre((1,0,1), t_3) &= (0,0,2)\\
pre((2,2,0), t_1) &= (2,1,0)\\
pre((2,2,0) \wedge R[t_2], t_2) &= pre((2,2,1), t_2) = (3,3,0)\\
pre((2,2,0), t_3) &= (1,2,1)
\end{aligned}$$

The new minimal marking set is now $m_3 = \{(0,0,2),(1,0,1),(2,1,0)\}$. Next we compute the predecessors for $(2,1,0)$:

$$\begin{aligned}
pre((2,1,0), t_1) &= (2,0,0)\\
pre((2,1,0) \wedge R[t_2], t_2) &= pre((2,1,1), t_2) = (3,2,0)\\
pre((2,1,0), t_3) &= (1,1,1)
\end{aligned}$$

The new set is $m_4 = \{(0,0,2),(1,0,1),(2,0,0)\}$. For $M' = (2,0,0) \in m_4$, we have $M_0 \geq M'$, therefore we can conclude that M is coverable from M_0.

If instead we would continue, we would compute the predecessors for $(2,0,0)$:

$$\begin{aligned}
pre((2,0,0) \wedge R[t_1], t_1) &= pre((2,1,0), t_1) = (2,0,0)\\
pre((2,0,0) \wedge R[t_2], t_2) &= pre((2,0,1), t_2) = (3,1,0)\\
pre((2,0,0), t_3) &= (1,0,1)
\end{aligned}$$

No new minimal markings are obtained, so we would have reached a fixpoint.

Solution R.2

(a) The only singleton trap is $\{p_4\}$. Siphons are defined as $R \subseteq P$ such that ${}^\bullet R \subseteq R^\bullet$. This implies constraints for any siphon R: $p_2 \in R \Rightarrow p_4, p_4 \in R \Rightarrow p_3$ and $p_3 \in R \Rightarrow p_1$. Using these, the siphons of the net are $\{p_1\}, \{p_1, p_3\}, \{p_1, p_3, p_4\}$ and $\{p_1, p_2, p_3, p_4\}$.

(b) Let $Q = \{p_1, p_3\}$. Q is a siphon. Since Q is not marked by M_0, we conclude that $(\mathcal{N}, M_0)$ is not live (by Prop 4.4.4 of the lecture notes). □

(c) The incidence matrix of $\mathcal{N}$ is:

$$\boldsymbol{N} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The marking equation is:

$$\begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \boldsymbol{x}.$$

The unique solution of the marking equation is

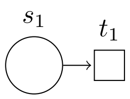
$$\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Since it is non negative, we cannot conclude whether the marking is reachable or not.

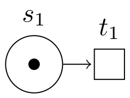
Let us consider the trap $Q = \{p_4\}$ which is marked by the initial marking. We can conclude that the target marking is not reachable since it does not mark Q.

Solution R.3

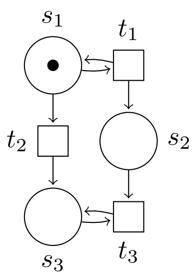
- In the following net, t_1 is L_0-live (dead). As t_1 is dead, it cannot be L_i-live for $i \geq 1$.



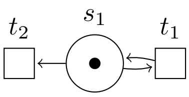
- In the following net, t_1 is L_1-live, as it is enabled at M_0, but not L_1-live, as it can fire at most once.



- In the following net, t_3 is L_2-live, as for any $k \in \mathbb{N}$, it occurs k times in the firing sequence $\sigma = t_1^k t_2 t_3^k$. However, it is not L_3-live, as it needs to be enabled by the occurrence of t_2, after which t_1 cannot occur anymore, so the number of times t_3 can occur is limited by the number of tokens in s_2 after the occurrence of t_2.



- In the following net, t_3 is L_3-live, as it occurs infinitely often in $\sigma = t_1 t_1 t_1 \ldots$. However, it is not L_4-live, as it is disabled by the occurrence of t_2.



Solution R.4

1. Any trap R satisfies $R^\bullet \subseteq {}^\bullet R$ and therefore $\forall t \in T : \exists s \in {}^\bullet t : s \in R \implies \exists s' \in t^\bullet : s' \in R$. This can be encoded with the following formula, which can be unrolled for a given net N:

$$\bigwedge_{t \in T} \left(\left(\bigvee_{s \in {}^\bullet t} r_s \right) \implies \left(\bigvee_{s' \in t^\bullet} r'_s \right) \right)$$

   Any assignment satisfying the formula gives rise to a set R which satisfies the trap condition and is therefore a trap.

2. The constraints are as follows:

$$\begin{aligned} r_{s_1} &\implies r_{s_2} \vee r_{s_3} \\ r_{s_1} &\implies r_{s_4} \vee r_{s_5} \\ r_{s_2} &\implies r_{s_6} \\ r_{s_3} &\implies r_{s_7} \\ r_{s_4} &\implies r_{s_6} \\ r_{s_5} &\implies r_{s_7} \\ r_{s_6} \vee r_{s_7} &\implies r_{s_1} \end{aligned}$$

3. To ensure that the trap is marked at M_0 and unmarked at M, we can add the following constraint:

$$\left(\bigvee_{s \in S : M_0(s) > 0} r_s \right) \wedge \left(\bigwedge_{s \in S : M(s) > 0} \neg r_s \right)$$

4. The additional constraints are:

$$r_{s_1} \wedge (\neg r_{s_2} \wedge \neg r_{s_5})$$

5. We obtain a satisying assignment A for the constraints by setting $A(r_{s_1}) = A(r_{s_3}) = A(r_{s_4}) = A(r_{s_6}) = A(r_{s_7}) = 1$ and $A(r_{s_2}) = A(r_{s_5}) = 0$. The trap obtained from these constraints is $R = \{s_1, s_3, s_4, s_6, s_7\}$. As the trap is marked at M_0, it needs to stay marked in any reachable marking, therefore the marking M is not reachable.

Solution R.5

We use the following approach: We replace each place in the Petri net with weighted arcs with a ring of places in the Petri net without weighted arcs. The size of the ring is given by the maximum input or output weight, and the sum of the tokens in the ring represent the number of tokens in the original place. The tokens can move around freely in the ring, and a transition with a weighted arc that puts or takes k tokens into or out of the original place is now connected with unweighted arcs to k of the places in the ring.

Formally, let $S = \{s_1, \ldots, s_n\}$ be the places of the Petri net N. For each $s_i \in S$, define an integer k_i by

$$k_i := \max(\{W(t, s_i) \mid t \in T\} \cup \{W(s_i, t) \mid t \in T\})$$

The places in the Petri net N' are the sets of ring places for each original place. The transitions are the original transition, plus a fresh set of ring transitions.

$$S' = \bigcup_{s_i \in S} \{s_{i,j} \mid 1 \leq j \leq k_i\} \qquad\qquad T' = T \uplus \{t_{s_{i,j}} \mid s_{i,j} \in S'\}$$

The flow relation connects each transition t to a number of ring places $s_{i,j}$ given by the weight between t and s_i. We also connect the ring places and transitions cyclically.

$$\begin{aligned} F' = &\{(s_{i,j}, t) \in S' \times T \mid W(s_i, t) \geq j\} \ \cup \ \{(t, s_{i,j}) \in T \times S' \mid W(t, s_i) \geq j\} \ \cup \\ &\{(s_{i,j}, t_{s_{i,j}}) \mid s_{i,j} \in S'\} \ \cup \ \{(t_{s_{i,j}}, t_{s_{i,1+(j \bmod k_i)}}) \mid s_{i,j} \in S'\} \end{aligned}$$

The initial and target marking are given by having all the tokens in the first place of each ring.

$$M'_0(s_{i,j}) = \begin{cases} M_0(s_i) & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \qquad M'(s_{i,j}) = \begin{cases} M(s_i) & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

By construction, if M is reachable in N, then there is a reachable marking M' in N' with $\sum_j M'(s_{i,j}) = M(s_i)$ for all $s_i \in S$. We can use the ring transitions to move all tokens in $s_{i,j}$ to $s_{i,1}$ to reach the target marking.

Applying the construction to the Petri net and marking above gives us the Petri net below, along with the marking M' given by $M'(s_{1,1}) = 2$ and $M'(s) = 0$ for all places $s \neq s_{1,1}$.