Technische Universität München **Summer Term 2019**
I7
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Petri nets — Exercise Sheet 6

Exercise 6.1

(a) Prove: If (N, M_0) is a live S-system and $M_0' \geq M_0$, then (N, M_0') is also live.

(b) Prove: If (N, M_0) is a live T-system and $M_0' \geq M_0$, then (N, M_0') is also live.

(c) Give an S-system $(\mathcal{N}, M_0)$ that is 1-bounded and such that $|M_0| > 1$.

(d) Give a strongly connected T-system $(\mathcal{N}, M_0)$ which is not live and such that $M_0 \neq \mathbf{0}$.

(e) Let $(\mathcal{N}, M_0)$ be a T-system. Show that if $(\mathcal{N}, M_0)$ is strongly connected and live, then it is bounded.

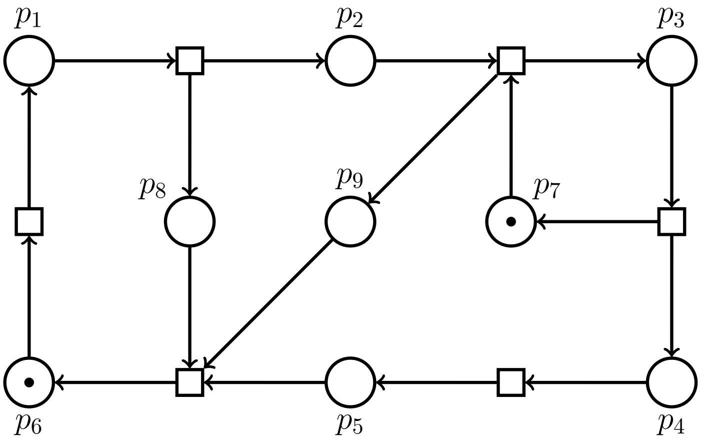
(f) Reprove (e), but this time without assuming that $(\mathcal{N}, M_0)$ is live.

Exercise 6.2

(a) Show that the problem of determining whether a T-system is *not live* belongs to NP.

(b) Give a polynomial time algorithm for deciding liveness of T-systems.

(c) Test whether the following T-system is live by using your previous algorithm:



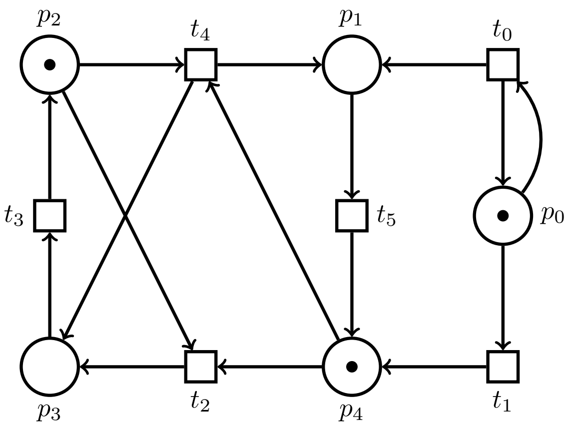
Exercise 6.3

For each $n \in \mathbb{N}$, give a 1-bounded T-system (N, M_0) with n transitions and a reachable marking M such that the minimal occurrence sequence σ with $M_0 \xrightarrow{\sigma} M$ has a length of $\frac{n(n-1)}{2}$.

Hint: First try find a Petri net and a marking for $n = 3$, where the minimal sequence has length 3. For this a net with 4 places suffices. Then try to generalize your solution.

Exercise 6.4

Consider the following free-choice system $(\mathcal{N}, M_0)$:



(a) Give all minimal proper siphons of $(\mathcal{N}, M_0)$.

(b) Use (a) to say whether $(\mathcal{N}, M_0)$ is live or not.

Exercise 6.5

(a) Let $\mathcal{N} = (P, T, F)$ be a Petri net, and let $s, t \in T$ be such that ${}^\bullet s \cap t^\bullet = \emptyset$. Show that if $M \xrightarrow{ts} M'$, then $M \xrightarrow{st} M'$.

(b) Let $\mathcal{N} = (P, T, F)$ be a Petri net which is not strongly connected. Show that $P \cup T$ can be partitioned into two disjoint sets $U, V \subseteq P \cup T$ such that $F \cap (V \times U) = \emptyset$.

(c) Let U and V be a partition as in (b). Show that if $M \xrightarrow{\sigma} M'$, then there exist $\sigma_U \in (T \cap U)^*$ and $\sigma_V \in (T \cap V)^*$ such that $\sigma = \sigma_U \sigma_V$ and $M \xrightarrow{\sigma_U \sigma_V} M'$.

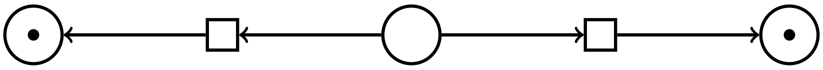
(d) Let $(\mathcal{N}, M_0)$ be live and bounded. Use (a), (b) and (c) to show that $\mathcal{N}$ is strongly connected.

Solution 6.1

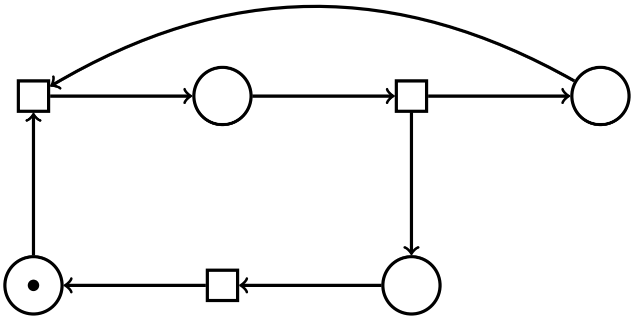
(a) By the Liveness Theorem for S-systems, (N, M_0) is live iff N is strongly connected and $M_0(S) > 0$, and as $M_0'(S) \geq M_0(S) > 0$, (N, M_0') is also live.

(b) By the Liveness Theorem for T-systems, (N, M_0) is live iff $M_0(\gamma) > 0$ for every circuit γ, and as $M_0'(\gamma) \geq M_0(\gamma) > 0$, (N, M_0') is also live.

(c)



(d)



(e) Let $\mathcal{N} = (P, T, F)$. Let $b = |M_0|$. We show that every place is b-bounded. Let $p \in P$. Since $\mathcal{N}$ is strongly connected, p lies on some circuit γ. Note that $M_0(\gamma) \leq b$ and that $(\mathcal{N}, M_0)$ is live. Therefore, by Theorem 5.2.4, p is b-bounded. ☐

(f) Let $\mathcal{N} = (P, T, F)$. Let $b = |M_0|$. We show that every place is b-bounded. Let $p \in P$. Since $\mathcal{N}$ is strongly connected, p lies on some circuit γ. By Proposition 5.2.2, for every reachable marking M, $M(\gamma) = M_0(\gamma)$. So there can be no reachable marking M in which $M(p) > b$ and p is b-bounded.

Solution 6.2

(a) By Theorem 5.2.3, $(\mathcal{N}, M_0)$ is not live if and only if $M_0(\gamma) = 0$ for some circuit γ. Note that every cycle γ contains a simple cycle γ'. Moreover, if $M_0(\gamma) = 0$, then $M_0(\gamma') = 0$. This implies that,

$$(\mathcal{N}, M_0) \text{ is not live } \iff M_0(\gamma) = 0 \text{ for some simple circuit } \gamma.$$

Therefore, to test whether $(\mathcal{N}, M_0)$ is not live, it suffices to test a circuit γ of size at most $|P \cup T|$ and check whether $M_0(\gamma) = 0$. ☐

(b) Since a graph may contain exponentially many simple cycles, we cannot directly use the approach of (a). Instead, we construct the subnet $\mathcal{N}'$ obtained from $\mathcal{N}$ by removing all places containing tokens. We then perform depth-first search to test whether $\mathcal{N}'$ contains a cycle. This procedure can be implemented as follows:

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Input: $T$-system $(\mathcal{N}, M_0)$ where $\mathcal{N} = (P, T, F)$
Output: $(\mathcal{N}, M_0)$ live?
while $\exists p \in P$ such that $\neg$visited($p$) and $M_0(p) = 0$ do
    if has-cycle($p$) then return false
return true

has-cycle($p$):
    visited($p$) $\leftarrow$ true
    onstack($p$) $\leftarrow$ true

    for $q \in (p^\bullet)^\bullet$ such that $M_0(q) = 0$ do
        if onstack($q$) then
            return true
        else if $\neg$visited($q$) then
            if has-cycle($q$) then return true

    onstack($p$) $\leftarrow$ false
    return false
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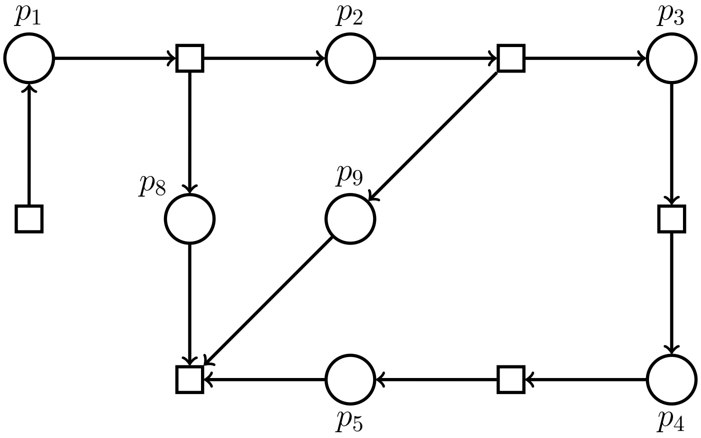
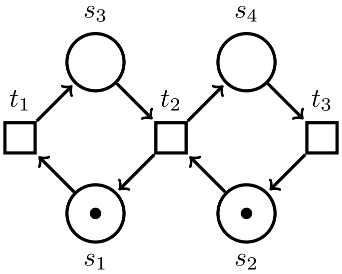
(c) We obtain the following subnet:



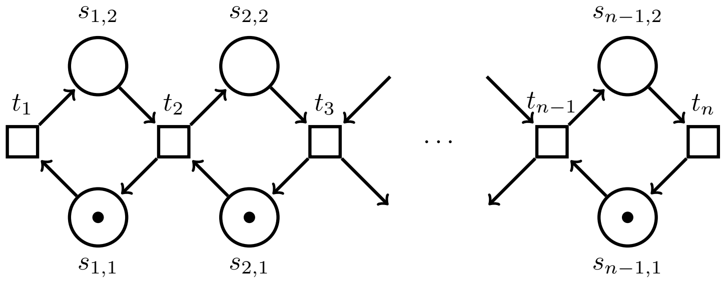
A depth-first search shows that this subnet contains no cycle. Therefore, the system is live.

Solution 6.3

For $n = 3$, we can take the following net with the marking $M = (0, 0, 1, 1)$. To reach this marking, we need to fire t_1 and t_2 to mark s_3 and s_4. However, firing t_2 undoes the effect of t_1 on s_3, so we need to fire t_1 twice. The minimal sequence is then $\sigma = t_1 t_2 t_1$ of length 3.



This construction can be repeated for arbitrary n, as shown in the following sketch of a Petri net. To reach the marking M with $M(s_{i,1}) = 0$ and $M(s_{i,2}) = 1$ for all $1 \leq i \leq n-1$ with a minimal sequence, we need to fire $\sigma = t_1 t_2 \dots t_{n-1}\ t_1 t_2 \dots t_{n-2}\ \dots\ t_1$, which has a length of $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$.



Solution 6.4

(a) We claim that the system has two minimal proper siphons: $\{p_0\}$ and $\{p_2, p_3\}$.

Let us show the claim. By inspecting ${}^\bullet p$ and $p^\bullet$ for every place p, we find a single siphon of size one: $\{p_0\}$. Moreover, we have ${}^\bullet\{p_2, p_3\} = \{t_2, t_3, t_4\} = \{p_2, p_3\}^\bullet$. Now, note that $t_0 \in {}^\bullet p_1$ and ${}^\bullet t_0 = \{p_0\}$. Therefore, any siphon containing p_1 must also contain p_0. Similarly, any siphon containing p_4 must also contain p_0. Thus, no minimal siphon contains p_1 or p_4, and we are done. ☐

(b) The system is not live. By Commoner's Theorem, the system is live if and only if every minimal proper siphon contains a trap marked at M_0. The minimal siphon $\{p_2, p_3\}$ is also a trap and it is marked at M_0. However, the minimal siphon $\{p_0\}$ is not a trap and hence it does not contain a marked trap.

Solution 6.5

(a) Let $X \in \mathbb{N}^P$ be such that $M \xrightarrow{t} X \xrightarrow{s} M'$. For the sake of contradiction, suppose s is not enabled in M. There exists $p \in P$ such that $p \in {}^\bullet s$ and $M(p) = 0$. Since s is enabled in X, we have $X(p) > 0$. Therefore, it must be the case that $p \in t^\bullet$. This implies that $p \in {}^\bullet s \cap t^\bullet$ which is a contradiction. Thus, s is enabled in M and $M \xrightarrow{s} Y$ for some marking $Y \in \mathbb{N}^P$.

Let us now show that t is enabled in Y. Let $q \in {}^\bullet t$. We must show that $Y(q) > 0$.

Case 1: $q \notin {}^\bullet s$. If $q \notin {}^\bullet s$, then $Y(q) \geq M(q) > 0$.

Case 2: $q \in {}^\bullet s$. If $q \in {}^\bullet s$, then

$$Y(q) = M(q) - 1. \tag{1}$$

Since s is enabled in X, we have $X(q) > 0$. Moreover, $q \notin t^\bullet$ since ${}^\bullet s \cap t^\bullet = \emptyset$. This implies that $M(q) > X(q)$, and hence $M(q) \geq 2$. By (1), we derive $Y(q) \geq 1$. ☐

(b) Since $\mathcal{N}$ is not strongly connected, there exist $u, v \in P \cup T$ such that there is no path from v to u. Let

$$U = \{x \in P \cup T : \text{there is a path from } x \text{ to } u\},$$
$$V = (P \cup T) \setminus U.$$

Note that both sets are non empty since $u \in U$ and $v \in V$. Moreover, $U \cap V = \emptyset$ and $U \cup V = P \cup T$ by definition.

Let us show that $F \cap (V \times U) = \emptyset$. Assume there exists $e \in F \cap (V \times U)$. There exist $x \in U$ and $y \in V$ such that $(y, x) \in F$. Since $x \in U$, there exists a path σ from x to u. Therefore, $(y, x)\sigma$ is a path from y to u. This implies that $y \in U$ which is a contradiction. ☐

(c) Let $U' = T \cap U$ and $V' = T \cap V$. Let us first show that ${}^\bullet(U') \cap (V')^\bullet = \emptyset$. For the sake of contradiction, assume there exist $s \in V'$, $t \in U'$ and $q \in P$ such that $q \in s^\bullet$ and $q \in {}^\bullet t$. We have $(s, q) \in F$ and $(q, t) \in F$. If $q \in U$, then by (b) and $(s, q) \in F$, we obtain a contradiction. Similarly, if $q \in V$, then $(q, t) \in F$ yields a contradiction.

We now prove the claim by induction of $|\sigma|$. If $|\sigma| = 0$, it follows trivially. Assume that $|\sigma| > 0$ and that the claim holds for firing sequences of length $|\sigma| - 1$. There exist $\sigma' \in T^*$, $s \in T$ and $Y \in \mathbb{N}^P$ such that $\sigma = \sigma' s$ and

$$M \xrightarrow{\sigma'} X \xrightarrow{s} M'.$$

By induction hypothesis, there exists $\pi_U \in (U')^*$ and $\pi_V \in (V')^*$ such that $M \xrightarrow{\pi_U \pi_V} X$. If $s \in V'$ or $|\pi_V| = 0$, then we are done. Otherwise, let $\pi'_V \in (V')^*$ and $t \in V'$ be such that $\pi_V = \pi'_V t$. Since ${}^\bullet(U') \cap (V')^\bullet = \emptyset$, we can apply (a) and obtain

$$M \xrightarrow{\pi_U \pi'_V s} Y \xrightarrow{t} M'$$

for some $Y \in \mathbb{N}^P$. By induction hypothesis, there exist $\gamma_U \in (U')^*$ and $\gamma_V \in (V')^*$ such that

$$M \xrightarrow{\gamma_U \gamma_V} Y.$$

Let $\sigma_U = \gamma_U$ and $\sigma_V = \gamma_V t$. We are done since $\sigma_U \in (U')^*$, $\sigma_V \in (V')^*$ and $M \xrightarrow{\sigma_U \sigma_V} M'$. ☐

(d) Let $\mathcal{N} = (P, T, F)$. For the sake of contradiction, assume $\mathcal{N}$ is not strongly connected. By (b), there exists a partition U, V of $P \cup T$ such that $F \cap (V \times U) = \emptyset$. Since $\mathcal{N}$ is connected, there exist $u \in U$ and $v \in V$ such that $(u, v) \in F$. Let $b \in \mathbb{N}$ be such that $(\mathcal{N}, M_0)$ is b-bounded. Since $(\mathcal{N}, M_0)$ is live, there exist $\sigma \in T^*$ and $M \in N^P$ such that $M_0 \xrightarrow{\sigma} M$ and (u, v) is taken $b+1$ times. By (c), there exist $\sigma_U \in U^*$ and $\sigma_V \in V^*$ such that $M_0 \xrightarrow{\sigma_U \sigma_V} M$. Let $X \in \mathbb{N}^P$ be such that $M_0 \xrightarrow{\sigma_U} X \xrightarrow{\sigma_V} M$.

Case 1: $u \in P$, $v \in T$. Since $F \cap (V \times U) = \emptyset$, there is no transition of V that puts tokens into places of U. Note that v decreases the amount of token of u by 1. Since $X \xrightarrow{\sigma_V} M$, these two observations imply that $X(u) \geq b + 1$. As X is reachable from M_0, this contradicts $(\mathcal{N}, M_0)$ being b-bounded.

Case 2: $u \in T$, $v \in P$. Since $F \cap (V \times U) = \emptyset$, there is no transition of U that consumes tokens from places of V. Note that u increases the amount of token of u by 1. Since $M_0 \xrightarrow{\sigma_U} X$, these two observations imply that $X(u) \geq b + 1$. This contradicts $(\mathcal{N}, M_0)$ being b-bounded. ☐