Petri nets — Exercise Sheet 6

Exercise 6.1

(a) Prove: If \((N, M_0)\) is a live S-system and \(M'_0 \geq M_0\), then \((N, M'_0)\) is also live.
(b) Prove: If \((N, M_0)\) is a live T-system and \(M'_0 \geq M_0\), then \((N, M'_0)\) is also live.
(c) Give an S-system \((N, M_0)\) that is 1-bounded and such that \(|M_0| > 1\).
(d) Give a strongly connected T-system \((N, M_0)\) which is not live and such that \(M_0 \neq 0\).
(e) Let \((N, M_0)\) be a T-system. Show that if \((N, M_0)\) is strongly connected and live, then it is bounded.
(f) Reprove (e), but this time without assuming that \((N, M_0)\) is live.

Exercise 6.2

(a) Show that the problem of determining whether a T-system is not live belongs to NP.
(b) Give a polynomial time algorithm for deciding liveness of T-systems.
(c) Test whether the following T-system is live by using your previous algorithm:

Exercise 6.3

For each \(n \in \mathbb{N}\), give a 1-bounded T-system \((N, M_0)\) with \(n\) transitions and a reachable marking \(M\) such that the minimal occurrence sequence \(\sigma\) with \(M_0 \xrightarrow{\sigma} M\) has a length of \(\frac{n(n-1)}{2}\).

Hint: First try find a Petri net and a marking for \(n = 3\), where the minimal sequence has length 3. For this a net with 4 places suffices. Then try to generalize your solution.
Exercise 6.4
Consider the following free-choice system \((\mathcal{N}, M_0)\):

(a) Give all minimal proper siphons of \((\mathcal{N}, M_0)\).

(b) Use (a) to say whether \((\mathcal{N}, M_0)\) is live or not.

Exercise 6.5

(a) Let \(\mathcal{N} = (P, T, F)\) be a Petri net, and let \(s, t \in T\) be such that \(*s \cap t^* = \emptyset\). Show that if \(M \xrightarrow{ts} M'\), then \(M \xrightarrow{st} M'\).

(b) Let \(\mathcal{N} = (P, T, F)\) be a Petri net which is not strongly connected. Show that \(P \cup T\) can be partitioned into two disjoint sets \(U, V \subseteq P \cup T\) such that \(F \cap (V \times U) = \emptyset\).

(c) Let \(U\) and \(V\) be a partition as in (b). Show that if \(M \xrightarrow{\sigma} M'\), then there exist \(\sigma_U \in (T \cap U)^*\) and \(\sigma_V \in (T \cap V)^*\) such that \(\sigma = \sigma_U \sigma_V\) and \(M \xrightarrow{\sigma_U \sigma_V} M'\).

(d) Let \((\mathcal{N}, M_0)\) be live and bounded. Use (a), (b) and (c) to show that \(\mathcal{N}\) is strongly connected.
Solution 6.1

(a) By the Liveness Theorem for S-systems, \((N, M_0)\) is live iff \(N\) is strongly connected and \(M_0(S) > 0\), and as \(M'_0(S) \geq M_0(S) > 0\), \((N, M'_0)\) is also live.

(b) By the Liveness Theorem for T-systems, \((N, M_0)\) is live iff \(M_0(\gamma) > 0\) for every circuit \(\gamma\), and as \(M'_0(\gamma) \geq M_0(\gamma) > 0\), \((N, M'_0)\) is also live.

(c) ![Diagram](image)

(d) ![Diagram](image)

(e) Let \(\mathcal{N} = (P, T, F)\). Let \(b = |M_0|\). We show that every place is \(b\)-bounded. Let \(p \in P\). Since \(\mathcal{N}\) is strongly connected, \(p\) lies on some circuit \(\gamma\). Note that \(M_0(\gamma) \leq b\) and that \((\mathcal{N}, M_0)\) is live. Therefore, by Theorem 5.2.4, \(p\) is \(b\)-bounded.

(f) Let \(\mathcal{N} = (P, T, F)\). Let \(b = |M_0|\). We show that every place is \(b\)-bounded. Let \(p \in P\). Since \(\mathcal{N}\) is strongly connected, \(p\) lies on some circuit \(\gamma\). By Proposition 5.2.2, for every reachable marking \(M\), \(M(\gamma) = M_0(\gamma)\). So there can be no reachable marking \(M\) in which \(M(p) > b\) and \(p\) is \(b\)-bounded.

Solution 6.2

(a) By Theorem 5.2.3, \((\mathcal{N}', M_0)\) is not live if and only if \(M_0(\gamma) = 0\) for some circuit \(\gamma\). Note that every cycle \(\gamma\) contains a simple cycle \(\gamma'\). Moreover, if \(M_0(\gamma) = 0\), then \(M_0(\gamma') = 0\). This implies that,

\[(\mathcal{N}', M_0)\] is not live \(\iff\) \(M_0(\gamma) = 0\) for some simple circuit \(\gamma\).

Therefore, to test whether \((\mathcal{N}', M_0)\) is not live, it suffices to test a circuit \(\gamma\) of size at most \(|P \cup T|\) and check whether \(M_0(\gamma) = 0\).

(b) Since a graph may contain exponentially many simple cycles, we cannot directly use the approach of (a). Instead, we construct the subnet \(\mathcal{N}'\) obtained from \(\mathcal{N}\) by removing all places containing tokens. We then perform depth-first search to test whether \(\mathcal{N}'\) contains a cycle. This procedure can be implemented as follows:
**Input:** T-system \((\mathcal{N}, M_0)\) where \(\mathcal{N} = (P, T, F)\)

**Output:** \((\mathcal{N}, M_0)\) live?

```plaintext
while \(\exists p \in P\) such that \(\neg\text{visited}(p)\) and \(M_0(p) = 0\) do
  if has-cycle\(p\) then return false
return true
```

```plaintext
has-cycle\(p\):
  visited\(p\) \(\leftarrow\) true
  onstack\(p\) \(\leftarrow\) true

  for \(q \in (p)^*\) such that \(M_0(q) = 0\) do
    if onstack\(q\) then
      return true
    else if \(\neg\text{visited}(q)\) then
      if has-cycle\(q\) then return true

  onstack\(p\) \(\leftarrow\) false
return false
```

(c) We obtain the following subnet:

![Petri Net Diagram]

A depth-first search shows that this subnet contains no cycle. Therefore, the system is live.

**Solution 6.3**

For \(n = 3\), we can take the following net with the marking \(M = (0, 0, 1, 1)\). To reach this marking, we need to fire \(t_1\) and \(t_2\) to mark \(s_3\) and \(s_4\). However, firing \(t_2\) undoes the effect of \(t_1\) on \(s_3\), so we need to fire \(t_1\) twice. The minimal sequence is then \(\sigma = t_1t_2t_1\) of length 3.

![Net Diagram for Solution 6.3]

This construction can be repeated for arbitrary \(n\), as shown in the following sketch of a Petri net. To reach the marking \(M\) with \(M(s_{i,1}) = 0\) and \(M(s_{i,2}) = 1\) for all \(1 \leq i \leq n - 1\) with a minimal sequence, we need to fire \(\sigma = t_1t_2\ldots t_{n-1} t_1t_2\ldots t_{n-2} \ldots t_1\), which has a length of \(\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}\).
Solution 6.4
(a) We claim that the system has two minimal proper siphons: \{p_0\} and \{p_2, p_3\}.

Let us show the claim. By inspecting \(\cdot p\) and \(p^*\) for every place \(p\), we find a single siphon of size one: \(\{p_0\}\). Moreover, we have \(\cdot \{p_2, p_3\} = \{t_2, t_3, t_4\} = \{p_2, p_3\}^*\). Now, note that \(t_0 \in \cdot p_1\) and \(\cdot t_0 = \{p_0\}\). Therefore, any siphon containing \(p_1\) must also contain \(p_0\). Similarly, any siphon containing \(p_4\) must also contain \(p_0\). Thus, no minimal siphon contains \(p_1\) or \(p_4\), and we are done.

(b) The system is not live. By Commoner’s Theorem, the system is live if and only if every minimal proper siphon contains a trap marked at \(M_0\). The minimal siphon \(\{p_2, p_3\}\) is also a trap and it is marked at \(M_0\). However, the minimal siphon \(\{p_0\}\) is not a trap and hence it does not contain a marked trap.

Solution 6.5
(a) Let \(X \in \mathbb{N}^P\) be such that \(M \overset{t}{\rightarrow} X \overset{\sigma}{\rightarrow} M'\). For the sake of contradiction, suppose \(s\) is not enabled in \(M\). There exists \(p \in P\) such that \(p \in \cdot s\) and \(M(p) = 0\). Since \(s\) is enabled in \(X\), we have \(X(p) > 0\). Therefore, it must be the case that \(p \in \cdot t^*\). This implies that \(p \in \cdot s \cap \cdot t^*\) which is a contradiction. Thus, \(s\) is enabled in \(M\) and \(M_0 \overset{\sigma}{\rightarrow} Y\) for some marking \(Y \in \mathbb{N}^P\).

Let us now show that \(t\) is enabled in \(Y\). Let \(q \in \cdot t\). We must show that \(Y(q) > 0\).

Case 1: \(q \not\in \cdot s\). If \(q \not\in \cdot s\), then \(Y(q) \geq M(q) > 0\).

Case 2: \(q \in \cdot s\). If \(q \in \cdot s\), then

\[
Y(q) = M(q) - 1. \tag{1}
\]

Since \(s\) is enabled in \(X\), we have \(X(q) > 0\). Moreover, \(q \not\in \cdot t^*\) since \(\cdot s \cap \cdot t^* = \emptyset\). This implies that \(M(q) \geq 2\). By (1), we derive \(Y(q) \geq 1\).

(b) Since \(\mathcal{N}\) is not strongly connected, there exist \(u, v \in P \cup T\) such that there is no path from \(v\) to \(u\). Let \(U = \{x \in P \cup T : \text{there is a path from } x \text{ to } u\}\), \(V = (P \cup T) \setminus U\).

Note that both sets are non-empty since \(u \in U\) and \(v \in V\). Moreover, \(U \cap V = \emptyset\) and \(U \cup V = P \cup T\) by definition.

Let us show that \(F \cap (V \times U) = \emptyset\). Assume there exists \(e \in F \cap (V \times U)\). There exist \(x \in U\) and \(y \in V\) such that \((y, x) \in F\). Since \(x \in U\), there exists a path \(\sigma\) from \(x\) to \(u\). Therefore, \((y, x)\sigma\) is a path from \(y\) to \(u\). This implies that \(y \in U\) which is a contradiction.

(c) Let \(U' = T \cap U\) and \(V' = T \cap V\). Let us first show that \(\cdot (U') \cap (V') = \emptyset\). For the sake of contradiction, assume there exist \(s \in V'\), \(t \in U'\) and \(q \in P\) such that \(q \in \cdot s\) and \(q \in \cdot t\). We have \((s, q) \in F\) and \((q, t) \in F\). If \(q \in U\), then by (b) and \((s, q) \in F\), we obtain a contradiction. Similarly, if \(q \in V\), then \((q, t) \in F\) yields a contradiction.

We now prove the claim by induction of \(|\sigma|\). If \(|\sigma| = 0\), it follows trivially. Assume that \(|\sigma| > 0\) and that the claim holds for firing sequences of length \(|\sigma| - 1\). There exist \(\sigma' \in T^*, s \in T\) and \(Y \in \mathbb{N}^P\) such that \(\sigma = \sigma's\) and

\[
M \overset{\sigma'}{\rightarrow} X \overset{\sigma}{\rightarrow} M'.
\]
By induction hypothesis, there exists \( \pi_U \in (U')^* \) and \( \pi_V \in (V')^* \) such that \( M \xrightarrow{\pi_U \pi_V} X \). If \( s \in V' \) or \( |\pi_V| = 0 \), then we are done. Otherwise, let \( \pi'_V \in (V')^* \) and \( t \in V' \) be such that \( \pi_V = \pi'_V t \). Since \( *U \cap (V')^* = \emptyset \), we can apply (a) and obtain

\[
M \xrightarrow{\pi_U \pi'_V s} Y \xrightarrow{t} M'
\]

for some \( Y \in \mathbb{N}^P \). By induction hypothesis, there exist \( \gamma_U \in (U')^* \) and \( \gamma_V \in (V')^* \) such that

\[
M \xrightarrow{\gamma_U \gamma_V} Y.
\]

Let \( \sigma_U = \gamma_U \) and \( \sigma_V = \gamma_V t \). We are done since \( \sigma_U \in (U')^*, \sigma_V \in (V')^* \) and \( M \xrightarrow{\sigma_U \sigma_V} M' \).

(d) Let \( \mathcal{N} = (P,T,F) \). For the sake of contradiction, assume \( \mathcal{N} \) is not strongly connected. By (b), there exists a partition \( U,V \) of \( P \cup T \) such that \( F \cap (V \times U) = \emptyset \). Since \( \mathcal{N} \) is connected, there exist \( u \in U \) and \( v \in V \) such that \( (u,v) \in F \). Let \( b \in \mathbb{N} \) be such that \( (\mathcal{N},M_0) \) is \( b \)-bounded. Since \( (\mathcal{N},M_0) \) is live, there exist \( \sigma \in T^* \) and \( M \in \mathbb{N}^P \) such that \( M_0 \xrightarrow{\sigma} M \) and \( (u,v) \) is taken \( b+1 \) times. By (c), there exist \( \sigma_U \in U^* \) and \( \sigma_V \in V^* \) such that \( M_0 \xrightarrow{\sigma_U \sigma_V} M \). Let \( X \in \mathbb{N}^P \) be such that \( M_0 \xrightarrow{\sigma_U} X \xrightarrow{\sigma_V} M \).

Case 1: \( u \in P, v \in T \). Since \( F \cap (V \times U) = \emptyset \), there is no transition of \( V \) that puts tokens into places of \( U \). Note that \( v \) decreases the amount of token of \( u \) by 1. Since \( X \xrightarrow{\sigma_V} M \), these two observations imply that \( X(u) \geq b + 1 \). As \( X \) is reachable from \( M_0 \), this contradicts \( (\mathcal{N},M_0) \) being \( b \)-bounded.

Case 2: \( u \in T, v \in P \). Since \( F \cap (V \times U) = \emptyset \), there is no transition of \( U \) that consumes tokens from places of \( V \). Note that \( u \) increases the amount of token of \( u \) by 1. Since \( M_0 \xrightarrow{\sigma_U} X \), these two observations imply that \( X(u) \geq b + 1 \). This contradicts \( (\mathcal{N},M_0) \) being \( b \)-bounded.