Exercise 5.1
Consider the following Petri net (with weights) $N$:

(graph image)

(a) Build the incidence matrix of $N$.
(b) Let $M_0 = \{p_1, p_4\}$. Try to determine whether

$$
M_0 \rightarrow \{p_1, p_1, p_1, p_4\},
$$

$$
M_0 \rightarrow \{p_1, p_1, p_1, p_1, p_2\},
$$

$$
M_0 \rightarrow \{p_1, p_2, p_5\},
$$

by solving the marking equation.

Exercise 5.2
For the following invariants, check if the net below has such an invariant. If yes, give one such invariant.

1. a semi-positive T-invariant
2. a positive T-invariant

(graph image)
Exercise 5.3
Exhibit counterexamples that disprove the following conjectures:

1. For a Petri net \((\mathcal{N}, M_0)\), an S-invariant \(I\) of \(\mathcal{N}\) and a marking \(M\), if \(I \cdot M_0 = I \cdot M\), then \(M\) is reachable from \(M_0\).

2. For a Petri net \((\mathcal{N}, M_0)\) and a place \(s\) of \(\mathcal{N}\), if \(s\) is bounded, then there is an S-invariant \(I\) of \(\mathcal{N}\) with \(I(s) > 0\).

Exercise 5.4
On exercise sheet 1, we saw Lamport’s algorithm for mutual exclusion. We used LoLA to show that it ensures mutual exclusion, which however needs state-space exploration. We now want to use invariants and traps to show that this holds. Below is a slightly smaller but equivalent model of Lamport’s algorithm as a Petri net.

The goal is to show that there is no reachable marking \(M\) such that \(M(p_3) \geq 1\) and \(M(q_5) \geq 1\).

(a) The net has the following S-invariants \(I_1, \ldots, I_6\), which form a basis of the space of all S-invariants:

\[
I_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
I_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
I_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
I_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix},
I_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},
I_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.
\]

Use these invariants to show that there is a unique marking \(M\) where \(M \sim M_0\), \(M(p_3) \geq 1\) and \(M(q_5) \geq 1\).

(b) Use traps to show that the marking \(M\) derived in (a) is not reachable from \(M_0\). For this, find the largest unmarked trap at \(M\) using the algorithm for the largest siphon, adapted to traps.

Exercise 5.5
1. Give a procedure that, given a net \(\mathcal{N}\), constructs a boolean formula \(\varphi\) satisfying the following properties:
   - The formula contains variables \(r_s\) for each place \(s \in S\),
• if $\varphi$ is satisfiable, then $\mathcal{N}$ has a trap,
• and if $\varphi$ is not satisfiable, then $\mathcal{N}$ has no trap.
• Additionally, if $A$ is a model of $\varphi$, then the set given by $R = \{s \mid A(r_s)\}$ is a trap of $\mathcal{N}$.

2. Apply your procedure to the Petri net on the left below and give the resulting constraints.

3. Adapt your procedure such that, given two marking $M_0$ and $M$, it adds additional constraints to ensure that any trap $R$ obtained as a solution by the constraints is marked at $M_0$ and unmarked at $M$. The constraints should be satisfiable iff a trap marked at $M_0$ and unmarked at $M$ exists.

4. Construct the constraints for the Petri net below with the markings $M_0$ and $M$.

5. Use your constraints and the trap property to show that $M$ is not reachable from $M_0$ in the net below.
Solution 5.1

(a) 

\[
\begin{array}{c|ccccc}
   & t_1 & t_2 & t_3 & t_4 & t_5 \\
\hline
   p_1 & -1 & 3 & 0 & 0 & 0 \\
p_2 & 1 & -1 & -1 & 2 & 0 \\
p_3 & 0 & 0 & 0 & 0 & 2 \\
p_4 & 0 & -1 & 0 & 1 & -1 \\
p_5 & 0 & 0 & 1 & -1 & 0 \\
\end{array}
\]

\[N = \begin{pmatrix}
-1 & 3 & 0 & 0 & 0 \\
1 & -1 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & -1 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 0
\end{pmatrix}
\]

This can be verified using APT by using the command `java -jar apt.jar matrices pn_4-1.apt`.

(b) Let us first write the markings as vectors:

\[M_0 = \begin{pmatrix}
2 \\
0 \\
0
\end{pmatrix}, \quad M_1 = \begin{pmatrix}
3 \\
0 \\
1
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
4 \\
1 \\
0 \\
0 \\
0
\end{pmatrix}, \quad M_3 = \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}.
\]

We need to solve \(M_i = M_0 + N \cdot X\), for each \(i \in \{1, 2, 3\}\), which is equivalent to solving \(N \cdot X = M_i - M_0\).

All three systems of equations can be solved simultaneously by using Gaussian elimination:

\[
\begin{pmatrix}
-1 & 3 & 0 & 0 & 0 \\
1 & -1 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & -1 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Markings \(M_1\) and \(M_3\) are not reachable from \(M_0\) since their associated (unique) solutions contain respectively negative and non integer values. Since the marking equation for \(M_2\) has a non negative integer solution, we cannot conclude whether \(M_2\) can be reached or not. In fact, a closer look at the Petri net shows that it is reachable since \(M_0 \xrightarrow{t_1, t_3} M_2\).

Solution 5.2

A vector \(J\) is a T-invariant if \(N \cdot J = 0\) or if \(\forall s \in S : \sum_{t \in \bullet s} J(t) = \sum_{t \in \bullet^* s} J(t)\) or \(J = \sigma\) for some occurrence sequence \(\sigma\) and marking \(M\) with \(M \xrightarrow{\sigma} M\) (fundamental property of T-invariants). By the second definition, we obtain the following equations for a T-invariant \(J = (t_1, t_2, t_3, t_4, t_5)\):

\[
\begin{align*}
t_1 &= t_2 \\
t_3 &= t_1 + t_4 \\
t_2 &= t_3 \\
t_5 &= t_2 \\
t_3 + t_4 &= t_5
\end{align*}
\]

These can be simplified and reduced to following equivalent set of equations:

\[
\begin{align*}
t_1 &= t_2 \\
t_2 &= t_3 \\
t_3 &= t_5 \\
t_4 &= 0
\end{align*}
\]

By specifying \(t_1\) the T-invariant is completely defined. By setting \(t_1 = 1\), we obtain the following semi-positive T-invariant:

\[J = (1, 1, 1, 1, 0)
\]

As \(t_4 = 0\) for all T-invariants, there can be no positive T-invariant.
Solution 5.3

1. In the following net, we have \( I = (0) \) as an S-invariant (the zero vector is an S-invariant for any net), and therefore we have \( I \cdot M_0 = 0 = I \cdot M \) for any markings \( M_0 \) and \( M \). However the marking \( M = (2) \) is not reachable from \( M_0 = (1) \).

2. In the following net, the place \( s_1 \) is bounded. We have \( N = (-1) \) and therefore \( I(s_1) = 0 \) for any S-invariant.

As a more interesting example, take the following Petri net, known from exercise 2.1(c), which is live and bounded.

![Petri Net Diagram]

Any S-invariant \( I \) would need to satisfy

\[
I(s_1) = I(s_2) + I(s_5)
\]

\[
I(s_2) = I(s_1) + I(s_5)
\]

\[
I(s_3) = I(s_4) + I(s_5)
\]

\[
I(s_4) = I(s_3) + I(s_5)
\]

and therefore \( I(s_5) = 0 \), so there is no positive S-invariant.

Solution 5.4

(a) Assume \( M \sim M_0, M(p_3) \geq 1 \) and \( M(q_5) \geq 1 \). We then need to have \( I_k \cdot M = I_k \cdot M_0 \) for \( k \in \{1, 2, 3, 4, 5, 6\} \). We can then derive

\[
\begin{align*}
I_1 \cdot M &= I_1 \cdot M_0 \\
\sum_{i=1}^{3} M(p_i) &= 1 & M(p_3) &\geq 1 \\
M(p_1) &= 0, M(p_2) &= 0, M(p_3) &= 1 \\
I_2 \cdot M &= I_2 \cdot M_0 \\
M(p_2) + M(p_3) + M(x_f) &= 1 \\
I_3 \cdot M &= I_3 \cdot M_0 \\
M(x_1) + M(x_f) &= 1 \\
I_4 \cdot M &= I_4 \cdot M_0 \\
\sum_{i=1}^{5} M(q_i) &= 1 & M(q_5) &\geq 1 \\
M((q_1, q_2, q_3, q_4)) &= 0, M(q_5) &= 1 \\
I_5 \cdot M &= I_5 \cdot M_0 \\
M(q_2) + M(q_3) + M(q_5) + M(y_f) &= 1 \\
I_6 \cdot M &= I_6 \cdot M_0 \\
M(y_t) + M(y_f) &= 1
\end{align*}
\]
from which it follows that $M = \{p_3, x_t, q_5, y_t\}$. We have that $M$ agrees with $M_0$ on $I_1, \ldots, I_6$, and as these are a basis, also on all invariants. Therefore we have $M \sim M_0$. This means using only invariants (or the marking equation over the rationals), we can not conclude that $M$ is not reachable from $M_0$.

★ A basis of the space of semi-positive $S$-invariants of this can be found with APT by using the command

```
java -jar apt.jar traplamport_4-4.spt
```

(b) As the invariants in (a) were inconclusive in determining if a marking $M$ with $M(p_3) \geq 1$ and $M(q_6)$ is reachable, we now use traps to show that the $M$ of (a) is not reachable. We want to find a trap $R$ such that $M_0(R) > 0$ and $M(R) = 0$, therefore violating the fundamental property of traps for reachable markings.

It suffices to look at the largest unmarked trap in $M$, derived with the fixed-point algorithm for siphons from the lecture, adapted to traps by switching pre- and postsets. Initially, we set $R = \{s \in S \mid M(s) = 0\}$. We then obtain:

$$R = \{p_1, p_2, x_f, q_1, q_2, q_3, q_4, y_f\} = S \setminus \{p_3, x_t, q_5, y_t\}$$

$$R^* = \{s_1, s_2, s_3, t_1, t_2, t_3, t_4, t_5, t_6, l_7\} = T$$

We have $R^* \subseteq R$, so $R$ is already a trap and our solution to the largest unmarked trap at $M$.

With $M_0 = \{p_1, x_f, q_1, y_f\}$ we have $M_0(R) = 4 > 0$. As $M(R) = 0$, by the fundamental property of traps, we then obtain that $M$ is not reachable from $M_0$. We showed in (a) that this $M$ is the only $M$ with $M(p_3) \geq 1$ and $M(q_6) \geq 1$ satisfying $M \sim M_0$, and thus also the only $M$ for which the marking equation has a solution. With our trap $R$ it then follows that there is no such $M$ where both $M \sim M_0$ and $M(R) > 0$ hold, so no such $M$ is reachable. This shows that the algorithm satisfies mutual exclusion.

★ The minimal trap $R' = \{p_2, q_2, q_3, x_t, y_f\} \subseteq R$ would also be sufficient to show this property. This trap can be found e.g. with APT by using the command

```
java -jar apt.jar traplamport_4-4.spt
```

Solution 5.5

1. Any trap $R$ satisfies $R^* \subseteq R$ and therefore $\forall t \in T : \exists s \in R : s \in R \implies \exists s' : s' \in R$. This can be encoded with the following formula, which can be unrolled for a given net $N$:

$$\bigwedge_{t \in T} \left( \bigvee_{s \in \text{pre } t} r_s \right) \implies \left( \bigvee_{s' \in \text{post } t} r'_s \right)$$

Any assignment satisfying the formula gives rise to a set $R$ which satisfies the trap condition and is therefore a trap.

2. The constraints are as follows:

$$r_{s_1} \implies r_{s_2} \lor r_{s_3}$$

$$r_{s_1} \implies r_{s_4} \lor r_{s_5}$$

$$r_{s_2} \implies r_{s_6}$$

$$r_{s_3} \implies r_{s_7}$$

$$r_{s_4} \implies r_{s_6}$$

$$r_{s_5} \implies r_{s_7}$$

$$r_{s_6} \lor r_{s_7} \implies r_{s_1}$$

3. To ensure that the trap is marked at $M_0$ and unmarked at $M$, we can add the following constraint:

$$\left( \bigvee_{s \in S : M_0(s) > 0} r_s \right) \land \left( \bigwedge_{s \in S : M(s) > 0} \neg r_s \right)$$

4. The additional constraints are:

$$r_{s_1} \land (\neg r_{s_2} \land \neg r_{s_3})$$
5. We obtain a satisfying assignment $A$ for the constraints by setting $A(r_{s_1}) = A(r_{s_3}) = A(r_{s_4}) = A(r_{s_6}) = A(r_{s_7}) = 1$ and $A(r_{s_2}) = A(r_{s_5}) = 0$. The trap obtained from these constraints is $R = \{s_1, s_3, s_4, s_6, s_7\}$. As the trap is marked at $M_0$, it needs to stay marked in any reachable marking, therefore the marking $M$ is not reachable.