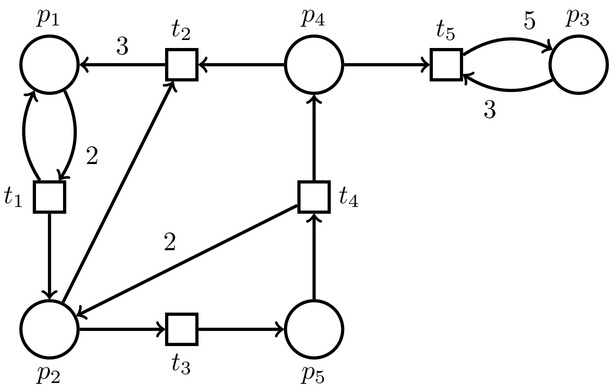
Petri nets — Exercise Sheet 5

Exercise 5.1
Consider the following Petri net (with weights) $\mathcal{N}$:



(a) Build the incidence matrix of $\mathcal{N}$.

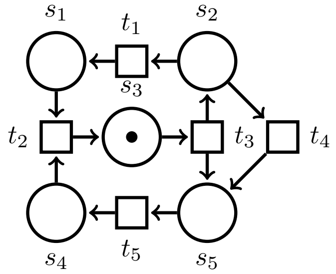
(b) Let $M_0 = \{p_1, p_1\}$. Try to determine whether

$$M_0 \xrightarrow{*} \{p_1, p_1, p_1, p_4\},$$
$$M_0 \xrightarrow{*} \{p_1, p_1, p_1, p_1, p_2\},$$
$$M_0 \xrightarrow{*} \{p_1, p_2, p_5\},$$

by solving the marking equation.

Exercise 5.2
For the following invariants, check if the net below has such an invariant. If yes, give one such invariant.

1. a semi-positive T-invariant
2. a positive T-invariant



Exercise 5.3
Exhibit counterexamples that disprove the following conjectures:

1. For a Petri net $(\mathcal{N}, M_0)$, an S-invariant I of $\mathcal{N}$ and a marking M, if $I \cdot M_0 = I \cdot M$, then M is reachable from M_0.
2. For a Petri net $(\mathcal{N}, M_0)$ and a place s of $\mathcal{N}$, if s is bounded, then there is an S-invariant I of $\mathcal{N}$ with $I(s) > 0$.

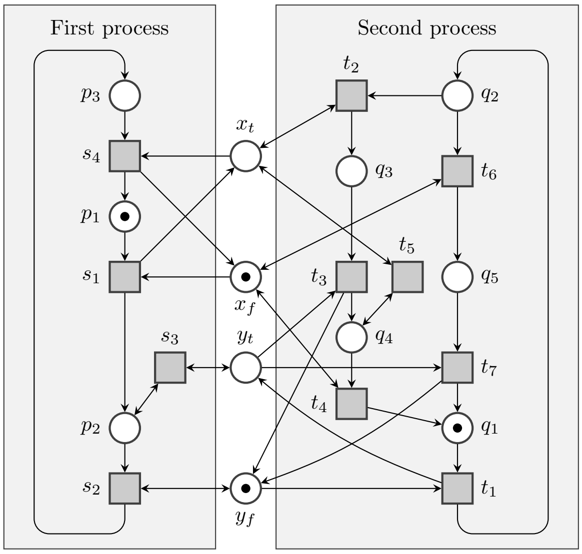
Exercise 5.4
On exercise sheet 1, we saw Lamport's algorithm for mutual exclusion. We used `LoLA` to show that it ensures mutual exclusion, which however needs state-space exploration. We now want to use invariants and traps to show that this holds. Below is a slightly smaller but equivalent model of Lamport's algorithm as a Petri net.

First process

```
     x = False
     while True:
p1:    x = True
p2:    while y:  pass
p3:    # critical section
       x = False
```

Second process

```
     y = False
     while True:
q1:    y = True
q2:    if x:
q3:      y = False
q4:      while x:  pass
         goto q1
q5:    # critical section
       y = False
```



The goal is to show that there is no reachable marking M such that $M(p_3) \geq 1$ and $M(q_5) \geq 1$.

(a) The net has the following S-invariants $I_1, \ldots, I_6$, which form a basis of the space of all S-invariants:

	p_1	p_2	p_3	x_t	x_f	q_1	q_2	q_3	q_4	q_5	y_t	y_f
$I_1 =$	**1**	**1**	**1**	0	0	0	0	0	0	0	0	0
$I_2 =$	0	**1**	**1**	0	**1**	0	0	0	0	0	0	0
$I_3 =$	0	0	0	**1**	**1**	0	0	0	0	0	0	0
$I_4 =$	0	0	0	0	0	**1**	**1**	**1**	**1**	**1**	0	0
$I_5 =$	0	0	0	0	0	0	**1**	**1**	0	**1**	0	**1**
$I_6 =$	0	0	0	0	0	0	0	0	0	0	**1**	**1**

Use these invariants to show that there is a unique marking M where $M \sim M_0$, $M(p_3) \geq 1$ and $M(q_5) \geq 1$.

(b) Use traps to show that the marking M derived in (a) is not reachable from M_0. For this, find the largest unmarked trap at M using the algorithm for the largest siphon, adapted to traps.

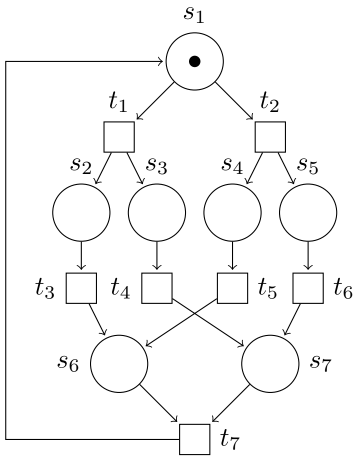
Exercise 5.5

1. Give a procedure that, given a net $\mathcal{N}$, constructs a boolean formula φ satisfying the following properties:
   - The formula contains variables r_s for each place $s \in S$,

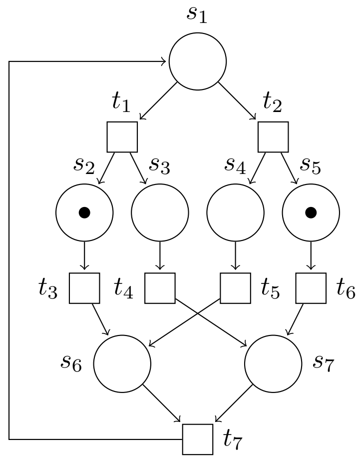
- if φ is satisfiable, then $\mathcal{N}$ has a trap,
- and if φ is not satisfiable, then $\mathcal{N}$ has no trap.
- Additionally, if A is a model of φ, then the set given by $R = \{s \mid A(r_s)\}$ is a trap of $\mathcal{N}$.

2. Apply your procedure to the Petri net on the left below and give the resulting constraints.
3. Adapt your procedure such that, given two marking M_0 and M, it adds additional constraints to ensure that any trap R obtained as a solution by the constraints is marked at M_0 and unmarked at M. The constraints should be satisfiable iff a trap marked at M_0 and unmarked at M exists.
4. Construct the constraints for the Petri net below with the markings M_0 and M.
5. Use your constraints and the trap property to show that M is not reachable from M_0 in the net below.

Marking M_0



Marking M



Solution 5.1

(a)

$$\boldsymbol{N} = \begin{array}{c|ccccc} & t_1 & t_2 & t_3 & t_4 & t_5 \\ \hline p_1 & -1 & 3 & 0 & 0 & 0 \\ p_2 & 1 & -1 & -1 & 2 & 0 \\ p_3 & 0 & 0 & 0 & 0 & 2 \\ p_4 & 0 & -1 & 0 & 1 & -1 \\ p_5 & 0 & 0 & 1 & -1 & 0 \end{array}$$

★ This can be verified using `APT` by using the command `java -jar apt.jar matrices pn_4-1.apt`.

(b) Let us first write the markings as vectors:

$$M_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We need to solve $M_i = M_0 + \boldsymbol{N} \cdot X$, for each $i \in \{1, 2, 3\}$, which is equivalent to solving $\boldsymbol{N} \cdot X = M_i - M_0$. All three systems of equations can be solved simultaneously by using Gaussian elimination:

$$\left(\begin{array}{ccccc|ccc} -1 & 3 & 0 & 0 & 0 & 1 & 2 & -1 \\ 1 & -1 & -1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

Markings M_1 and M_3 are not reachable from M_0 since their associated (unique) solutions contain respectively negative and non integer values. Since the marking equation for M_2 has a non negative integer solution, we cannot conclude whether M_2 can be reached or not. In fact, a closer look at the Petri net shows that it is reachable since $M_0 \xrightarrow{t_1 t_3 t_4 t_2} M_2$.

Solution 5.2

A vector J is a T-invariant if $\mathbf{N} \cdot J = 0$ or if $\forall s \in S : \sum_{t \in {}^\bullet s} J(t) = \sum_{t \in s^\bullet} J(t)$ or $J = \boldsymbol{\sigma}$ for some occurrence sequence σ and marking M with $M \xrightarrow{\sigma} M$ (fundamental property of T-invariants). By the second definition, we obtain the following equations for a T-invariant $J = (t_1, t_2, t_3, t_4, t_5)$:

$$\begin{aligned} t_1 &= t_2 \\ t_3 &= t_1 + t_4 \\ t_2 &= t_3 \\ t_5 &= t_2 \\ t_3 + t_4 &= t_5 \end{aligned}$$

These can be simplified and reduced to following equivalent set of equations:

$$\begin{aligned} t_1 &= t_2 = t_3 = t_5 \\ t_4 &= 0 \end{aligned}$$

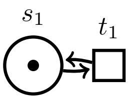
By specifying t_1 the T-invariant is completely defined. By setting $t_1 = 1$, we obtain the following semi-positive T-invariant:

$$J = (1, 1, 1, 1, 0)$$

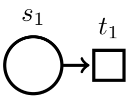
As $t_4 = 0$ for all T-invariants, there can be no positive T-invariant.

Solution 5.3

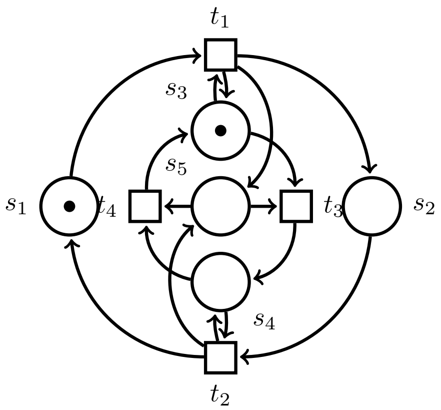
1. In the following net, we have $I = (0)$ as an S-invariant (the zero vector is an S-invariant for any net), and therefore we have $I \cdot M_0 = 0 = I \cdot M$ for any markings M_0 and M. However the marking $M = (2)$ is not reachable from $M_0 = (1)$.



2. In the following net, the place s_1 is bounded. We have $\mathbf{N} = (-1)$ and therefore $I(s1) = 0$ for any S-invariant.



As a more interesting example, take the following Petri net, known from exercise 2.1(c), which is live and bounded.



Any S-invariant I would need to satisfy

$$\begin{aligned} I(s_1) &= I(s_2) + I(s_5) \\ I(s_2) &= I(s_1) + I(s_5) \\ I(s_3) &= I(s_4) + I(s_5) \\ I(s_4) &= I(s_3) + I(s_5) \end{aligned}$$

and therefore $I(s_5) = 0$, so there is no positive S-invariant.

Solution 5.4

(a) Assume $M \sim M_0$, $M(p_3) \geq 1$ and $M(q_5) \geq 1$. We then need to have $I_k \cdot M = I_k \cdot M_0$ for $k \in \{1, 2, 3, 4, 5, 6\}$. We can then derive

$$\frac{\dfrac{\dfrac{\dfrac{I_1 \cdot M = I_1 \cdot M_0}{\sum_{i=1}^{3} M(p_i) = 1} \qquad M(p_3) \geq 1}{M(p_1) = 0,\ M(p_2) = 0,\ M(p_3) = 1} \qquad \dfrac{I_2 \cdot M = I_2 \cdot M_0}{M(p_2) + M(p_3) + M(x_f) = 1}}{M(x_f) = 0} \qquad \dfrac{I_3 \cdot M = I_3 \cdot M_0}{M(x_t) + M(x_f) = 1}}{M(x_t) = 1}$$

and

$$\frac{\dfrac{\dfrac{\dfrac{I_4 \cdot M = I_4 \cdot M_0}{\sum_{i=1}^{5} M(q_i) = 1} \qquad M(q_5) \geq 1}{M(\{q_1, q_2, q_3, q_4\}) = 0,\ M(q_5) = 1} \qquad \dfrac{I_5 \cdot M = I_5 \cdot M_0}{M(q_2) + M(q_3) + M(q_5) + M(y_f) = 1}}{M(y_f) = 0} \qquad \dfrac{I_6 \cdot M = I_6 \cdot M_0}{M(y_t) + M(y_f) = 1}}{M(y_t) = 1}$$

from which it follows that $M = \{p_3, x_t, q_5, y_t\}$. We have that M agrees with M_0 on $I_1, \ldots, I_6$, and as these are a basis, also on all invariants. Therefore we have $M \sim M_0$. This means using only invariants (or the marking equation over the rationals), we can not conclude that M is not reachable from M_0.

★ A basis of the space of *semi-positive* S-invariants of this can be found with `APT` by using the command `java -jar apt.jar invariants lamport_4-4.apt s`.

(b) As the invariants in (a) were inconclusive in determining if a marking M with $M(p_3) \geq 1$ and $M(q_5)$ is reachable, we now use traps to show that the M of (a) is not reachable. We want to find a trap R such that $M_0(R) > 0$ and $M(R) = 0$, therefore violating the fundamental property of traps for reachable markings.

It suffices to look at the largest unmarked trap in M, derived with the fixed-point algorithm for siphons from the lecture, adapted to traps by switching pre- and postsets. Initially, we set $R = \{s \in S \mid M(s) = 0\}$. We then obtain:

$$\begin{aligned} R &= \{p_1, p_2, x_f, q_1, q_2, q_3, q_4, y_f\} = S \setminus \{p_3, x_t, q_5, y_t\} \\ {}^\bullet R &= \{s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4, t_5, t_6, t_7\} = T \\ R^\bullet &= \{s_1, s_2, s_3, t_1, t_2, t_3, t_4, t_5, t_6\} = T \setminus \{s_4, t_7\} \end{aligned}$$

We have $R^\bullet \subseteq {}^\bullet R$, so R is already a trap and our solution to the largest unmarked trap at M.

With $M_0 = \{p_1, x_f, q_1, y_f\}$ we have $M_0(R) = 4 > 0$. As $M(R) = 0$, by the fundamental property of traps, we then obtain that M is not reachable from M_0. We showed in (a) that this M is the only M with $M(p_3) \geq 1$ and $M(q_5) \geq 1$ satisfying $M \sim M_0$, and thus also the only M for which the marking equation has a solution. With our trap R it then follows that there is no such M where both $M \sim M_0$ and $M(R) > 0$ hold, so no such M is reachable. This shows that the algorithm satisfies mutual exclusion.

★ The minimal trap $R' = \{p_2, q_2, q_3, x_f, y_f\} \subseteq R$ would also be sufficient to show this property. This trap can be found e.g. with `APT` by using the command `java -jar apt.jar traps lamport_4-4.apt`.

Solution 5.5

1. Any trap R satisfies $R^\bullet \subseteq {}^\bullet R$ and therefore $\forall t \in T : \exists s \in {}^\bullet t : s \in R \implies \exists s' \in t^\bullet : s' \in R$. This can be encoded with the following formula, which can be unrolled for a given net N:

$$\bigwedge_{t \in T} \left(\left(\bigvee_{s \in pre t} r_s \right) \implies \left(\bigvee_{s' \in post t} r'_s \right) \right)$$

Any assignment satisfying the formula gives rise to a set R which satisfies the trap condition and is therefore a trap.

2. The constraints are as follows:

$$\begin{aligned} r_{s_1} &\implies r_{s_2} \vee r_{s_3} \\ r_{s_1} &\implies r_{s_4} \vee r_{s_5} \\ r_{s_2} &\implies r_{s_6} \\ r_{s_3} &\implies r_{s_7} \\ r_{s_4} &\implies r_{s_6} \\ r_{s_5} &\implies r_{s_7} \\ r_{s_6} \vee r_{s_7} &\implies r_{s_1} \end{aligned}$$

3. To ensure that the trap is marked at M_0 and unmarked at M, we can add the following constraint:

$$\left(\bigvee_{s \in S : M_0(s) > 0} r_s \right) \wedge \left(\bigwedge_{s \in S : M(s) > 0} \neg r_s \right)$$

4. The additional constraints are:

$$r_{s_1} \wedge (\neg r_{s_2} \wedge \neg r_{s_5})$$

5. We obtain a satisying assignment A for the constraints by setting $A(r_{s_1}) = A(r_{s_3}) = A(r_{s_4}) = A(r_{s_6}) = A(r_{s_7}) = 1$ and $A(r_{s_2}) = A(r_{s_5}) = 0$. The trap obtained from these constraints is $R = \{s_1, s_3, s_4, s_6, s_7\}$. As the trap is marked at M_0, it needs to stay marked in any reachable marking, therefore the marking M is not reachable.