Petri nets — Exercise Sheet 4

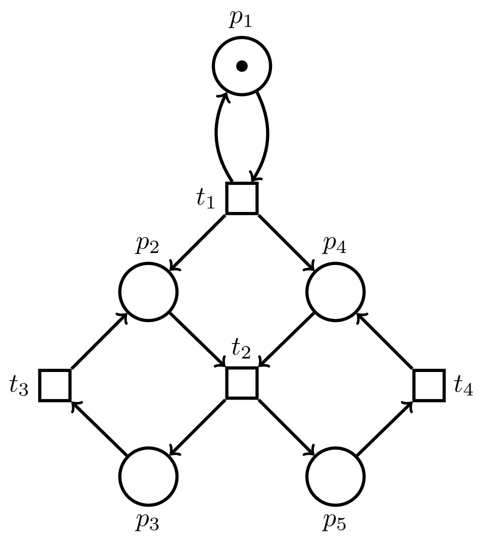
Exercise 4.1

(a) Show that

$$X = \{(x_1, x_2, x_3) \in \mathbb{N}^3 : (x_1 + 3 \leq x_2 \leq x_3 + 1) \vee (x_2 = 2x_1 + x_3 + 5)\}$$

is semilinear by giving its representation as a finite set of roots and periods.

(b) Consider the following Petri net, and define its set of reachable markings. Show that the number of tokens per place of these markings is describable by a semi-linear set.



Exercise 4.2

(a) Reduce the coverability problem to the reachability problem.

For that, describe an algorithm that, given a Petri net $(\mathcal{N}, M_0)$ and a marking M, constructs a Petri net $(\mathcal{N}', M_0')$ and a marking M' such that M' is reachable in $\mathcal{N}'$ from M_0' **if and only if** M is coverable in $\mathcal{N}$ from M_0. The algorithm should run in polynomial time.

(b) Consider problem **P**:

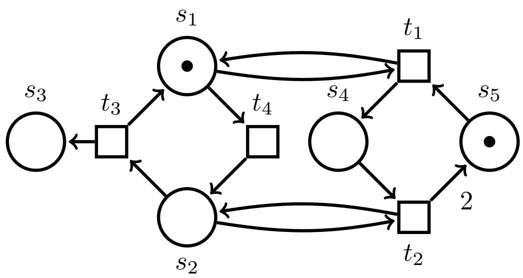
INPUT: A Petri net $(\mathcal{N}, M_0)$ and a transition t of $\mathcal{N}$.
QUESTION: Is there an infinite run σ in $(\mathcal{N}, M_0)$ such that t occurs infinitely many times in σ ?

You are given an algorithm that, given a Petri net, returns its coverability graph. Using this algorithm, devise an algorithm to solve problem **P**.

Then prove this algorithm correct : prove that there exists an infinite run σ in $(\mathcal{N}, M_0)$ such that t occurs infinitely many times in σ **if and only if** the algorithm gives the correct answer.

Exercise 4.3

We want to show that the following Petri net with weighted arcs has a non-semilinear reachability set.



Consider the following sets of markings, given as $M = (s_1, s_2, s_3, s_4, s_5)$:

$$\mathcal{M}_1 = \{(1, 0, x_1, x_2, x_3) \mid 0 < x_2 + x_3 \leq 2^{x_1}\}$$
$$\mathcal{M}_2 = \{(0, 1, x_1, x_2, x_3) \mid 0 < 2x_2 + x_3 \leq 2^{x_1+1}\}$$
$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$$

The set $\mathcal{M}$ is non-semilinear. We are going to show that $\mathcal{M}$ is equal to the set of reachable markings for the above Petri net.

1. Show that if $M_0 \xrightarrow{*} M$, then $M \in \mathcal{M}$. For this, show that $M_0 \in \mathcal{M}$ and if $M \in \mathcal{M}$ and $M \xrightarrow{t} M'$ for some transition t, then also $M' \in \mathcal{M}$.
2. Show that if $M \in \mathcal{M}$, then $M_0 \xrightarrow{*} M$.

   Note: This is a rather hard exercise. *Hint*: Do this by induction on $x_1 = M(s_3)$ for $M \in \mathcal{M}$. In the induction step at x_1, do a case distinction between $M \in \mathcal{M}_1$ and $M \in \mathcal{M}_2$. In each case, find an M' for which you can apply the induction hypothesis and from which M is reachable.

Exercise 4.4

(a) Show that the upward closed sets ($\subseteq \mathbb{N}^k$ for some positive constant k) are semi-linear.

(b) The dual notion of an upward closed set is called a downward closed set. Downward closed sets ($\subseteq \mathbb{N}^k$ for some positive constant k) are sets $\mathcal{M}$ such that $\forall M, M' \in \mathbb{N}^k$, if $M \in \mathcal{M}$ and $M' \leq M$ then $M' \in \mathcal{M}$.

- Show that the complement of a downward closed set is upward closed.
- Show that downward closed sets are also semi-linear, using the fact that a finite intersection of semi-linear sets is semi-linear.