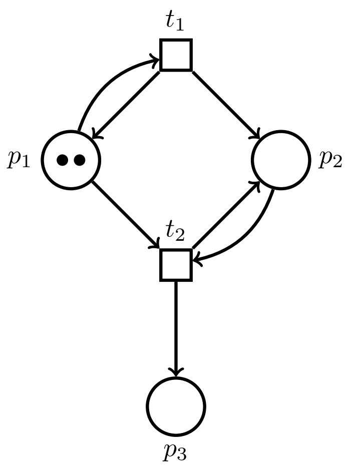
Petri nets — Endterm

- You have **75 minutes** to complete the exam.
- Answers must be written in a **separate booklet**. Do not answer on the exam.
- Please let us know if you need more paper.
- Write your name and Matrikelnummer on every sheet.
- Write with a non-erasable **pen**. Do not use red or green.
- You are not allowed to use auxiliary means other than pen and paper.
- You can obtain **40 points**. You need **17 points** to pass.
- Note that we sometimes represent a marking M by the tuple $(M(p_1), M(p_2), \ldots, M(p_n))$.

Question 1 (8 points)

Apply the backwards reachability algorithm to decide if the marking $M = (0, 0, 2)$ is coverable by the initial marking $M_0 = (2, 0, 0)$. Record all intermediate sets of markings with their finite representation of minimal elements.



Question 2 ($4 + 5 = 9$ points)

(a) Prove: Let (N, M_0) be a live T-system. For every marking M, if M is reachable from M_0, then M_0 is reachable from M.

(b) Consider a T-system (N, M_0). We say (N, M_0) is *floodable* if for all $k \in \mathbb{N}$, there exists a marking M reachable from M_0 with at least k tokens in each place, i.e. such that $M(s) \geq k$ for all $s \in S$.

Give an algorithm that runs in polynomial time and decides whether or not a given T-system (N, M_0) is floodable.

Question 3 (4 + 4 = 8 points)

Let (N, M_0) be a Petri net with $N = (S, T, F)$, and let M_t be a marking of N. We define

$$L(N, M_0, M_t) = \{w \in T^* | M_0 \xrightarrow{w} M_t\}$$

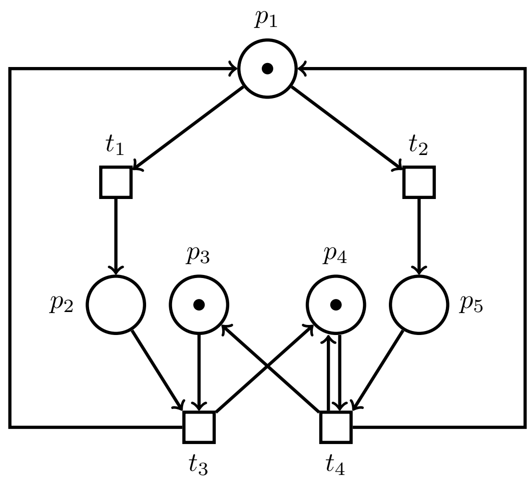
the *terminal language* for (N, M_0, M_t), where the transition set T is the alphabet of the language and the words are the transition sequences leading to the *terminal marking* M_t.

(a) Give a Petri net (N, M_0) and a terminal marking M_t such that the transition set of N is $T = \{a, b, c\}$ and the terminal language for (N, M_0, M_t) is $L = \{aba, aca\}$.

(b) Give a Petri net (N, M_0) such that the transition set of N is $T = \{a, b, c\}$ and the terminal language for (N, M_0, M_0) is $L = L((abcb)^*) = \{\epsilon, abcb, abcbabcb, \ldots\}$.

Question 4 (3 + 4 + 2 = 9 points)

Consider the following free-choice system:



(a) Give all minimal proper siphons of the net.
Hint: There are four minimal proper siphons.

(b) Which of the siphons of (a) contain a proper trap? Justify your answer by giving the traps if they contain one, or showing why no proper subset of the siphon is a trap.

(c) Use the results from (a) and (b) to decide if the system is live.

Question 5 (6 points)

Consider the class of Petri nets N where the following holds:

For all markings M, M' and vectors $X : T \to \mathbb{N}$, if $M' = M + \boldsymbol{N} \cdot X$ then there exists a sequence σ such that $\boldsymbol{\sigma} = X$ and $M \xrightarrow{\sigma} M'$

For this class of Petri nets, give an algorithm to decide the following problem by a reduction to the problem of deciding if a linear system of equations has an integer solution:

Given a system (N, M_0) and a transition t of N, is there an infinite run σ in (N, M_0) such that t occurs infinitely many times in σ?

The algorithm should construct a linear system of equations such that the system has an integer solution if and only if the answer to the problem is positive. Further, the reduction should run in polynomial time.

Solution 1 (8 points)

We start with $m_0 = \{(0,0,2)\}$, which is the set of minimal elements for $\{M\}$. Recall that the backwards reachability algorithm iteratively updates m to

$$m = min(m \cup \bigcup_{t \in T} pre(R[t] \wedge m, t)).$$

We have $R[t_1] = (1,1,0)$ and $R[t_2] = (0,1,1)$.

$$pre(R[t_1] \wedge (0,0,2), t_1) = pre((1,1,2), t_1) = (1,0,2)$$
$$pre(R[t_2] \wedge (0,0,2), t_2) = pre((0,1,2), t_2) = (1,1,1)$$

After adding the new markings to m_0 and eliminating non-minimal markings, our new set is $m_1 = \{(0,0,2),(1,1,1)\}$. For the new marking $(1,1,1)$, we compute the predecessors:

$$pre(R[t_1] \wedge (1,1,1), t_1) = pre((1,1,1), t_1) = (1,0,1)$$
$$pre(R[t_2] \wedge (1,1,1), t_2) = pre((1,1,1), t_2) = (2,1,0)$$

We add the new markings, take the minimal elements and obtain $m_2 = \{(0,0,2),(1,0,1),(2,1,0)\}$. For $(2,1,0)$, we compute the predecessors:

$$pre(R[t_1] \wedge (2,1,0), t_1) = pre((2,1,0), t_1) = (2,0,0)$$

Here we stop because M_0 now covers (actually, is equal to) a minimal marking that reaches a marking covering M. So M_0 does cover M.

Solution 2 (4 + 5 = 9 points)

(a) In a live T-system, a marking M is reachable from M_0 iff $M_0 \sim M$. Let M be a reachable marking. Then $M_0 \sim M$ and, as the relation is symmetric, $M \sim M_0$, so M_0 is reachable from M.

(b) We check if the T-system contains a circuit by a depth-first search from all empty places. This can be done in polynomial time. If the system contains a circuit, then the system is not floodable, otherwise it is floodable.

Argument of correctness: If the system contains a circuit γ, then by the fundamental property of T-systems, we have $M(\gamma) = M_0(\gamma)$ for any reachable marking M and therefore $M(s) \leq M_0(\gamma)$ for any place $s \in \gamma$, so these s are bounded and the net is not floodable.

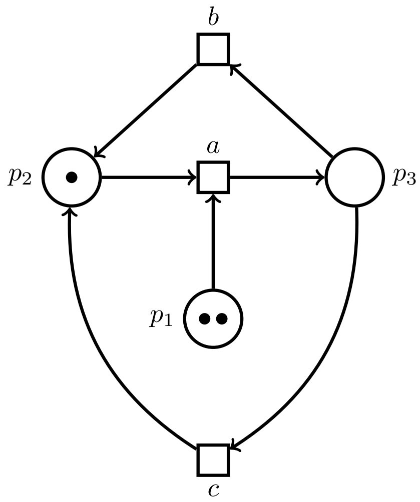
If the system contains no circuit, then the net is acyclic. We prove that any T-system without circuits can be flooded by induction on the number of places $n = |S|$.

If $n = 1$, then there is only one $s \in S$ and one $t \in {}^\bullet s \setminus s^\bullet$. For any k, we can fire the sequence t^k to put k tokens in s.

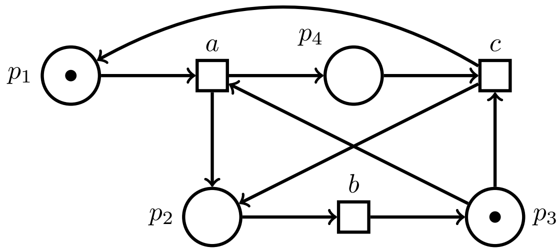
Now fix $n \in \mathbb{N}$ and assume that any T-system without circuits and n places is floodable. Let (N, M_0) with $N = (S,T,F)$ be a T-system without circuits and $n+1$ places. As N is acyclic, there exists a place s and transitions t, u such that ${}^\bullet s = \{t\}, s^\bullet = \{u\}$ and $u^\bullet = \emptyset$. With $S' = S \setminus \{s\}$, $F' = F \cap (S' \times T \cup T \times S')$, $N' = (S', T, F')$ and $M_0'(s') = M_0(s')$ for any $s' \in S'$, the system (N', M_0') is a T-system without circuits and n places. By the induction hypothesis, (N', M_0') is floodable. Now let $k \in \mathbb{N}$. By floodability, there is a sequence σ' in (N', M_0') leading to a marking M' with $M'(s') \geq 2k$ for all $s' \in S'$. We obtain σ from σ' by removing any occurrence of u. Then σ is fireable in (N, M_0) and leads to a marking M with $M(s') \geq 2k$ for all $s' \in S'$. Since the net is acyclic, we have $s \notin {}^\bullet t$. From M we can fire the sequence t^k, reaching a marking M'' where $M''(s) \geq k$ and $M''(s') \geq k$ for each $s' \in S'$. This shows that (N, M_0) is floodable by the sequence σt^k.

Solution 3 (4 + 4 = 8 points)

(a) The following net with initial marking $M_0 = (2,1,0)$ and terminal marking $M_t = (0,0,1)$ answers the question.



(b) The following net with $M_0 = (1, 0, 1, 0)$ and terminal marking M_0 answers the question.



Solution 4 (3 + 4 + 2 = 9 points)

(a) Any siphon R of the net needs to satisfy the following:

$$p_2 \in R \rightarrow p_1 \in R$$
$$p_5 \in R \rightarrow p_1 \in R$$
$$(p_1 \in R \vee p_4 \in R) \rightarrow (p_2 \in R \vee p_3 \in R)$$
$$(p_1 \in R \vee p_3 \in R \vee p_4 \in R) \rightarrow (p_4 \in R \vee p_5 \in R)$$

The only proper siphon not containing p_1 is $\{p_3, p_4\}$. If $p_1 \in R$, then we need to additionally choose one of $\{p_2, p_3\}$ and one of $\{p_4, p_5\}$ to obtain a siphon. Of these combinations, $\{p_1, p_3, p_4\}$ is not minimal, but the other three are. In total we get four minimal siphons:

$$R_1 = \{p_3, p_4\}$$
$$R_2 = \{p_1, p_2, p_5\}$$
$$R_3 = \{p_1, p_2, p_4\}$$
$$R_4 = \{p_1, p_3, p_5\}$$

(b) The siphons R_1 and R_2 are traps by themselves, as

$$R_1^\bullet = {}^\bullet R_1 = \{t_1, t_2, t_3, t_4\}$$
$$R_2^\bullet = {}^\bullet R_2 = \{t_3, t_4\}$$

For R_3, we have that $\{p_4\} \subseteq R_3$ is a trap, as

$$p_4^\bullet = \{t_4\} \subseteq \{t_3, t_4\} = {}^\bullet p_4$$

For any trap Q, we have

$$p_1 \in Q \rightarrow p_2 \in Q$$
$$p_1 \in Q \rightarrow p_5 \in Q$$
$$(p_2 \in Q \vee p_3 \in Q) \rightarrow (p_1 \in Q \vee p_4 \in Q)$$
$$(p_4 \in Q \vee p_5 \in Q) \rightarrow (p_1 \in Q \vee p_3 \in Q \vee p_4 \in Q)$$

Therefore if one of p_1, p_3, p_5 is in Q, then also $p_2 \in Q$ or $p_4 \in Q$, so there can be no trap Q with $\emptyset \neq Q \subseteq R_4$.

(c) As the system is free-choice, we can apply Commoner's theorem. We have that R_4 is a proper siphon that contain no proper trap at all, so especially it contains no initially marked trap, therefore the system is not live.

Solution 5 (6 points)
With variables $M, M' : S \to \mathbb{N}$ and $X, Y : T \to \mathbb{N}$, the algorithm constructs the following linear system of equations:

$$\begin{aligned} M &= M_0 + \boldsymbol{N} \cdot X \\ M' &= M + \boldsymbol{N} \cdot Y \\ M' &\geq M \\ Y(t) &\geq 1 \end{aligned}$$

By the assumption on the net, if the system has a solution M, M', X, Y then there exist σ, τ with $\boldsymbol{\sigma} = X$, $\boldsymbol{\tau} = Y$, t occurs in τ and $M_0 \xrightarrow{\sigma} M \xrightarrow{\tau} M'$. As $M' \geq M$, the sequence $\sigma\tau\tau\tau\ldots$ is enabled at M_0 and t occurs infinitely often along this sequence

In the other direction, if t occurs infinitely often along some sequence, then by Dickson's lemma, we obtain markings M, M' and sequences σ, τ such that $M_0 \xrightarrow{\sigma} M \xrightarrow{\tau} M'$, $M' \geq M$ and t occurs in τ. Then $M, M', X = \boldsymbol{\sigma}, Y = \boldsymbol{\tau}$ is a solution to above system of equations.