Comparing place invariants and the marking equation

Definition

Let N be a net and let L, M be two markings of N.
We say that L and M agree on all P-invariants if
$I \cdot L = I \cdot M$ for every P-invariant I.

Theorem Let N be a net and let L, M be two markings of N. L and M agree on all P-invariants iff the marking equation $L = M + \mathbf{C} \cdot X$ has a rational solution $X \in \mathbb{Q}^{|T|}$

Proof: ($\Leftarrow$) Easy

($\Rightarrow$) Let V_C be the vector space generated by the columns of C.

Let V_P be the vector space of P-invariant

By definition of P-invariant we have

$$X \in V_P \text{ iff } X \cdot Y = 0 \text{ for every } Y \in V_C$$

A well-known theorem of linear algebra yields:

$$X \in V_C \text{ iff } X \cdot Y = 0 \text{ for every } Y \in V_P$$

Since $Y \cdot L = Y \cdot M$ holds for every $Y \in V_P$, we have
$Y \cdot (L - M) = 0$ for every $Y \in V_P$, and so, by the theorem above,
$(L-M) \in V_C$. So $(L-M)$ is a linear combination of the columns of C, which means

$(L - M) = C \cdot X_0$ for some $X_0 \in \mathbb{Q}^{|T|}$, and so

$$M + C \cdot X_0 = L$$