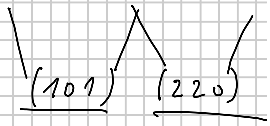
Backwards coverability algorithm.

Definition Upward-closed set of markings



A set $\mathcal{M}$ of markings is <u>upward-closed</u> if $M \in \mathcal{M}$ and $M' \geq M$ then $M' \in \mathcal{M}$

A marking M of an upward-closed $\mathcal{M}$ is <u>minimal</u> if there is no $M' \in \mathcal{M}$, $M' \neq M$ such that $M' \leq M$.

Lemma Every upward-closed set has finitely many minimal elements

Proof Assume the contrary. Then there exists an infinite sequence of minimal markings, pairwise different. By Dickson's lemma, there are $i < j$ s.t. $M_i \leq M_j$. But then M_j is not minimal ↯

Definition Let $\mathcal{M}$ be a set of markings. Let t be a transition. We define

$$\mathrm{pre}(\mathcal{M}, t) = \{ M' \mid M' \xrightarrow{t} M \text{ for some } M \in \mathcal{M} \}$$

$$\mathrm{pre}(\mathcal{M}) = \bigcup_{t \in T} \mathrm{pre}(\mathcal{M}, t)$$

Lemma If $\mathcal{M}$ is upward-closed, then $\mathrm{pre}(\mathcal{M})$ is also upward-closed

Proof

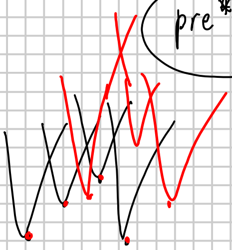
$$\begin{tikzcd} \mathrm{pre}(\mathcal{M}) \ni M \arrow[r, "t"] \arrow[d, phantom, sloped, "\leq"] & M'' \in \mathcal{M} \arrow[d, phantom, sloped, "\leq"] \\ \mathrm{pre}(\mathcal{M}) \ni M' \arrow[r, "t"] & M''' \in \mathcal{M} \end{tikzcd}$$

$$\begin{array}{ccc} M & \leq & M' \\ \cap & & \cap \\ \mathcal{M} & & \mathcal{M} \end{array}$$

Definition Define

$$\mathrm{pre}^0(\mathcal{M}) = \mathcal{M} \qquad \mathrm{pre}^{i+1}(\mathcal{M}) = \mathrm{pre}(\mathrm{pre}^i(\mathcal{M}))$$

$$\mathrm{pre}^*(\mathcal{M}) = \bigcup_{i=0}^{\infty} \mathrm{pre}^i(\mathcal{M})$$



Theorem There is $i \geq 0$ such that $pre^*(\mathcal{M}) = \bigcup_{j=0}^{i} pre^j(\mathcal{M})$

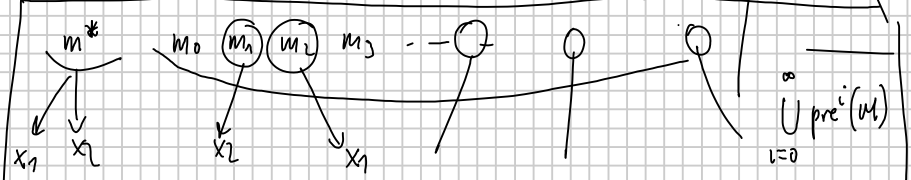
Proof a) $pre^*(\mathcal{M})$ is upward-closed.

Because union of upward-closed set is upward-closed

Let m^* be the set of minimal elements of $pre^*(\mathcal{M})$

We know that m^* is finite

Let m_i be the set of minimal elements of $pre^i(\mathcal{M})$



$\bigcup_{i=0}^{\infty} pre^i(\mathcal{M})$

$x \in \bigcup_{i=0}^{\infty} pre^i(\mathcal{M}) \longrightarrow x \in pre^j(\mathcal{M})$

$\wedge|$

$x' \in \bigcup_{i=0}^{\infty} pre^j(\mathcal{M})$ $\quad x' \in pre^j(\mathcal{M})$

Let i be the smallest index such that $m^* \subseteq \bigcup_{j=0}^{i} m_j$

We then have $pre^*(\mathcal{M}) = \bigcup_{j=0}^{i} pre^j(\mathcal{M})$

$M_1 \quad \{M_1\}\uparrow = pre^2(\mathcal{M})$

$\downarrow \quad \cup$

$M_2 \quad \{M_2\}\uparrow = pre(\mathcal{M})$

$\downarrow \quad \cup$

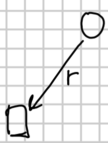
$M_3 \quad \{M_3\}\uparrow = \mathcal{M}$

Backwards coverability algorithm

Goal marking M

$\mathcal{M} := \{ M' \mid M' \geq M \}$

$Old\mathcal{M} := \emptyset$



while ($\mathcal{M} \neq Old\mathcal{M}$) and $M_0 \notin \mathcal{M}$

$\quad Old\mathcal{M} := \mathcal{M}$

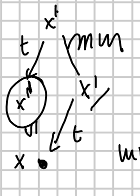
$\longrightarrow \quad \mathcal{M} := \mathcal{M} \cup pred(\mathcal{M})$ $\qquad pre^0(\mathcal{M})$

$\qquad pre^0(\mathcal{M}) \cup pre^1(\mathcal{M})$

if $M_0 \in \mathcal{M}$ then answer "covered"

else answer "not covered"

$min(pre(\mathcal{M})) = \underbrace{min(pre(min(\mathcal{M})))}$ $\qquad min(\mathcal{M})$



$min(\mathcal{M}_1 \cup \mathcal{M}_2) = min(min(\mathcal{M}_1) \cup min(\mathcal{M}_2))$