Commoner's Theorem

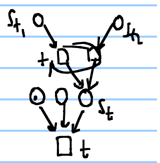
Theorem If every syphon of N (a free-choice net) contains a trap marked at M_0, then (N, M_0) is live.

Proof Let M be a marking of N. A transition is

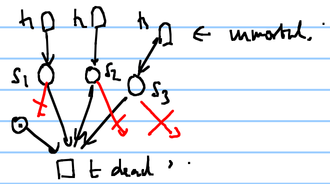
- dead at M if it is not enabled at any marking reachable from M.
- live at M (immortal) if t "cannot die", i.e. it is not dead at any marking reachable from M.
- mortal at M if there is $M \xrightarrow{*} M'$ such that t is dead at M'.

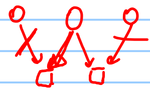
. Claim: there is a reachable marking M such that every transition is either dead or immortal at M.

. Claim: for every t dead at M there exists $s_t \in {}^\bullet t$ s.t.
  - $M(s) = 0$ and
  - every $t' \in {}^\bullet s_t$ is dead at M.

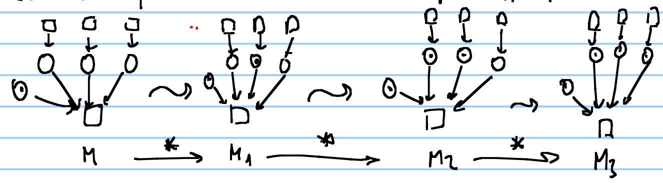


Proof of the claim: Let t be dead at M and let $\{s_1, \ldots, s_n\}$ be the places in ${}^\bullet t$ not marked at M. Assume that for every s_i there is $t_i \in {}^\bullet s_i$ live at M

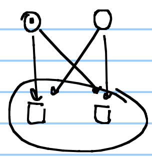




Since N is free-choice there are markings $M_1, \ldots, M_n$ s.t.



Contradicting that t is dead at M.



Theorem If (N, M_0) is free-choice and live, then every siphon of N contains a trap marked at M_0.

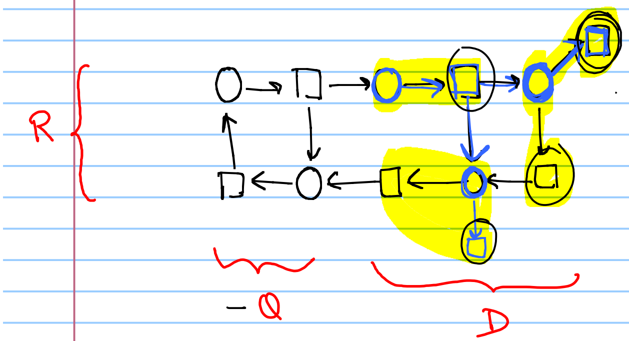
Proof sketch Let R be a siphon, and let Q be the maximal trap included in R. Let $D = R \setminus Q$

Assume Q is initially unmarked

We find a firing sequence σ that "empties" D without adding tokens to Q. After the execution of σ "R is empty" which implies that (N, M_0) is not live.

The sequence σ is constructed as follows:

- We construct an allocation that assigns to each place $s \in D$ a transition of $s^\bullet$. The firing sequence σ only executes allocated transitions.



We need to guarantee:

- the allocation does not define cycles
- the allocation does not allocate any transition of $^\bullet Q$ (this would mark the trap)
- while there are tokens in D we can always fire allocated transitions again and again

Clusters

Definition Let x be a place or transition of a net $N = (S, T, F)$. The cluster of x, denoted by $[x]$ is the minimal set of nodes such that

- $x \in [x]$
- if $s \in [x] \cap S$ then $s^\bullet \subseteq [x]$
- if $t \in [x] \cap T$ then ${}^\bullet t \subseteq [x]$

Proposition Every node of a net belongs to exactly one cluster

→ The set of clusters is a partition of $S \cup T$

Definition
Let $N = (S, T, F)$ be a net, and let C be a set of clusters of N. An allocation of C is a function $\alpha : C \to T$ satisfying $\alpha(c) \in c$

Proposition Let N be a free-choice net, let R be a set of places. Let Q be the maximal trap included in R, and let $D = Q \setminus R$.
Let $C = \{ [t] \mid t \in D^\bullet \}$
There exists a circuit-free allocation $\alpha : C \to T$ such that $\alpha(C) \cap {}^\bullet Q = \emptyset$.
(circuit-free: the set of arcs $\{ (s, \alpha([s])) \mid s \in D \} \cup F \cap (T \times S)$ does not contain any circuit)

Proof By induction on $|R|$.

- $|R| = 0$. Then $C = \emptyset$ ✓
- $|R| > 0$. If R is a trap then $C = \emptyset$ "way-out"
  If R is not a trap: there is $t \in R^\bullet \setminus {}^\bullet R$
  Let $R' = R \setminus {}^\bullet t$, Q' be the maximal trap of R', $D' = R' \setminus Q'$, $C' = \{ [t] \mid t \in D'^\bullet \}$
  By induction hypothesis there exists $\alpha' : C' \to T$ circuit-free for D' such that $\alpha'(C') \cap {}^\bullet Q' = \emptyset$.
  Define $\alpha : C \to T$ by

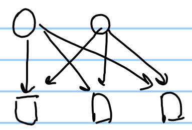
$$\alpha(c) = \begin{cases} \alpha'(c) & \text{if } c \neq [t] \\ t & \text{if } c = [t] \end{cases}$$

We have to prove:

- $\alpha(C) \subseteq \alpha'(C') \cup \{t\}$ (α is well defined)
- α is circuit-free for D
- α puts no tokens in Q.

Proposition Let α be an allocation of a
live free-choice system
with domain C. There exist an infinite occurrence
sequence σ that
- fires allocated transitions infinitely often
AND
- never fires any non-allocated transitions of C

Proof Immediate consequence of



Corollary The non-liveness problem for free-choice nets
is NP-complete

Proof
- NP-hardness: by reduction from SAT
- Membership in NP: - guess a siphon (or get it as advice)
  - compute the largest trap in the siphon
  - check the trap is empty at M_0