**Petri nets — Homework 6**

Due 27.06.2018

Exercise 6.1

(a) Give an $S$-system $(\mathcal{N}, M_0)$ that is 1-bounded and such that $|M_0| > 1$.

(b) Give a strongly connected $T$-system $(\mathcal{N}, M_0)$ which is not live and such that $M_0 \neq 0$.

(c) Give a bounded $T$-system $(\mathcal{N}, M_0)$ which is not strongly connected and such that $M_0 \neq 0$.

(d) Let $(\mathcal{N}, M_0)$ be a $T$-system. Show that if $(\mathcal{N}, M_0)$ is strongly connected and live, then it is bounded.

(e) ★ Reprove (d), but this time without assuming that $(\mathcal{N}, M_0)$ is live.

Exercise 6.2

(a) Let $\mathcal{N} = (P, T, F)$ be a Petri net, and let $s, t \in T$ be such that $s \cap t^* = \emptyset$. Show that if $M \xrightarrow{t} M'$, then $M \xrightarrow{s} M'$.

(b) Let $\mathcal{N} = (P, T, F)$ be a Petri net which is not strongly connected. Show that $P \cup T$ can be partitioned into two disjoint sets $U, V \subseteq P \cup T$ such that $F \cap (V \times U) = \emptyset$.

(c) Let $U$ and $V$ be a partition as in (b). Show that if $M \xrightarrow{(T \cap V)^*} M'$, then there exist $\sigma_U \in (T \cap U)^*$ and $\sigma_V \in (T \cap V)^*$ such that $\sigma = \sigma_U \sigma_V$ and $M \xrightarrow{\sigma_U \sigma_V} M'$.

(d) Let $(\mathcal{N}, M_0)$ be live and bounded. Use (a), (b) and (c) to show that $\mathcal{N}$ is strongly connected.

Exercise 6.3

(a) Show that the problem of determining whether a $T$-system is not live belongs to NP.

(b) Give a polynomial time algorithm for deciding liveness of $T$-systems.

(c) Test whether the following $T$-system is live by using your previous algorithm:
Exercise 6.4
Consider the following free-choice system \((\mathcal{N}, M_0)\):

(a) Give all minimal proper siphons of \((\mathcal{N}, M_0)\).

(b) Use (a) to say whether \((\mathcal{N}, M_0)\) is live or not.
Solution 6.1

(a) 

(b) 

(c) 

(d) Let $\mathcal{N} = (P,T,F)$. Let $b = M_0(P)$. We show that every place is $b$-bounded. Let $p \in P$. Since $\mathcal{N}$ is strongly connected, $p$ lies on some circuit $\gamma$. Note that $M_0(\gamma) \leq b$ and that $(\mathcal{N},M_0)$ is live. Therefore, by Theorem 5.2.4, $p$ is $b$-bounded.

(e) Let $\mathcal{N} = (P,T,F)$. For the sake of contradiction, assume $(\mathcal{N},M_0)$ is unbounded. Since $(\mathcal{N},M_0)$ is unbounded, there exist $M,M' \in \mathcal{N}P$, $\sigma \in T^+$ and $p \in P$ such that 

$$M_0 \xrightarrow{\sigma} M', \ M' \geq M \text{ and } M'(p) > M(p). \quad (1)$$

Let $p = \{s\}$ and $p^* = \{t\}$. Since $\mathcal{N}$ is strongly connected, there exists a path $\pi$ from $t$ to $s$. Therefore, $p$ lies on the circuit $\gamma = (p,t)\pi(s,p)$.

Let $n = M_0(\gamma)$. By (1), we may fire $\sigma$ arbitrarily many times from $M$, thus increasing the amount of tokens in $p$ by at least one each time. Therefore, there exists $M'' \in \mathcal{N}P$ such that

$$M \xrightarrow{\sigma^{n+1}} M'' \text{ and } M''(p) > n.$$ 

By the fundamental property of $T$-systems, $M''(\gamma) = M(\gamma) = n$, which is a contradiction since $M''(\gamma) \geq M''(p) > n$.

Solution 6.2

(a) Let $X \in \mathbb{N}P$ be such that $M \xrightarrow{\sigma} X \xrightarrow{\sigma} M'$. For the sake of contradiction, suppose $s$ is not enabled in $M$. There exists $p \in P$ such that $p \in \bullet s$ and $M(p) = 0$. Since $s$ is enabled in $X$, we have $X(p) > 0$. Therefore, it must be the case that $p \in \bullet t$. This implies that $p \in \bullet s \cap \bullet t$ which is a contradiction. Thus, $s$ is enabled in $M$ and $M \xrightarrow{\sigma} Y$ for some marking $Y \in \mathbb{N}P$. 

Let us now show that \( t \) is enabled in \( Y \). Let \( q \in \textbf{M}t \). We must show that \( Y(q) > 0 \).

Case 1: \( q \notin \textbf{M}s \). If \( q \notin \textbf{M}s \), then \( Y(q) \geq M(q) > 0 \).

Case 2: \( q \in \textbf{M}s \). If \( q \in \textbf{M}s \), then

\[
Y(q) = M(q) - 1.
\]  

(2)

Since \( s \) is enabled in \( X \), we have \( X(q) > 0 \). Moreover, \( q \notin t^* \) since \( \textbf{M}s \cap t^* = \emptyset \). This implies that \( M(q) > X(q) \), and hence \( M(q) \geq 2 \). By (2), we derive \( Y(q) \geq 1 \).

(b) Since \( N \) is not strongly connected, there exist \( u, v \in P \cup T \) such that there is no path from \( v \) to \( u \). Let

\[
U = \{x \in P \cup T : \text{there is a path from } x \text{ to } u\},
\]

\[
V = (P \cup T) \setminus U.
\]

Note that both sets are non empty since \( u \in U \) and \( v \in V \). Moreover, \( U \cap V = \emptyset \) and \( U \cup V = P \cup T \) by definition.

Let us show that \( F \cap (V \times U) = \emptyset \). Assume there exists \( e \in F \cap (V \times U) \). There exist \( x \in U \) and \( y \in V \) such that \( (y, x) \in F \). Since \( x \in U \), there exists a path \( \sigma \) from \( x \) to \( u \). Therefore, \( (y, x)\sigma \) is a path from \( y \) to \( u \). This implies that \( y \in U \) which is a contradiction.

(c) Let \( U' = T \cap U \) and \( V' = T \cap V \). Let us first show that \( \textbf{M}(U') \cap \textbf{M}(V') = \emptyset \). For the sake of contradiction, assume there exist \( s \in V' \), \( t \in U' \) and \( q \in P \) such that \( q \in s^* \) and \( q \in t^* \). We have \( s, q \in F \) and \( (q, t) \in F \). If \( q \in U \), then by (b) and \( (s, q) \in F \), we obtain a contradiction. Similarly, if \( q \in V \), then \( (q, t) \in F \) yields a contradiction.

We now prove the claim by induction of \(|\sigma|\). If \(|\sigma| = 0\), it follows trivially. Assume that \(|\sigma| > 0 \) and that the claim holds for firing sequences of length \(|\sigma| - 1 \). There exist \( \sigma' \in T^* \), \( s \in T \) and \( Y \in \mathbb{N}^P \) such that \( \sigma = \sigma's \) and

\[
M \xrightarrow{\sigma'} X \xrightarrow{s} M'.
\]

By induction hypothesis, there exists \( \pi_U \in (U')^* \) and \( \pi_V \in (V')^* \) such that \( M \xrightarrow{\pi_U \pi_V} X \). If \( s \in V' \) or \(|\pi_V| = 0 \), then we are done. Otherwise, let \( \pi'_V \in (V')^* \) and \( t \in V' \) be such that \( \pi_V = \pi'_V t \). Since \( (U') \cap (V')^* = \emptyset \), we can apply (a) and obtain

\[
M \xrightarrow{\pi_U \pi'_V s} Y \xrightarrow{t} M'.
\]

for some \( Y \in \mathbb{N}^P \). By induction hypothesis, there exist \( \gamma_U \in (U')^* \) and \( \gamma_V \in (V')^* \) such that

\[
M \xrightarrow{\gamma_U \gamma_V} Y.
\]

Let \( \sigma_U = \gamma_U \) and \( \sigma_V = \gamma_V t \). We are done since \( \sigma_U \in (U')^* \), \( \sigma_V \in (V')^* \) and \( M \xrightarrow{\sigma_U \sigma_V} M' \).

(d) Let \( N = (P, T, F) \). For the sake of contradiction, assume \( N \) is not strongly connected. By (b), there exists a partition \( U, V \) of \( P \cup T \) such that \( F \cap (V \times U) = \emptyset \). Since \( N \) is connected, there exist \( u \in U \) and \( v \in V \) such that \( u, v \in F \). Let \( b \in \mathbb{N} \) be such that \( (N, M_0) \) is \( b \)-bounded. Since \( (N, M_0) \) is live, there exist \( \sigma \in T^* \) and \( M \in \mathbb{N}^P \) such that \( M \xrightarrow{\sigma} M \) and \( u, v \) is taken \( b + 1 \) times. By (c), there exist \( \sigma_U \in U^* \) and \( \sigma_V \in V^* \) such that \( M \xrightarrow{\sigma_U \sigma_V} M \). Let \( X \in \mathbb{N}^P \) be such that \( M_0 \xrightarrow{\sigma_U} X \xrightarrow{\sigma_V} M \).

Case 1: \( u \in P \), \( v \in T \). Since \( F \cap (V \times U) = \emptyset \), there is no transition of \( V \) that puts tokens into places of \( U \). Note that \( v \) decreases the amount of token of \( u \) by \( 1 \). Since \( X \xrightarrow{\sigma_U} M \), these two observations imply that \( X(u) \geq b + 1 \). As \( X \) is reachable from \( M_0 \), this contradicts \( (N, M_0) \) being \( b \)-bounded.

Case 2: \( u \in T \), \( v \in P \). Since \( F \cap (V \times U) = \emptyset \), there is no transition of \( U \) that consumes tokens from places of \( V \). Note that \( u \) increases the amount of token of \( u \) by \( 1 \). Since \( M_0 \xrightarrow{\sigma_U} X \), these two observations imply that \( X(u) \geq b + 1 \). This contradicts \( (N, M_0) \) being \( b \)-bounded.
Solution 6.3

(a) By Theorem 5.2.3, \((\mathcal{N}, M_0)\) is not live if and only if \(M_0(\gamma) = 0\) for some circuit \(\gamma\). Note that every cycle \(\gamma\) contains a simple cycle \(\gamma'\). Moreover, if \(M_0(\gamma) = 0\), then \(M_0(\gamma') = 0\). This implies that,

\[\mathcal{N}, M_0\) is not live \iff M_0(\gamma) = 0 \text{ for some simple circuit } \gamma.\]

Therefore, to test whether \((\mathcal{N}, M_0)\) is not live, it suffices to test a circuit \(\gamma\) of size at most \(|P \cup T|\) and check whether \(M_0(\gamma) = 0\).

(b) Since a graph may contain exponentially many simple cycles, we cannot directly use the approach of (a). Instead, we construct the subnet \(\mathcal{N}'\) obtained from \(\mathcal{N}\) by removing all places containing tokens. We then perform depth-first search to test whether \(\mathcal{N}'\) contains a cycle. This procedure can be implemented as follows:

```plaintext
Input: T-system \((\mathcal{N}, M_0)\) where \(\mathcal{N} = (P, T, F)\)
Output: \((\mathcal{N}, M_0)\) live?
while \(\exists p \in P\) such that \(\neg\text{visited}(p)\) and \(M_0(p) = 0\) do
    if has-cycle(p) then return false
return true
has-cycle(p):
    visited(p) ← true
    onstack(p) ← true
    for \(q \in (p)^*\) such that \(M_0(q) = 0\) do
        if onstack(q) then return true
        else if \(\neg\text{visited}(q)\) then
            if has-cycle(q) then return true
        onstack(p) ← false
    return false
```

(c) We obtain the following subnet:

![Diagram](image)

A depth-first search shows that this subnet contains no cycle. Therefore, the system is live.

Solution 6.4

(a) We claim that the system has two minimal proper siphons: \(\{p_0\}\) and \(\{p_2, p_3\}\).

Let us show the claim. By inspecting \(\bullet p\) and \(p^*\) for every place \(p\), we find a single siphon of size one: \(\{p_0\}\). Moreover, we have \(\bullet \{p_2, p_3\} = \{t_2, t_3, t_4\} = \{p_2, p_3\}^*\). Now, note that \(t_0 \in \bullet p_1\) and \(\bullet t_0 = \{p_0\}\). Therefore, any siphon containing \(p_1\) must also contain \(p_0\). Similarly, any siphon containing \(p_4\) must also contain \(p_0\). Thus, no minimal siphon contains \(p_1\) or \(p_4\), and we are done.
(b) The system is not live. By Commoner’s Theorem, the system is live if and only if every minimal proper siphon contains a trap marked at $M_0$. The minimal siphon $\{p_2, p_3\}$ is also a trap and it is marked at $M_0$. However, the minimal siphon $\{p_0\}$ is not a trap and hence it does not contain a marked trap.