Exercise 5.1
Consider the following Petri net $\mathcal{N} = (P,T,F)$:

(a) Give a basis of the vector space of $T$-invariants of $\mathcal{N}$. [Hint: use a characterization of $T$-invariants.]

(b) Let $M = \{p_1,p_2,p_4,p_5\}$ and $M' = \{p_1,p_3,p_5\}$. We have shown that $\mathcal{N}$ is bounded from any initial marking in #4.3(b). Can you tell whether $(\mathcal{N},M)$ and $(\mathcal{N},M')$ are live?

Exercise 5.2
Consider the following Petri net (with weights) $\mathcal{N}$:

(a) Use siphons/traps to prove or disprove that $\mathcal{N}$ is live from $M_0 = \{p_2,3 \cdot p_4\}$,
(b) Can the marking equation be used to prove or disprove that \( \{p_2, p_4\} \xrightarrow{*} \{p_1, 3p_2, p_3\} \)? Is so, why? If not, can traps or siphons help?

**Exercise 5.3**

Let \( \mathcal{N} = (P, T, W) \) be a net with weights and let \( M_0, M \) be markings. We say that a trap \( Q \) is **minimal** if it is non-empty and every non-empty \( Q' \subset Q \) is not a trap.

(a) Disprove the following statement:

There exists a trap \( Q \) of \( \mathcal{N} \) such that \( M_0(Q) > 0 \) and \( M(Q) = 0 \) if and only if there exists a minimal trap \( R \) of \( \mathcal{N} \) such that \( M_0(R) > 0 \) and \( M(R) = 0 \).

(b) Show that Petri nets can have exponentially many minimal traps. More formally, exhibit an infinite family of (distinct) nets \( \mathcal{N}_1 = (P_1, T_1, F_1), \mathcal{N}_2 = (P_2, T_2, F_2), \ldots \), some \( c > 1 \), and a function \( f \in \Omega(c^n) \) such that \( \mathcal{N}_i \) has \( f(|P_i|) \) minimal traps for every \( i \geq 1 \).

(c) For every \( x \in \{\text{false, true}\}^P \), let \( Q_x = \{p : p \in P, x_p = \text{true}\} \).

(i) Give a Boolean formula \( \varphi_{\text{trap}} \) over variables \( \{x_p : p \in P\} \) such that \( \varphi_{\text{trap}}(x) \) holds if and only if \( Q_x \) is a trap of \( \mathcal{N} \).

(ii) Give a quantified Boolean formula \( \varphi_{\text{mintrap}} \) over variables \( \{x_p : p \in P\} \) such that \( \varphi_{\text{mintrap}}(x) \) holds if and only if \( Q_x \) is a minimal trap of \( \mathcal{N} \).

(iii) Give a Boolean formula \( \varphi_{\text{constraints}} \) over variables \( \{x_p : p \in P\} \) such that \( \varphi_{\text{constraints}}(x) \) holds if and only if \( M_0(Q_x) > 0 \) implies \( M(Q_x) > 0 \).

(d) Construct \( \varphi_{\text{trap}}(x) \) for the following Petri net:

(e) ★ Use the SMT solver Z3 to prove that the Petri net above cannot reach a marking \( M \) such that \( M(p_2) = 2 \) and \( M(p_3) \geq 1 \). See instructions on the webpage of the course. You may start from the given partial solution if you need help.
Solution 5.1
(a) Recall that $J$ is a $T$-invariant if and only if $\sum_{t \in \bullet p} I(t) = \sum_{t \in p} I(t)$ for every $p \in P$. This gives rise to the following system of equations:

\[
\begin{align*}
J(t_1) &= J(t_3), \\
J(t_2) &= J(t_1) + J(t_2), \\
0 &= J(t_2), \\
J(t_1) + J(t_3) + J(t_4) &= 0, \\
J(t_1) + J(t_5) &= J(t_4) + J(t_5), \\
J(t_2) &= J(t_3), \\
J(t_2) + J(t_5) &= 0.
\end{align*}
\]

This system of equations is equivalent to $J(t_1) = J(t_2) = J(t_3) = J(t_4) = J(t_5) = 0$. Therefore, the vector space of $T$-invariants of $N$ is trivial, i.e. it only contains the null vector.

★ This can be verified using PIPE by loading the Petri net and clicking on “Invariant Analysis” in the left menu.

(b) Yes. We can actually now show that $N$ is not live from any initial marking $M_0$. Assume, $(N, M_0)$ is live. Since $(N, M_0)$ is also bounded by #4.3(b), then it is well-formed. We have seen that every well-formed net has a positive $T$-invariant. This is a contradiction since the only $T$-invariant of $N$ is the trivial invariant which is not positive. Therefore, $(N, M_0)$ is not live. In particular, this implies that both $(N, M)$ and $(N', M')$ are not live.

Solution 5.2
(a) Let $Q = \{p_1, p_3\}$. We have

\[
\begin{align*}
\bullet Q &= \{t_1, t_2\} \\
&\subseteq \{t_1, t_2, t_3\} \\
&= Q^*.
\end{align*}
\]

Therefore $Q$ is a siphon. Since $Q$ is not marked by $M_0$, we conclude that $(N, M_0)$ is not live.

(b) The incidence matrix of $N$ is:

\[
N = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 \\
1 & 1 & -1 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]

The marking equation is:

\[
\begin{pmatrix}
1 \\
3 \\
1 \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
1 \\
0 \\
1
\end{pmatrix}
+ \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 \\
1 & 1 & -1 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
\cdot \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
\]

The unique solution of the marking equation is

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \begin{pmatrix}
2 \\
1 \\
2 \\
3
\end{pmatrix}
\]

Since it is non negative, we cannot conclude whether the marking is reachable or not.

Let us consider the trap $Q = \{p_4\}$ which is marked by the initial marking. We can conclude that the target marking is not reachable since it does not mark $Q$.
Solution 5.3

(a) Consider the following Petri net:

Let \( M_0 = \{p_1\} \), \( M = \emptyset \) and \( Q = \{p_1, p_2, p_3\} \). Note that \( Q \) is a trap such that \( M_0(Q) > 0 \) and \( M(Q) = 0 \). Moreover, observe that \( Q \) is not a minimal trap since it contains the trap \( \{p_2, p_3\} \). We claim that no minimal trap marks \( M_0 \). For the sake of contradiction, suppose there exists a minimal trap \( R \) such that \( M_0(R) > 0 \). We have \( p_1 \in R \). Moreover, \( R \neq \{p_1\} \) and \( R \neq Q \) since the former is not a trap and the latter is not minimal. Thus, \( R = \{p_1, p_2\} \) or \( R = \{p_1, p_3\} \), but both are not traps, which yields a contradiction.

(b) For every \( n \in \mathbb{N}_{\geq 1} \), let \( N_n = (P_n, T_n, F_n) \) be the following net:

Note that every set \( Q \subseteq P_n \) that contains at least one of \( p_i \) or \( q_i \) for every \( 1 \leq i \leq n \) is a trap. Thus, \( N_n \) has at least \( 2^n = (\sqrt{2})|P_n| \) traps.

(c) (i) We have

\[
Q_x \text{ is a trap } \iff Q_x^* \subseteq Q_x
\]

\[
\quad \iff \forall t \in T \ [(t \in Q_x^*) \Rightarrow (t \in Q_x)]
\]

\[
\quad \iff \forall t \in T \ [(\exists p \in t : p \in Q_x) \Rightarrow (\exists p \in t^* : p \in Q_x)]
\]

\[
\quad \iff \forall t \in T \ [(\exists p \in t^* : x_p) \Rightarrow (\exists p \in t^* : x_p)]
\]
(d) $\varphi_{\text{trap}}$ is the conjunction of the following five constraints:

\[
(x_{p_1} \lor x_{p_2}) \rightarrow x_{p_1},
\]

\[
x_{p_1} \rightarrow (x_{p_1} \lor x_{p_2}),
\]

\[
x_{p_2} \rightarrow x_{p_4},
\]

\[
x_{p_4} \rightarrow (x_{p_2} \lor x_{p_3} \lor x_{p_4}),
\]

\[
x_{p_4} \rightarrow (x_{p_1} \lor x_{p_4}).
\]

(e) We could try to disprove the existence of $M$ by using the state equation, i.e. by checking whether there exist no $M$ and $x$ such that

\[
M_0 + \mathcal{N} : x = M, M(p_2) = 2 \text{ and } M(p_3) \geq 1.
\]

However, there exist infinitely many solutions:

\[
M = (0, 2, 1, 0) \text{ and } x = (1, 3, m_1 + m_0, 0) \text{ for any } m \in \mathbb{N}.
\]

Observe that $M_0$ marks the minimal traps $\{p_1\}$ and $\{p_4\}$, but $M(p_1) = M(p_4) = 0$. By combining the state equation with the fact that minimal trap cannot be emptied, it is possible to conclude that there exist no solution $(M,x)$. This can be verified automatically with Z3 using the formulas obtained in (c):

; trap(x) holds iff Q.x is a trap
(define-fun trap ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool)) Bool
    (and
        (=> (or x1 x2) (or x1))
        (=> (or x1) (or x1 x2))
        (=> (or x2) (or x4))
        (=> (or x4) (or x2 x3 x4)))
(=> (or x4) (or x1 x4))
)

; mintrap(x) holds iff Q_x is a minimal trap
(define-fun mintrap ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool)) Bool
(and
(trap x1 x2 x3 x4) ; Q_x is a trap
(or x1 x2 x3 x4) ; Q_x is non empty
(forall ((y1 Bool) (y2 Bool) (y3 Bool) (y4 Bool)) ; Every strict subset of Q_x is not a trap
(=>
  (and
   (or y1 y2 y3 y4) ; Q_y is non empty
   (=> y1 x1) (=> y2 x2) (=> y3 x3) (=> y4 x4) ; Q_y subset of Q_y
   (and (not y1) x1) (and (not y2) x2)
   (and (not y3) x3) (and (not y4) x4))
  (not (trap y1 y2 y3 y4))) ; Q(y) is not a trap
)
)
)

; constraints(x, M) holds iff "Q_x marks M_0" implies "Q_x marks M"
(define-fun constraints ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool) (m1 Int) (m2 Int) (m3 Int) (m4 Int)) Bool
(=>
  (or x1 x4) ; Q_x is marked in M_0, i.e. contains place p1 or p4
  (or ; Some place of Q_x is marked in M
   (and x1 (> m1 0))
   (and x2 (> m2 0))
   (and x3 (> m3 0))
   (and x4 (> m4 0))
  )
)
)

; mreach(m) holds iff there is a solution of the marking equation
; from (2, 0, 0, 1) thats leads to m
(define-fun mreach ((m1 Int) (m2 Int) (m3 Int) (m4 Int)) Bool
(exists ((t1 Int) (t2 Int) (t3 Int) (t4 Int) (t5 Int))
(and
  (> t1 0)
  (> t2 0)
  (> t3 0)
  (> t4 0)
  (> t5 0)
  (= m1 (+ 2 t1 (- 0 t2) t5))
  (= m2 (+ (* -2 t1) t2 (- 0 t3) t4))
  (= m3 t4)
  (= m4 (+ 1 t3 (- 0 t4) (- 0 t5)))
)
)
)

; There exists a solution to the marking equation
(declare-const m1 Int)
(declare-const m2 Int)
(declare-const m3 Int)
(declare-const m4 Int)
(assert (> m1 0))
(assert (= m2 2))
(assert (> m3 1))
(assert (> m4 0))
(assert (mreach m1 m2 m3 m4))

; The reachable marking satisfies minimal trap constraints
;; (In general, non minimal trap constraints may be necessary
;; to disprove reachability, but here it's not the case.
;; You may simply replace "mintrap" by "trap" below.)
(assert (forall ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool))
  =>
  (mintrap x1 x2 x3 x4)
  (constraints x1 x2 x3 x4 m1 m2 m3 m4)
)

(check-sat)