Technische Universität München
I7
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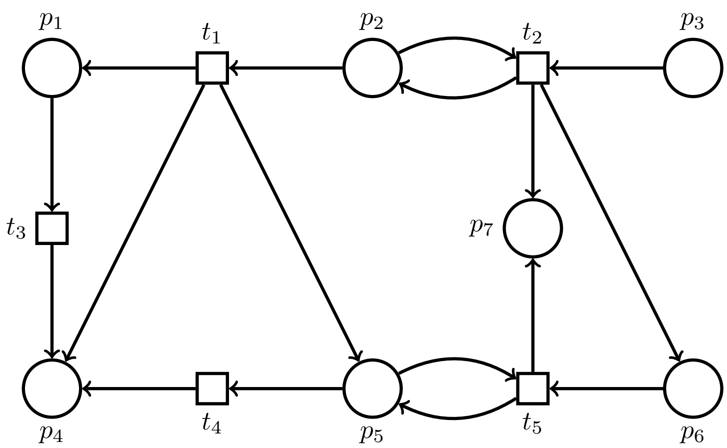
Petri nets — Homework 5

Due 13.06.2018

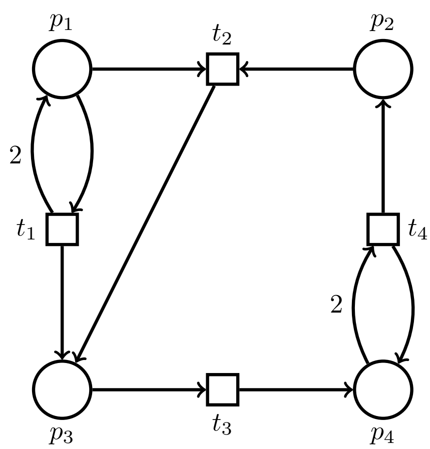
Exercise 5.1
Consider the following Petri net $\mathcal{N} = (P, T, F)$:



(a) Give a basis of the vector space of T-invariants of $\mathcal{N}$. [Hint:]

(b) Let $M = \{p_1, p_2, p_4, p_4\}$ and $M' = \{p_1, p_3, p_5\}$. We have shown that $\mathcal{N}$ is bounded from any initial marking in #4.3(b). Can you tell whether $(\mathcal{N}, M)$ and $(\mathcal{N}, M')$ are live?

Exercise 5.2
Consider the following Petri net (with weights) $\mathcal{N}$:



(a) Use siphons/traps to prove or disprove that $\mathcal{N}$ is live from $M_0 = \{p_2, 3 \cdot p_4\}$,

(b) Can the marking equation be used to prove or disprove that $\{p_2, p_4\} \xrightarrow{*} \{p_1, 3 \cdot p_2, p_3\}$? Is so, why? If not, can traps or siphons help?

Exercise 5.3

Let $\mathcal{N} = (P, T, W)$ be a net with weights and let M_0, M be markings. We say that a trap Q is *minimal* if it is non empty and every non empty $Q' \subset Q$ is not a trap.

(a) Disprove the following statement:

> There exists a trap Q of $\mathcal{N}$ such that $M_0(Q) > 0$ and $M(Q) = 0$ if and only if there exists a minimal trap R of $\mathcal{N}$ such that $M_0(R) > 0$ and $M(R) = 0$.

(b) Show that Petri nets can have exponentially many minimal traps. More formally, exhibit an infinite family of (distinct) nets $\mathcal{N}_1 = (P_1, T_1, F_1), \mathcal{N}_2 = (P_2, T_2, F_2), \ldots$, some $c > 1$, and a function $f \in \Omega(c^n)$ such that $\mathcal{N}_i$ has $f(|P_i|)$ minimal traps for every $i \geq 1$.

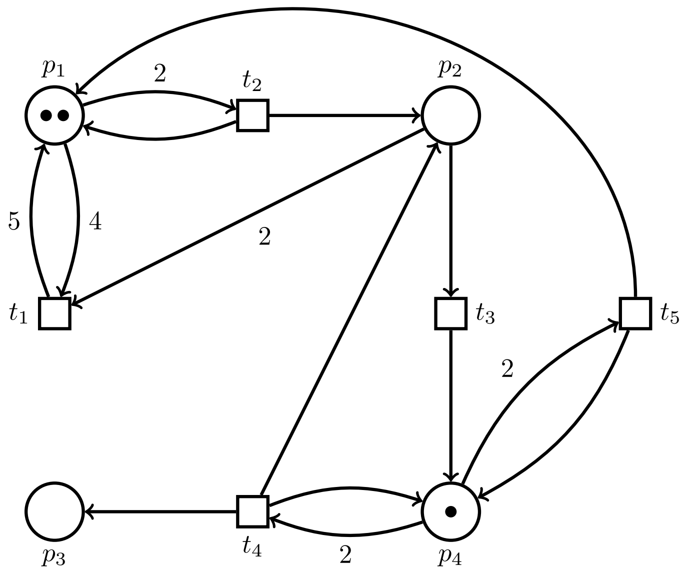
(c) For every $x \in \{\texttt{false}, \texttt{true}\}^P$, let $Q_x = \{p : p \in P, x_p = \texttt{true}\}$.

(i) Give a Boolean formula φ_{trap} over variables $\{x_p : p \in P\}$ such that $\varphi_{\text{trap}}(x)$ holds if and only if Q_x is a trap of $\mathcal{N}$.

(ii) Give a quantified Boolean formula φ_{mintrap} over variables $\{x_p : p \in P\}$ such that $\varphi_{\text{mintrap}}(x)$ holds if and only if Q_x is a minimal trap of $\mathcal{N}$.

(iii) Give a Boolean formula $\varphi_{\text{constraints}}$ over variables $\{x_p : p \in P\}$ such that $\varphi_{\text{constraints}}(x)$ holds if and only if $M_0(Q_x) > 0$ implies $M(Q_x) > 0$.

(d) Construct $\varphi_{\text{trap}}(x)$ for the following Petri net:



(e) ★ Use the SMT solver `Z3` to prove that the Petri net above cannot reach a marking M such that $M(p_2) = 2$ and $M(p_3) \geq 1$. See instructions on the webpage of the course. You may start from the given partial solution if you need help.

Solution 5.1

(a) Recall that J is a T-invariant if and only if $\sum_{t \in {}^\bullet p} I(t) = \sum_{t \in p^\bullet} I(t)$ for every $p \in P$. This gives rise to the following system of equations:

$$\begin{aligned} J(t_1) &= J(t_3), \\ J(t_2) &= J(t_1) + J(t_2), \\ 0 &= J(t_2), \\ J(t_1) + J(t_3) + J(t_4) &= 0, \\ J(t_1) + J(t_5) &= J(t_4) + J(t_5), \\ J(t_2) &= J(t_5), \\ J(t_2) + J(t_5) &= 0. \end{aligned}$$

This system of equations is equivalent to $J(t_1) = J(t_2) = J(t_3) = J(t_4) = J(t_5) = 0$. Therefore, the vector space of T-invariants of $\mathcal{N}$ is trivial, i.e. it only contains the null vector.

★ This can be verified using `PIPE` by loading the Petri net and clicking on "Invariant Analysis" in the left menu.

(b) Yes. We can actually now show that $\mathcal{N}$ is not live from any initial marking M_0. Assume, $(\mathcal{N}, M_0)$ is live. Since $(\mathcal{N}, M_0)$ is also bounded by #4.3(b), then it is well-formed. We have seen that every well-formed net has a positive T-invariant. This is a contradiction since the only T-invariant of $\mathcal{N}$ is the trivial invariant which is not positive. Therefore, $(\mathcal{N}, M_0)$ is not live. In particular, this implies that both $(\mathcal{N}, M)$ and $(\mathcal{N}, M')$ are not live.

Solution 5.2

(a) Let $Q = \{p_1, p_3\}$. We have

$$\begin{aligned} {}^\bullet Q &= \{t_1, t_2\} \\ &\subseteq \{t_1, t_2, t_3\} \\ &= Q^\bullet. \end{aligned}$$

Therefore Q is a siphon. Since Q is not marked by M_0, we conclude that $(\mathcal{N}, M_0)$ is not live. □

(b) The incidence matrix of $\mathcal{N}$ is:

$$\boldsymbol{N} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The marking equation is:

$$\begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \boldsymbol{x}.$$

The unique solution of the marking equation is

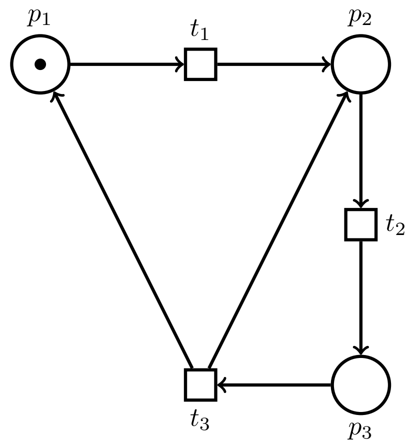
$$\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Since it is non negative, we cannot conclude whether the marking is reachable or not.

Let us consider the trap $Q = \{p_4\}$ which is marked by the initial marking. We can conclude that the target marking is not reachable since it does not mark Q.

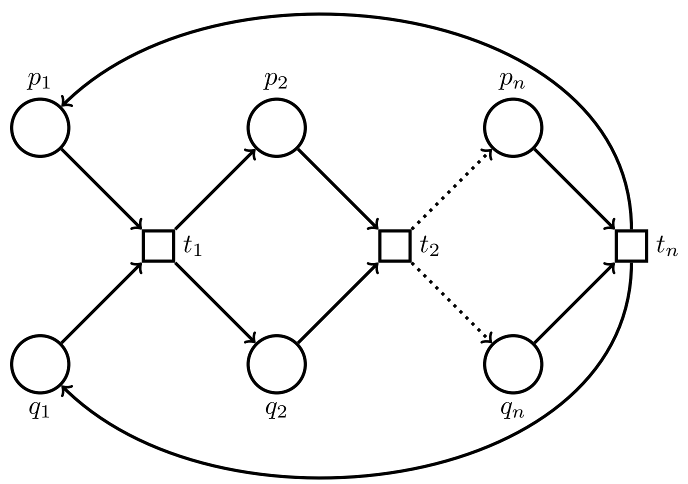
Solution 5.3

(a) Consider the following Petri net:



Let $M_0 = \{p_1\}$, $M = \emptyset$ and $Q = \{p_1, p_2, p_3\}$. Note that Q is a trap such that $M_0(Q) > 0$ and $M(Q) = 0$. Moreover, observe that Q is not a minimal trap since it contains the trap $\{p_2, p_3\}$. We claim that no minimal trap marks M_0. For the sake of contradiction, suppose there exists a minimal trap R such that $M_0(R) > 0$. We have $p_1 \in R$. Moreover, $R \neq \{p_1\}$ and $R \neq Q$ since the former is not a trap and the latter is not minimal. Thus, $R = \{p_1, p_2\}$ or $R = \{p_1, p_3\}$, but both are not traps, which yields a contradiction.

(b) For every $n \in \mathbb{N}_{\geq 1}$, let $\mathcal{N}_n = (P_n, T_n, F_n)$ be the following net:



Note that every set $Q \subseteq P_n$ that contains at least one of p_i or q_i for every $1 \leq i \leq n$ is a trap. Thus, $\mathcal{N}_n$ has at least $2^n = (\sqrt{2})^{|P_n|}$ traps.

(c) (i) We have

$$\begin{aligned}
Q_x \text{ is a trap} &\iff Q_x^\bullet \subseteq {}^\bullet Q_x \\
&\iff \forall t \in T\ [(t \in Q_x^\bullet) \implies (t \in {}^\bullet Q_x)] \\
&\iff \forall t \in T\ [(\exists p \in {}^\bullet t : p \in Q_x) \implies (\exists p \in t^\bullet : p \in Q_x)] \\
&\iff \forall t \in T\ [(\exists p \in {}^\bullet t : x_p) \implies (\exists p \in t^\bullet : x_p)].
\end{aligned}$$

Therefore, we construct the following formula:

$$\varphi_{\text{trap}}(x) = \bigwedge_{t \in T} \left[\left(\bigvee_{p \in {}^\bullet t} x_p \right) \rightarrow \left(\bigvee_{p \in t^\bullet} x_p \right) \right].$$

(ii) We have

$$\begin{aligned} Q_x \text{ is a min. trap} &\iff Q_x \text{ is a non empty trap} \land (\forall \emptyset \subset Q' \subset Q : Q' \text{ is not a trap}) \\ &\iff \varphi_{\text{trap}}(x) \land Q_x \neq \emptyset \land [\forall y : (Q_y \neq \emptyset \land Q_y \subseteq Q_x \land Q_y \neq Q_x) \implies \neg\varphi_{\text{trap}}(y)] \\ &\iff \varphi_{\text{trap}}(x) \land (\exists p \in P : x_p) \land [\forall y : ((\exists p \in P : y_p) \land (\forall p \in P : x_p \implies y_p) \land \\ &\qquad\qquad (\exists p \in P : \neg y_p \land x_p)) \implies \neg\varphi_{\text{trap}}(y)]. \end{aligned}$$

Therefore, we construct the following formula:

$$\varphi_{\text{mintrap}}(x) = \varphi_{\text{trap}}(x) \land \left(\bigvee_{p \in P} x_p \right) \land \left[\forall y \ \left(\left(\bigvee_{p \in P} y_p \right) \land \left(\bigwedge_{p \in P} (y_p \rightarrow x_p) \right) \land \left(\bigvee_{p \in P} (\neg y_p \land x_p) \right) \right) \rightarrow \neg\varphi_{\text{trap}}(y) \right].$$

(iii) We have

$$\begin{aligned} M_0(Q_x) > 0 \implies M(Q_x) > 0 &\iff (\exists p \in Q_x : M_0(p) > 0) \implies (\exists p \in Q_x : M(p) > 0) \\ &\iff (\exists p \in P : (x_p \land M_0(p) > 0)) \implies (\exists p \in P : (x_p \land M(p) > 0)). \end{aligned}$$

Therefore, we construct the following formula:

$$\varphi_{\text{constraints}}(x) = \left(\bigwedge_{p \in P} (x_p \land M_0(p) > 0) \right) \rightarrow \left(\bigwedge_{p \in P} (x_p \land M(p) > 0) \right).$$

(d) φ_{trap} is the conjunction of the following five constraints:

$$\begin{aligned} (x_{p_1} \lor x_{p_2}) &\rightarrow x_{p_1} \\ x_{p_1} &\rightarrow (x_{p_1} \lor x_{p_2}) \\ x_{p_2} &\rightarrow x_{p_4} \\ x_{p_4} &\rightarrow (x_{p_2} \lor x_{p_3} \lor x_{p_4}) \\ x_{p_4} &\rightarrow (x_{p_1} \lor x_{p_4}) \end{aligned}$$

(e) We could try to disprove the existence of M by using the state equation, i.e. by checking whether there exist no M and $\boldsymbol{x}$ such that

$$M_0 + \boldsymbol{N} \cdot \boldsymbol{x} = M, M(p_2) = 2 \text{ and } M(p_3) \geq 1.$$

However, there exist infinitely many solutions:

$$M = (0, 2, 1, 0) \text{ and } \boldsymbol{x} = (1, 3, m, 1 + m, 0) \text{ for any } m \in \mathbb{N}.$$

Observe that M_0 marks the minimal traps $\{p_1\}$ and $\{p_4\}$, but $M(p_1) = M(p_4) = 0$. By combining the state equation with the fact that minimal trap cannot be emptied, it is possible to conclude that there exist no solution $(M, \boldsymbol{x})$. This can be verified automatically with Z3 using the formulas obtained in (c):

```
; trap(x) holds iff Q_x is a trap
(define-fun trap ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool)) Bool
  (and
    (=> (or x1 x2) (or x1))
    (=> (or x1) (or x1 x2))
    (=> (or x2) (or x4))
    (=> (or x4) (or x2 x3 x4))
```

```
    (=> (or x4) (or x1 x4))
  )
)

; mintrap(x) holds iff Q_x is a minimal trap
(define-fun mintrap ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool)) Bool
  (and
    (trap x1 x2 x3 x4)                                    ; Q_x is a trap
    (or x1 x2 x3 x4)                                      ; Q_x is non empty
    (forall ((y1 Bool) (y2 Bool) (y3 Bool) (y4 Bool))     ; Every strict subset of Q_x is not a trap
      (=>
        (and
          (or y1 y2 y3 y4)                                ; Q_y is non empty
          (=> y1 x1) (=> y2 x2) (=> y3 x3) (=> y4 x4)     ; Q_y subset of Q_y
          (or (and (not y1) x1) (and (not y2) x2)         ; Q_y != Q_x
              (and (not y3) x3) (and (not y4) x4))
        )
        (not (trap y1 y2 y3 y4))                          ; Q(y) is not a trap
      )
    )
  )
)

; constraints(x, M) holds iff "Q_x marks M_0" implies "Q_x marks M"
(define-fun constraints ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool)
                         (m1 Int) (m2 Int) (m3 Int) (m4 Int)) Bool
  (=>
    (or x1 x4)          ; Q_x is marked in M_0, i.e. contains place p1 or p4
    (or                 ; Some place of Q_x is marked in M
      (and x1 (> m1 0))
      (and x2 (> m2 0))
      (and x3 (> m3 0))
      (and x4 (> m4 0))
    )
  )
)

; mreach(m) holds iff there is a solution of the marking equation
; from (2, 0, 0, 1) thats leads to m
(define-fun mreach ((m1 Int) (m2 Int) (m3 Int) (m4 Int)) Bool
  (exists ((t1 Int) (t2 Int) (t3 Int) (t4 Int) (t5 Int))
    (and
      (>= t1 0)
      (>= t2 0)
      (>= t3 0)
      (>= t4 0)
      (>= t5 0)
      (= m1 (+ 2 t1 (- 0 t2) t5))
      (= m2 (+ (* -2 t1) t2 (- 0 t3) t4))
      (= m3 t4)
      (= m4 (+ 1 t3 (- 0 t4) (- 0 t5)))
    )
  )
)

; There exists a solution to the marking equation
(declare-const m1 Int)
(declare-const m2 Int)
(declare-const m3 Int)
(declare-const m4 Int)

(assert (>= m1 0))
(assert (= m2 2))
(assert (>= m3 1))
(assert (>= m4 0))
```

```
(assert (mreach m1 m2 m3 m4))

; The reachable marking satisfies minimal trap constraints
;; (In general, non minimal trap constraints may be necessary
;;  to disprove reachability, but here it's not the case.
;;  You may simply replace "mintrap" by "trap" below.)
(assert
  (forall ((x1 Bool) (x2 Bool) (x3 Bool) (x4 Bool))
    (=>
      (mintrap x1 x2 x3 x4)
      (constraints x1 x2 x3 x4 m1 m2 m3 m4)
    )
  )
)

(check-sat)
```