Exercise 5.1
Consider the following Petri net $\mathcal{N} = (P, T, F)$:

(a) Give a basis of the vector space of $T$-invariants of $\mathcal{N}$. [Hint: use a characterization of $T$-invariants.]

(b) Let $M = \{p_1, p_2, p_4, p_4\}$ and $M' = \{p_1, p_3, p_5\}$. We have shown that $\mathcal{N}$ is bounded from any initial marking in #4.3(b). Can you tell whether $(\mathcal{N}, M)$ and $(\mathcal{N}, M')$ are live?

Exercise 5.2
Consider the following Petri net (with weights) $\mathcal{N}$:

(a) Use siphons/traps to prove or disprove that $\mathcal{N}$ is live from $M_0 = \{p_2, 3 \cdot p_4\}$,
(b) Can the marking equation be used to prove or disprove that \( \{p_2, p_4\} \xrightarrow{*} \{p_1, 3 \cdot p_2, p_3\}? \) Is so, why? If not, can traps or siphons help?

**Exercise 5.3**

Let \( \mathcal{N} = (P, T, W) \) be a net with weights and let \( M_0, M \) be markings. We say that a trap \( Q \) is *minimal* if it is non empty and every non empty \( Q' \subset Q \) is not a trap.

(a) Disprove the following statement:

There exists a trap \( Q \) of \( \mathcal{N} \) such that \( M_0(Q) > 0 \) and \( M(Q) = 0 \) if and only if there exists a minimal trap \( R \) of \( \mathcal{N} \) such that \( M_0(R) > 0 \) and \( M(R) = 0 \).

(b) Show that Petri nets can have exponentially many minimal traps. More formally, exhibit an infinite family of (distinct) nets \( \mathcal{N}_1 = (P_1, T_1, F_1), \mathcal{N}_2 = (P_2, T_2, F_2), \ldots \), some \( c > 1 \), and a function \( f \in \Omega(c^n) \) such that \( \mathcal{N}_i \) has \( f(|P_i|) \) minimal traps for every \( i \geq 1 \).

(c) For every \( x \in \{\text{false}, \text{true}\}^P \), let \( Q_x = \{p : p \in P, x_p = \text{true}\} \).

   (i) Give a Boolean formula \( \varphi_{\text{trap}} \) over variables \( \{x_p : p \in P\} \) such that \( \varphi_{\text{trap}}(x) \) holds if and only if \( Q_x \) is a trap of \( \mathcal{N} \).

   (ii) Give a quantified Boolean formula \( \varphi_{\text{mintrap}} \) over variables \( \{x_p : p \in P\} \) such that \( \varphi_{\text{mintrap}}(x) \) holds if and only if \( Q_x \) is a minimal trap of \( \mathcal{N} \).

   (iii) Give a Boolean formula \( \varphi_{\text{constraints}} \) over variables \( \{x_p : p \in P\} \) such that \( \varphi_{\text{constraints}}(x) \) holds if and only if \( M_0(Q_x) > 0 \) implies \( M(Q_x) > 0 \).

(d) Construct \( \varphi_{\text{trap}}(x) \) for the following Petri net:

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\begin{center}
\begin{tikzpicture}
  \node[place] (p1) at (2,3) {};  \node[place] (p2) at (6,3) {};  \node[place] (p3) at (2,-3) {};  \node[place] (p4) at (6,-3) {};
  \node[transition] (t1) at (1,1) {};  \node[transition] (t2) at (4,1) {};  \node[transition] (t3) at (4,-1) {};  \node[transition] (t4) at (1,-1) {};  \node[transition] (t5) at (6,-1) {};
  \draw (p1) edge[loop above] node{2} (p1) (p1) edge node{5} (t1) (t1) edge node{4} (p3) (p2) edge node{2} (t2) (t2) edge node{2} (p4) (p2) edge[loop above] node{2} (p2) (p3) edge node{2} (t4) (t4) edge node{2} (p4) (t3) edge node{2} (p4) (p4) edge[loop below] node{2} (p4) (t5) edge node{2} (p4);
\end{tikzpicture}
\end{center}
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(e) Use the SMT solver \( Z3 \) to prove that the Petri net above cannot reach a marking \( M \) such that \( M(p_2) = 2 \) and \( M(p_3) \geq 1 \). See instructions on the webpage of the course. You may start from the given partial solution if you need help.