Petri nets — Homework 3

Due 23.05.2018

Exercise 3.1
The algorithm COVERABILITY-GRAPH does not specify how the coverability graph should be traversed during its construction. Show that different traversal strategies can lead to different coverability graphs. More precisely, exhibit a marking \(M\) and two different coverability graphs for \((N, M)\), where \(N\) is the following net:

![Diagram](attachment:diagram.png)

Exercise 3.2
Let \(N\) and \(N'\) be respectively the left and right Petri nets below.

![Diagram](attachment:diagram.png)

Use the backward algorithm to answer the following questions.

(a) Describe the set of initial markings from which \(\{p_2, p_2\}\) is coverable in \(N\). Illustrate this set.

(b) Determine whether \(\{q_1, q_3\}\) is coverable from \(\{q_1\}\) in \(N'\).

(c) Determine whether \(\{q_1, q_2\}\) is coverable from \(\{q_1\}\) in \(N'\).
Exercise 3.3
A net with reset, doubling and transfer arcs is a tuple \((P, T, F, R, D, Tr)\) where \((P, T, F)\) is a net,

\[ R \subseteq P \times T, \quad D \subseteq T \times P, \quad Tr \subseteq (P \times T) \cup (T \times P), \]

and \(F, R, D\) and \(Tr\) are pairwise disjoint. Let \(M \in \mathbb{N}^P\) and \(t \in T\). We say that \(t\) is enabled at \(M\) if \(M(p) > 0\) for every \((p, t) \in F\). Firing \(t\) at \(M\) has the following effect:

- every arc \((p, t) \in F\) consumes a token from \(p\);
- every arc \((t, p) \in F\) produces a token in \(p\);
- every arc \((p, t) \in R\) empties \(p\);
- every arc \((t, p) \in D\) doubles the amount of tokens in \(p\);
- every arc \((p, t) \in Tr\) empties \(p\);
- every arc \((t, p) \in Tr\) adds \(\sum_{(q,t) \in Tr} M(q)\) tokens to \(p\).

Show that the backward algorithm works for this extended class of nets by showing that it is monotonic, i.e. show that for every markings \(X, X', Y \in \mathbb{N}^P\), if \(X \rightarrow Y\) and \(X' \geq X\), then \(X' \rightarrow Y'\) for some \(Y' \geq Y\).

Exercise 3.4

(a) Show that nets with inhibitor arcs are not monotonic.

(b) Give a net with reset arcs \(N\) and a marking \(M\) such that \((N, M)\) is bounded, but such that there exists a sequence \(M \xrightarrow{a} M' \xrightarrow{a'} M''\) with \(M'' \geq M'\) and \(M'' \neq M'\).
Solution 3.1
Let $M = \{p_1\}$. We exhibit two coverability graphs for $(\mathcal{N}, M)$, where nodes are labeled with respect to the total order $p_1 < p_2$. We construct the first coverability graph by first exploring the path $t_2t_3t_1t_1$:

For the second coverability graph, we first explore the path $t_2t_1t_3$:

Note that the subprocedure AddOmegas generates $(\omega, \omega)$ after exploring $t_2t_1t_3$ because, at this point, both $(1,0)$ and $(0,1)$ are “ancestors” of the current node labeled by $(1,1)$. 
Solution 3.2

(a) We execute the backward algorithm from $M = (0, 2)$. In order to build the whole set of initial markings, we ignore the stopping criterion based on $M_0$.

The set of initial markings is \( \{ M \in \mathbb{N}^2 : M \geq (0, 2) \text{ or } M \geq (2, 1) \text{ or } M \geq (3, 0) \} \).
(b) We want to determine whether $M = (1,0,1)$ is coverable from $M_0 = (1,0,0)$. It is not the case, since executing the backward algorithm from $M$ does not generate any marking less or equal to $M_0$:

<table>
<thead>
<tr>
<th>Iteration</th>
<th>pre($m$)</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>—</td>
<td>${(1,0,1)}$</td>
</tr>
<tr>
<td>1</td>
<td>$\text{pre}_1(1,0,1) = (1,0,1)$  [1,0,1]  [0,0,2]  [2,1,0]</td>
<td>${(1,0,1),(0,0,2),(2,1,0)}$</td>
</tr>
<tr>
<td>2</td>
<td>$\text{pre}_1(1,0,1) = (1,0,1)$ [1,0,1] $\text{pre}_2(1,0,1) = (0,0,2)$ [0,0,2] [2,1,0]</td>
<td>${(1,0,1),(0,0,2),(2,0,0)}$</td>
</tr>
<tr>
<td>3</td>
<td>$\text{pre}_1(2,0,0) = (2,0,0)$ [2,0,0] $\text{pre}_2(2,0,0) = (1,0,1)$ [1,0,1] [3,1,0]</td>
<td>${(1,0,1),(0,0,2),(2,0,0)}$ (unchanged)</td>
</tr>
</tbody>
</table>

(c) We want to determine whether $M' = (1,1,0)$ is coverable from $M_0 = (1,0,0)$. It is the case, since executing the backward algorithm from $M'$ yields $M_0$ after one iteration:

<table>
<thead>
<tr>
<th>Iteration</th>
<th>pre($m$)</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>—</td>
<td>${(1,1,0)}$</td>
</tr>
<tr>
<td>1</td>
<td>$\text{pre}_1(1,1,0) = (1,0,0)$ [1,0,0] $\text{pre}_2(1,1,0) = (0,1,1)$ [0,1,1] [2,2,0]</td>
<td>${(1,0,0),(0,1,1)}$</td>
</tr>
</tbody>
</table>

Solution 3.3

Let $X, X', Y \in \mathbb{N}^P$ and $t \in T$ be such that $X \xrightarrow{t} Y$ and $X' \geq X$. Let us first argue that $t$ is enabled at $X'$:


\(t\) is enabled at $X' \iff \quad X'(p) > 0 \text{ for every } (p,t) \in F$ 
\(\iff \quad X'(p) > 0 \text{ for every } (p,t) \in F \) (since $X' \geq X$) 
\(\iff \quad t \text{ is enabled at } X' \).

Let $Y' \in \mathbb{N}^P$ be the marking such that $X' \xrightarrow{t} Y'$. Let $p \in P$. It remains to show that $Y'(p) \geq Y(p)$. We only prove the case where $p$ is not in both the preset and postset of $t$:

<table>
<thead>
<tr>
<th>Case</th>
<th>Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(p,t) \in F$</td>
<td>$Y'(p) = X'(p) - 1 \geq X(p) - 1 = Y(p)$</td>
</tr>
<tr>
<td>$(t,p) \in F$</td>
<td>$Y'(p) = X'(p) + 1 \geq X(p) + 1 = Y(p)$</td>
</tr>
<tr>
<td>$(p,t) \in R$</td>
<td>$Y'(p) = 0 = Y(p)$</td>
</tr>
<tr>
<td>$(t,p) \in D$</td>
<td>$Y'(p) = 2 \cdot X'(p) \geq 2 \cdot X'(p) = Y(p)$</td>
</tr>
<tr>
<td>$(p,t) \in Tr$</td>
<td>$Y'(p) = 0 = Y(p)$</td>
</tr>
<tr>
<td>$(t,p) \in Tr$</td>
<td>$Y'(p) = X'(p) + \sum_{(q,t) \in Tr} X'(q) \geq X(p) + \sum_{(q,t) \in Tr} X(q) = Y(p)$</td>
</tr>
</tbody>
</table>
Solution 3.4
(a) Let $N$ be the following Petri net with inhibitor arcs:

![Petri Net Diagram](image)

We have $(0, 0) \xrightarrow{0} (0, 1)$, but $t$ is not enabled at $(1, 0)$.

(b) Consider the following Petri net $N$ where the arc from $p_3$ to $t_3$ is a reset arc.

![Petri Net Diagram](image)

It is bounded since its reachability graph is finite:

![Reachability Graph](image)

Moreover, we have $(0, 1, 1, 1, 0) \xrightarrow{t_3} (0, 1, 1, 1, 0) \xrightarrow{t_3^{t_1}t_2} (0, 1, 2, 1, 0)$. 