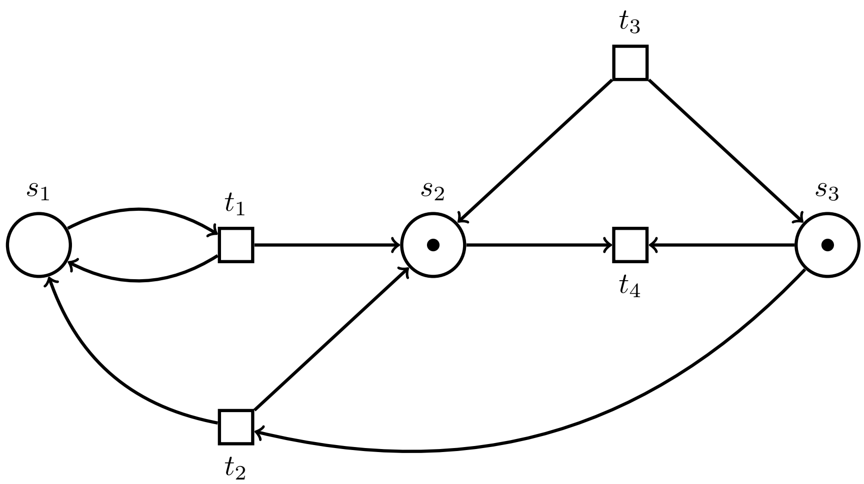
Petri nets — Endterm

- You have **75 minutes** to complete the exam.
- Answers must be written in a **separate booklet**. Do not answer on the exam.
- Please let us know if you need more paper.
- Write your name and Matrikelnummer on every sheet.
- Write with a non-erasable **pen**. Do not use red or green.
- You are not allowed to use auxiliary means other than pen and paper.
- You can obtain **40 points** plus **3 bonus points**. You need **17 points** to pass.
- Note that we sometimes represent a marking M by the tuple $(M(s_1), M(s_2), \ldots, M(s_n))$.

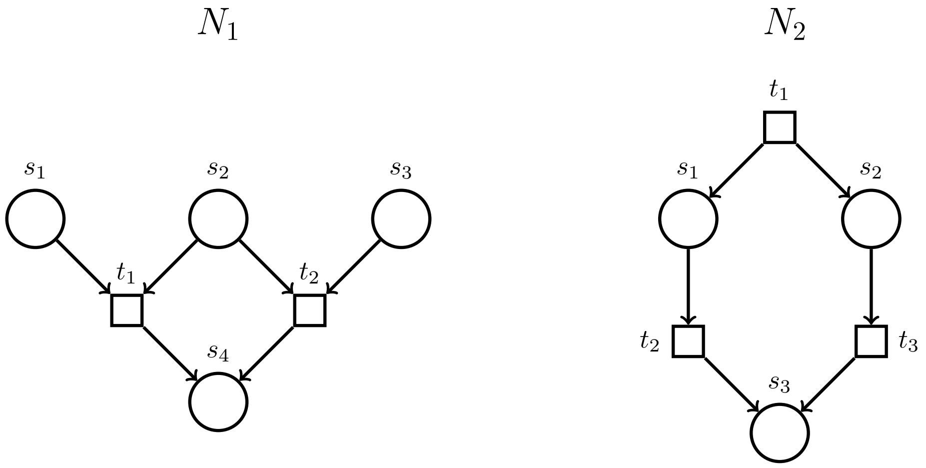
Question 1 (6 points)

Construct the coverability graph of this Petri net:



Question 2 (4 + 4 = 8 points)

Consider the following two nets:



(a) Construct a live and 1-bounded Petri net (N_1', M_1) such that N_1 is a subnet of N_1'.

> Recall: by the definition of a subnet, the arcs of N_1' between places of $\{s_1, s_2, s_3, s_4\}$ and transitions of $\{t_1, t_2\}$ are exactly those of N_1, no less and no more.

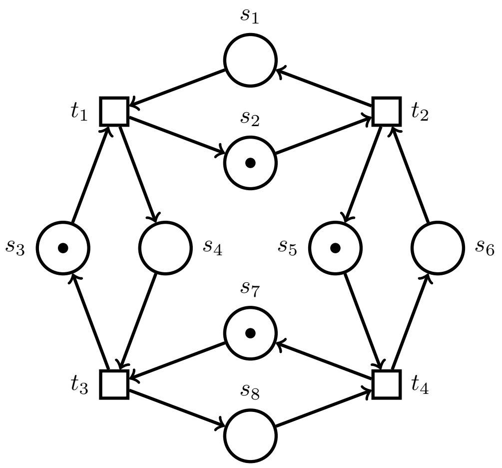
(b) Construct a live and 1-bounded Petri net (N_2', M_2) such that N_2 is a subnet of N_2'.

> Recall: by the definition of a subnet, the arcs of N_2' between places of $\{s_1, s_2, s_3\}$ and transitions of $\{t_1, t_2, t_3\}$ are exactly those of N_2, no less and no more.

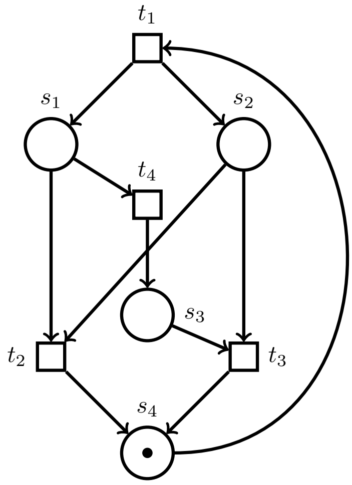
Question 3 **($2+2+2+2=8$ points)**

Consider the following Petri nets:

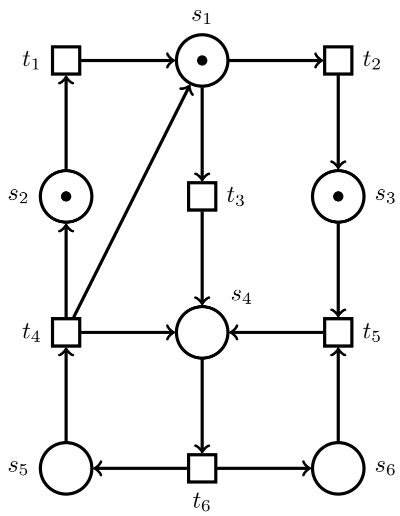
(N_1, M_1)



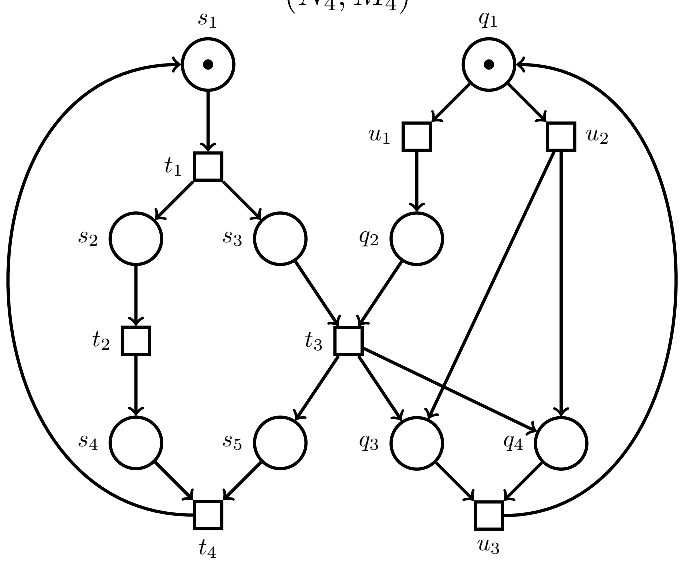
(N_2, M_2)



(N_3, M_3)



(N_4, M_4)



(a) Which ones are free-choice? For those that are not free-choice, explain why.

(b) Which ones are live? For those that are not live, explain why.

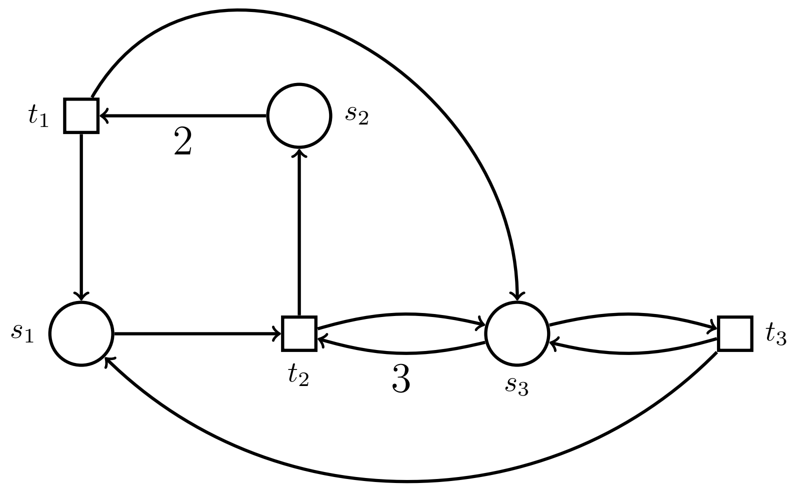
(c) Which ones are bounded? For those that are not bounded, explain why.

(d) Which ones are covered by S-components? For those that do, give the set of places of each S-component in a cover.

(e) **3 Bonus points.** Which ones have a siphon containing an initially unmarked proper trap? For those that do, give the siphon and the trap.

Question 4 (2 + 2 + 2 + 4= 10 points)

Consider the following net with weights $N = (S, T, W)$:



(a) Give all proper traps of N.

(b) Give all proper siphons of N.

(c) Give the incidence matrix of N.

(d) Say whether $M = (150, 80, 35)$ is reachable from $M_0 = (140, 70, 25)$ or not. Prove your answer.

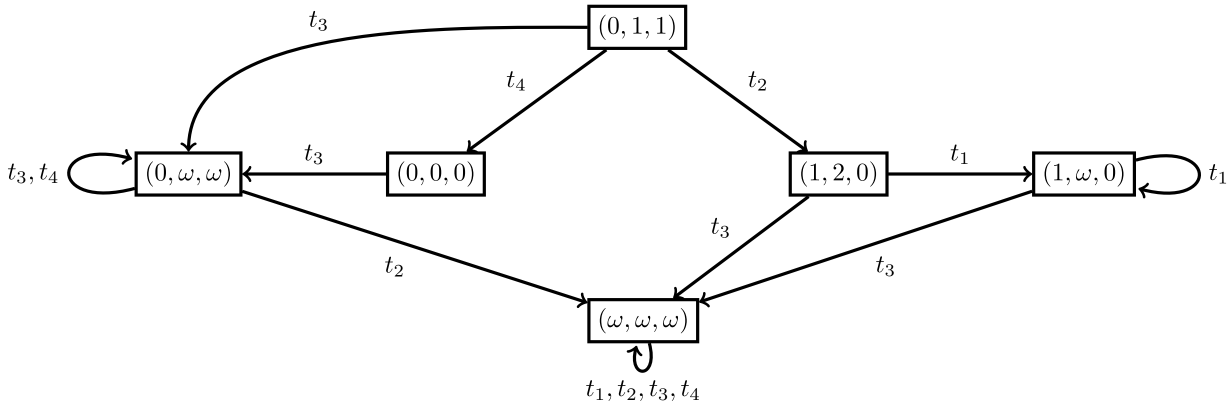
Question 5 (4 + 4 = 8 points)

Let $N = (S, T, F)$ be a strongly connected net with at least one place and one transition. Assume further that (N, M_0) is live and bounded for **every** marking M_0 such that $\sum_{s \in S} M_0(s) \geq 1$. Prove that N is an S-net in two steps:

(a) Prove that every transition of N has exactly one input place.

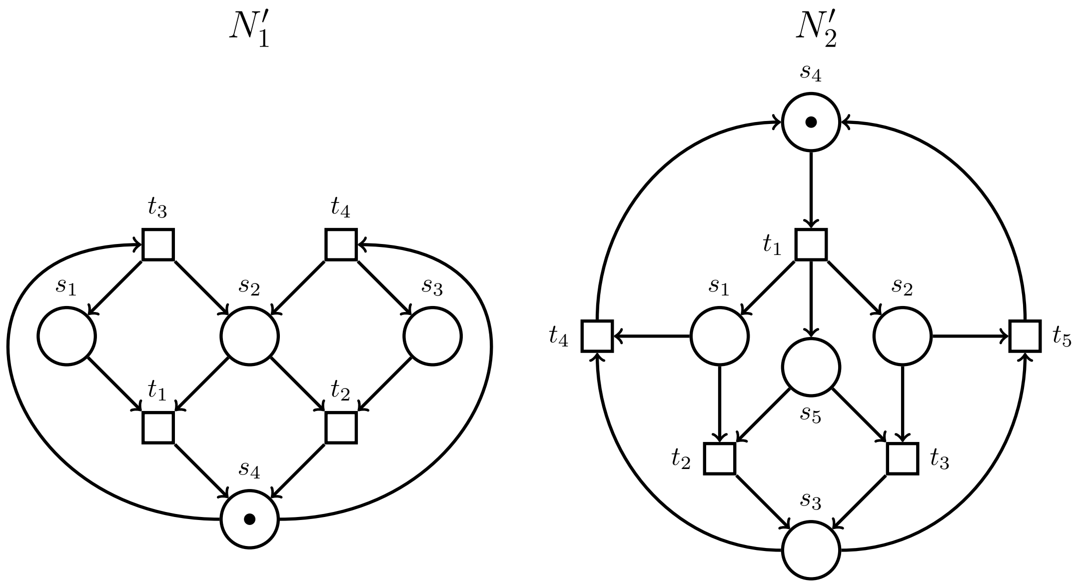
(b) Prove that every transition of N has exactly one output place.

Solution 1 (6 points)



Solution 2 $(4 + 4 = 8$ points)

Many different solutions are possible. For example:



Solution 3 $(2 + 2 + 2 + 2 = 8$ points)

(a) N_1, N_3, and N_4 are free-choice. N_2 is not free-choice because of $(s_1, t_2), (s_2, t_2), (s_2, t_3)$ and the absence of (s_1, t_3).

(b) (N_2, M_2) and (N_4, M_4) are live. (N_1, M_1) is not live because M_1 does not enable any transition. (N_3, M_3) is not live because firing $t_2\, t_1\, t_2$ leads to a deadlock.

(c) (N_1, M_1), (N_2, M_2), and (N_4, M_4) are bounded. (N_3, M_3) is not bounded. Since

$$(1, 1, 1, 0, 0, 0) \xrightarrow{t_3} (0, 1, 1, 1, 0, 0) \xrightarrow{t_6\, t_4} (1, 2, 1, 2, 1, 0)\,,$$

and $(1, 2, 1, 2, 1, 0) \geq (0, 1, 1, 1, 0, 0)$, we have $(0, 1, 1, 1, 0, 0) \xrightarrow{(t_6\, t_4)^n} (n, n+1, 1, n+1, n, 0)$ for every $n \geq 0$, and so (N_3, M_3) is not bounded.

(d) (N_3, M_3) is not covered by S-components. The others are.
Cover for (N_1, M_1): $\{\, \{s_1, s_2\}, \{s_3, s_4\}, \{s_5, s_6\}, \{s_7, s_8\} \,\}$.
Cover for (N_2, M_2): $\{\, \{s_1, s_3, s_4\}, \{s_2, s_4\} \,\}$.
Cover for (N_4, M_4): $\{\, \{s_1, s_2, s_4\}, \{s_1, s_3, s_5\}, \{q_1, q_2, q_3\}, \{q_1, q_2, q_4\} \,\}$.

(e) (N_1, M_1) and (N_3, M_3) have such a siphon, the others do not.
For (N_1, M_1): $\{s_1, s_4, s_8, s_6\}$ is both a siphon and a proper trap that is initially unmarked.

For (N_3, M_3): $\{s_1, s_2, s_4, s_5, s_6\}$ is a siphon, and $\{s_4, s_5, s_6\}$ is a proper trap contained in it, initially unmarked.

Solution 4 (2 + 2 + 2 + 4= 10 points)

(a) $\{s_3\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}$.

(b) $\{s_2, s_3\}, \{s_1, s_2, s_3\}$.

(c)

$$\boldsymbol{N} = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 1 & -2 & 0 \end{pmatrix}$$

(d) No. Let us consider the marking equation $M = M_0 + \boldsymbol{N} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ which is equivalent to

$$\begin{aligned} x - y + z &= 10 \\ -2x + y &= 10 \\ x - 2y &= 10 \end{aligned}$$

From the last two equations we obtain $x = y = -10$ (in the exam you were expected to correctly compute this). This implies that any solution[1] to the marking equation has negative entries, and hence that M is not reachable.

Solution 5 (4 + 4 = 8 points)

(a) Since N is strongly connected, every transition has at least one input place. We prove that it has at most one input place by contradiction. Assume that some transition $t \in T$ has two distinct input places s_1, s_2. Let M be the marking satisfying $M(s_1) = 1$ and $M(s) = 0$ for every $s \neq s_1$ (in particular $M(s_2) = 0$). Since (N, M) is live, there is a firing sequence $M \xrightarrow{\sigma} M'$ such that M' enables t, i.e., satisfies $M'(s_1) > 0$ and $M'(s_2) > 0$. It follows that $M \leq M'$ and $M \neq M'$, and so (N, M) is not bounded, contradiction.

(b) Since N is strongly connected, every transition has at least one output place. We prove that it has at most one output place by contradiction. Assume that a transition $t \in T$ has at least two distinct output places s_1, s_2.

First proof : Let M be the marking satisfying $M(s_1) = 1$ and $M(s) = 0$. Since (N, M) is live, we can reach a marking M' that enables t, and then fire t reaching a marking M''. This marking satisfies $M''(s_1) > 0$ and $M''(s_2) > 0$. Proceed now as in (a).

Second proof: Let M be any marking satisfying $\sum_{s \in S} M(s) \geq 1$. Since (N, M) is live, M enables an infinite firing sequence σ containing infinitely many occurrences of t. By (a), firing t increases the total number of tokens in the net by at least one, and firing any other transition does not decrease the total number of tokens. So during the execution of σ the number of tokens in N grows without bound. So (N, M) is not bounded, contradiction.

[1] The solution is unique.