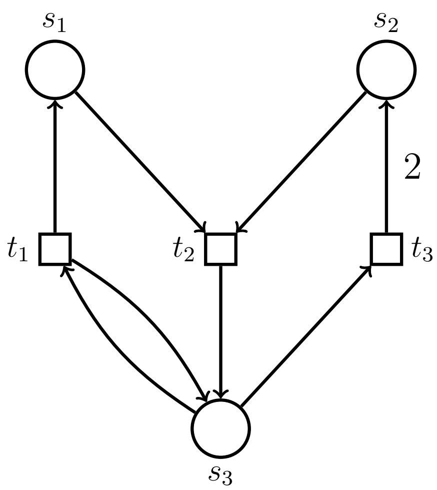
Petri nets — Endterm

- You have **90 minutes** to complete the exam.
- Answers must be written in a **separate booklet**. Do not answer on the exam.
- Please let us know if you need more paper.
- Write your name and Matrikelnummer on every sheet.
- Write with a non-erasable **pen**. Do not use red or green.
- You are not allowed to use auxiliary means other than pen and paper.
- You can obtain **40 points**. You need **17 points** to pass.
- Note that we sometimes represent a marking M by the tuple $(M(s_1), M(s_2), \ldots, M(s_n))$.

Question 1 (4 + 2 = 6 points)
Consider the following Petri net with weights $N = (S, T, W)$:



Let $M_0 = (0, 1, 0)$ and $M = (1, 0, 1)$. We wish to determine whether M is coverable from M_0. After one iteration of the backward reachability algorithm, we obtain the minimal basis $X = \{(2, 1, 0), (0, 0, 1)\}$.

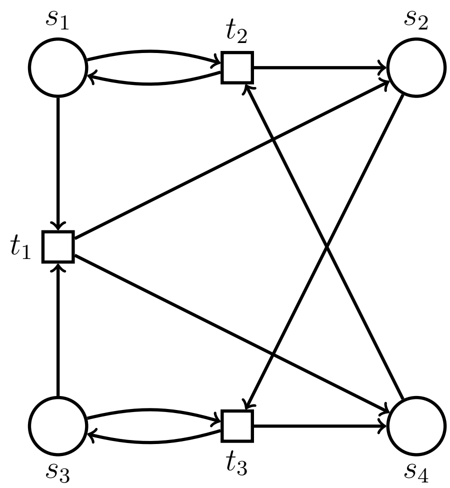
(a) Give the *minimal* basis Y obtained by executing the next iteration of the backward reachability algorithm from X.

(b) What can you conclude from Y obtained in (a)?

1. M is coverable from M_0.
2. M is not coverable from M_0.
3. None of the above, another iteration must be executed.

Justify your answer.

Question 2 $(2+2+2+2=8$ points)
Consider the following Petri net $N = (S, T, F)$:



(a) Give all of the minimal proper traps of N. Explain briefly why no other proper trap is minimal.

(b) Does N have a positive S-invariant? If so, exhibit one, if not, explain why.

(c) Prove that $M = (5, 13, 7, 15)$ is *not* reachable from $M_0 = (15, 3, 17, 4)$.

(d) Prove that $M = (0, 20, 0, 0)$ is *not* reachable from $M_0 = (10, 0, 10, 0)$.

Question 3 (5 points)
Recall that 3-SAT is the problem of determining the satisfiabillity of a Boolean formula in conjunctive normal form where clauses have at most three literals.

Give a polynomial time reduction from 3-SAT to the following reachability problem for 1-safe Petri nets:

Given:	1-safe Petri net (N, M_0) and a place s of N.
Determine:	does there exist a marking M such that $M_0 \xrightarrow{*} M$ and $M(s) = 1$?

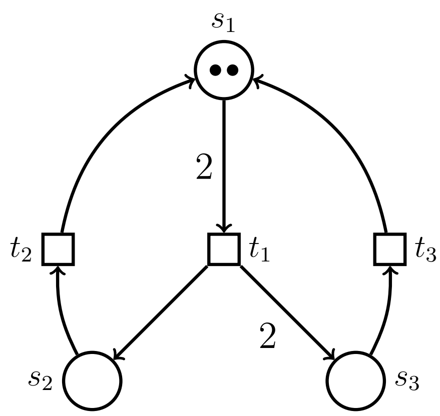
It suffices to explain your reduction informally and to illustrate it for the following formula:

$$\varphi(x_1, x_2, x_3, x_4) = (x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee x_4).$$

Question 4 (4 points)
The *projection* of a firing sequence $\sigma \in T^*$ onto $U \subseteq T$ is the sequence $h_U(\sigma)$ obtained by deleting all transitions of σ which do not belong to U. For example, $h_{\{u,v\}}(uvtvt) = uvv$, $h_{\{t\}}(uvtvt) = tt$ and $h_{\{u,v\}}(ttt) = \varepsilon$. The U-*traces* of a Petri net (N, M_0) is the set

$$L_U(N, M_0) = \{h_U(\sigma) : \sigma \text{ is a firing sequence enabled at } M_0\}.$$

Consider the following deadlock-free Petri net with weights (N, M_0):



Give a new Petri net (N', M_0') such that

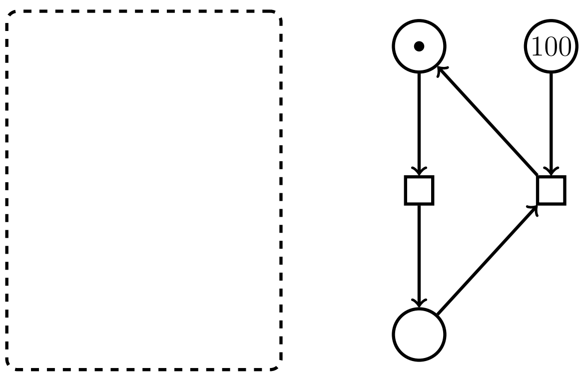
- N' has *no weights*,
- (N', M_0') is deadlock-free, and
- $L_{\{t_1,t_2,t_3\}}(N', M_0') = L_{\{t_1,t_2,t_3\}}(N, M_0)$.

Question 5 (3 + 3 + 3= 9 points)

(a) Give a Petri net (N, M_0) and connect it with arcs to the Petri net (N', M_0') shown below so that the resulting Petri net (N'', M_0'') satisfies the following properties:

- (N'', M_0'') is bounded, and
- (N'', M_0'') has a reachable marking with at least 2^{100} tokens.

(N, M_0) (N', M_0')



(b) Exhibit a connected Petri net $N = (S, T, F)$ such that $I = (1, -1, 0)$ is an S-invariant and $J = (1, 0, 1)$ is a T-invariant of N. Justify your answer.

(c) Exhibit a deadlock-free Petri net (N, M_0) and a marking $M \geq M_0$ such that (N, M) is *not* deadlock-free.

Question 6 (4 + 4 = 8 points)

(a) Let (N, M_0) be a live T-system. Prove that (N, M_0) is cyclic.

(b) Let N be a T-net and let M_0, M be markings. Prove that if (N, M_0) is live and $2M_0 \xrightarrow{*} 2M$, then $M_0 \xrightarrow{*} M$.

Solution 1

(a) By applying the algorithm, we obtain:

$$\begin{aligned}
\text{pre}_{t_1}(2,1,0) &= (1,1,1),\\
\text{pre}_{t_1}(0,0,1) &= (0,0,1),\\
\text{pre}_{t_2}(2,1,0) &= (3,2,0),\\
\text{pre}_{t_2}(0,0,1) &= (1,1,0),\\
\text{pre}_{t_3}(2,1,0) &= (2,0,1),\\
\text{pre}_{t_3}(0,0,1) &= (0,0,2).
\end{aligned}$$

The only marking which is not covered by X is $(1,1,0)$. Since $(1,1,0) < (2,1,0)$, we obtain the new basis $Y = \{(1,1,0), (0,0,1)\}$.

(b) We cannot conclude anything, another iteration is required. Indeed, we cannot conclude that M is coverable since M_0 is not larger or equal to any marking of Y. Moreover, we cannot conclude that M is uncoverable since $Y \neq X$ which means that at least one other iteration must be performed.

Solution 2

(a) $\{s_1, s_4\}$, $\{s_2, s_3\}$ and $\{s_2, s_4\}$.

Explanation. An inspection of the presets/postsets shows that none of the place is a trap on its own, and that the traps of size two are: $\{s_1, s_4\}$, $\{s_2, s_3\}$ and $\{s_2, s_4\}$. We claim that these traps are minimal. Indeed, any subset of size three or four must contain one of $\{s_1, s_4\}$, $\{s_2, s_3\}$ and $\{s_2, s_4\}$.

(b) Yes. A vector I is an S-invariant of N if and only if it is a solution of the following system:

$$\begin{aligned}
I(s_1) + I(s_2) &= I(s_3) + I(s_4),\\
I(s_1) + I(s_4) &= I(s_1) + I(s_3),\\
I(s_2) + I(s_3) &= I(s_2) + I(s_4).
\end{aligned}$$

This system is equivalent to

$$\begin{aligned}
I(s_2) &= 2 \cdot I(s_3) + I(s_1),\\
I(s_4) &= I(s_3).
\end{aligned}$$

Therefore, the vector space of S-invariants is described by $\{x \cdot (0,2,1,1) + y \cdot (1,-1,0,0) : x, y \in \mathbb{R}\}$. By taking $x = y = 1$, we obtain the positive S-invariant $I = (1,1,1,1)$.

Alternative solution. The vector $(1,1,1,1)$ is immediately seen as a positive S-invariant since transitions do not change the amount of tokens.

(c) The marking equation for M_0 and M is:

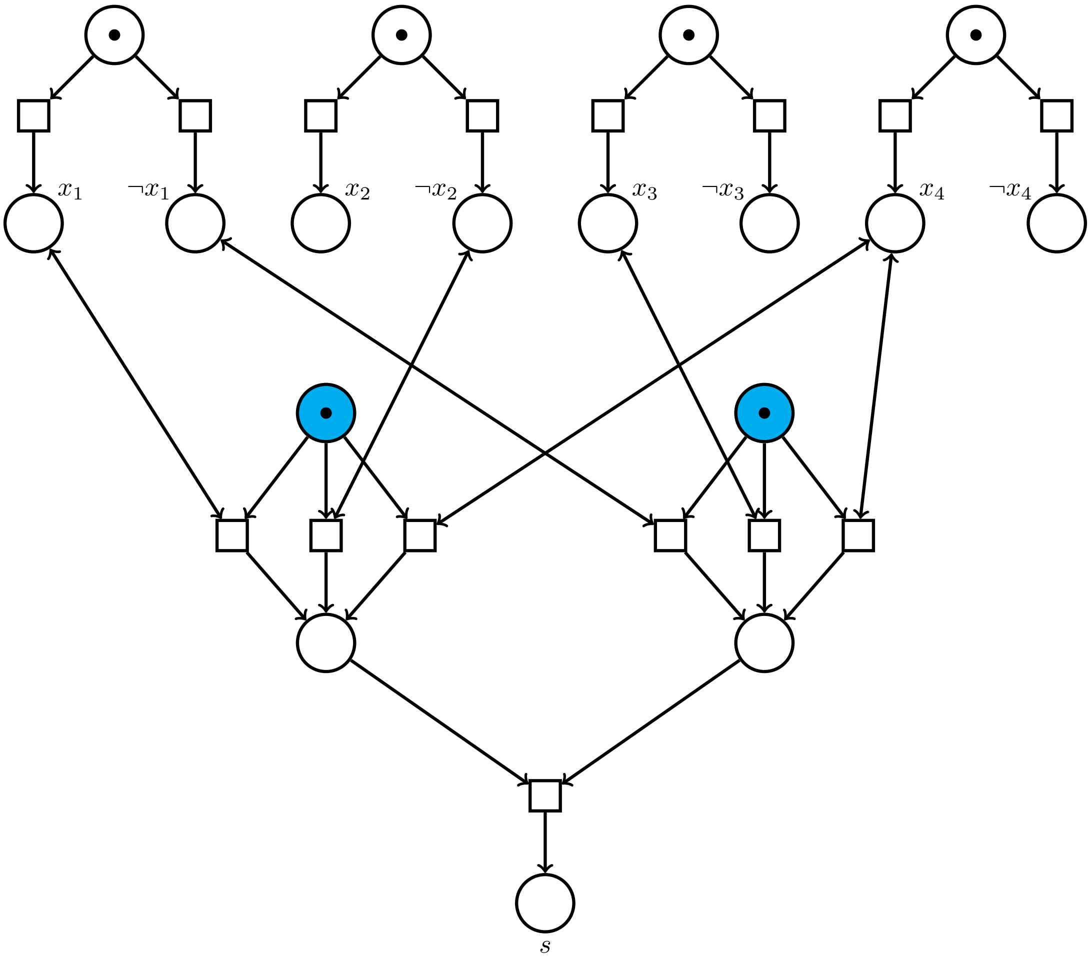
$$\begin{aligned}
15 - x_1 &= 5,\\
3 + x_1 + x_2 - x_3 &= 13,\\
17 - x_1 &= 7,\\
4 + x_1 - x_2 + x_3 &= 15.
\end{aligned}$$

The first equation implies that $x_1 = 10$. By the second equation, we obtain $x_2 = x_3$. By the fourth equation, we obtain $0 = 1$ which is a contradiction. Therefore, M is not reachable from M_0. □

Alternative solution. We have $(1,1,1,1) \cdot M_0 = 39 \neq 40 = (1,1,1,1) \cdot M$. Since $(1,1,1,1)$ is an S-invariant, M is not reachable from M_0. □

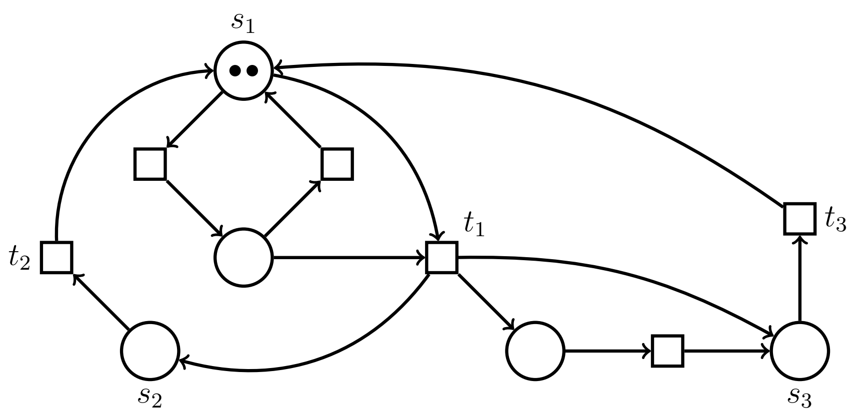
(d) The trap $\{s_2, s_3\}$ is initially marked at M_0, but not marked at M. Therefore, M is not reachable from M_0. □

Solution 3



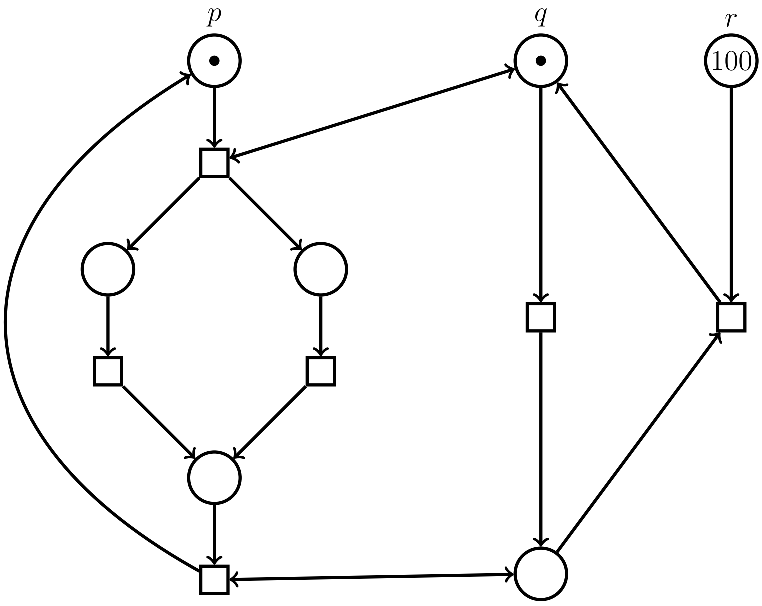
Note that the places colored in blue are crucial to make the Petri net 1-safe. Without these places, the net would only be 3-safe.

Solution 4



Solution 5

(a)



We have $\{p, q, 100 \cdot r\} \xrightarrow{*} \{2^{100} \cdot p, q, r\}$.

(b) By the given dimensions, the Petri net $N = (S, T, F)$ must have three places and three transitions. Therefore, without loss of generality, $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$. Let the incidence matrix of N be

	t_1	t_2	t_3
s_1	a	b	c
s_2	d	e	f
s_3	g	h	i

Since $I = (1, -1, 0)$ is an S-invariant, we have

$$a - d = 0,$$
$$b - e = 0,$$
$$c - f = 0.$$

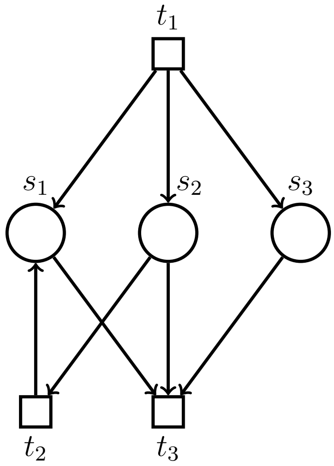
Moreover, since $J = (1, 0, 1)$ is a T-invariant, we have

$$a + c = 0,$$
$$d + f = 0,$$
$$g + i = 0.$$

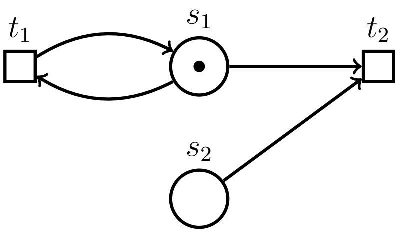
Therefore, we must have $a = d = -c = -f$, $b = e$ and $g = -i$. The assignment $a = b = d = e = g = 1$, $c = f = i = -1$ and $h = 0$ satisfies the above constraints. Therefore, it suffices to construct a connected Petri net whose incidence matrix is

	t_1	t_2	t_3
s_1	1	1	−1
s_2	1	−1	−1
s_3	1	0	−1

The following Petri net is such a Petri net:



(c) The following Petri net is deadlock-free from $(1, 0)$ since s_1 is always enabled. However, it is not deadlock-free from $(1, 1)$, since $(1, 1) \xrightarrow{t_2} (0, 0)$ and $(0, 0)$ is dead.



Solution 6

(a) Let M be a marking such that $M_0 \xrightarrow{*} M$. Since (N, M_0) is live, the reachability theorem for T-systems implies that $M_0 \sim M$. Note that $\sim$ is symmetric, and in particular that $M \sim M_0$. Moreover, (N, M) is live. Therefore, by the reachability theorem, we have $M \xrightarrow{*} M_0$. □

(b) Since (N, M_0) is live, every circuit of N is marked by M_0. Thus, since $2M_0 \geq M_0$, every circuit of N is also marked by $2M_0$. This implies that $(N, 2M_0)$ is live. By the reachability theorem for T-systems, we have $2M_0 \sim 2M$. Note that

$$\begin{aligned} 2M_0 \sim 2M &\iff I \cdot 2M_0 = I \cdot 2M \text{ for every } S\text{-invariant } I \\ &\iff I \cdot M_0 = I \cdot M \text{ for every } S\text{-invariant } I \\ &\iff M_0 \sim M. \end{aligned}$$

Therefore, $M_0 \sim M$ and by the reachability theorem we have $M_0 \xrightarrow{*} M$. □