Technische Universität München **Summer term 2017**
I7
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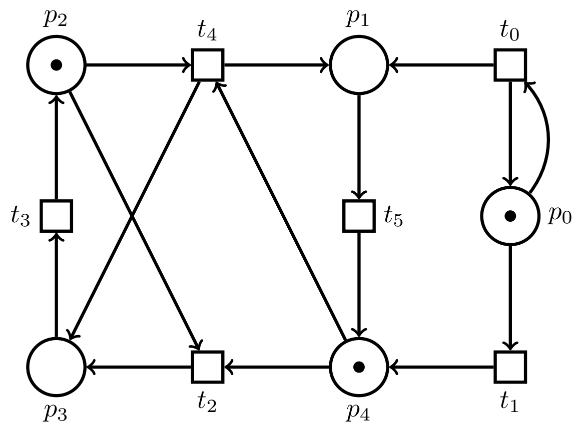
Petri nets — Homework 7

Due 26.07.2017

Exercise 7.1
Consider the following free-choice system $(\mathcal{N}, M_0)$:



(a) Give all minimal proper siphons of $(\mathcal{N}, M_0)$.

(b) Use (a) to say whether $(\mathcal{N}, M_0)$ is live or not.

Exercise 7.2

(a) Exhibit a non live system $(\mathcal{N}, M_0)$ for which every proper siphon contains a trap marked at M_0.

(b) Exhibit a live system $(\mathcal{N}, M_0)$ with a proper siphon that does not contain a trap marked at M_0. [Hint:]

(c) A system $(\mathcal{N}, M_0)$ is monotonously live if $(\mathcal{N}, M)$ is live for every marking $M \geq M_0$. In exercise #2.1, we have seen that Petri nets are generally not monotonously live. Show that live free-choice systems are monotonously live.

Exercise 7.3
Let $(\mathcal{N}, M_0)$ be a bounded and strongly connected free-choice system which is deadlock-free, where $\mathcal{N} = (P, T, F)$. For every $M \in \mathbb{N}^P$, let $d(M)$ be the number of transitions dead at M. Let $K \in \mathbb{N}^P$ be such that $d(K) = \max\{d(M) : M_0 \xrightarrow{*} M\}$.

(a) Let $u \in T$ be a transition not dead at K. Show that there exists an infinite firing sequence $\sigma \in T^\omega$ enabled at K and containing infinitely many occurrences of u. [Hint:]

(b) Let $u, v \in T$ be such that u is not dead at K and $v \in (u^\bullet)^\bullet$. Show that v is not dead at K. [Hint:]

(c) Show that there exists a path $\gamma \in (T \cup P)^*$ of $\mathcal{N}$ such that γ contains all transitions of T and γ starts with a transition enabled at K.

(d) Use (b) and (c) to show that $d(K) = 0$, and hence that $(\mathcal{N}, M_0)$ is live.

Exercise 7.4

Exercise #7.1 shows the following theorem:

> Every bounded, strongly connected and free-choice system is live if and only if it is deadlock-free.

Show that this theorem does not hold anymore if we remove any of its three conditions. More precisely,

(a) Exhibit a bounded and free-choice system which is deadlock-free, but not live;

(b) Exhibit a bounded and strongly connected system which is deadlock-free, but not live;

(c) ★ Exhibit a strongly connected and free-choice system which is deadlock-free, but not live. [Hint:]

Solution 7.1

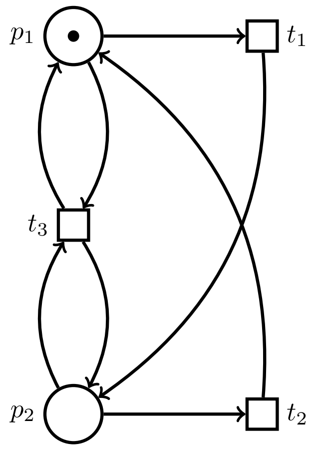
(a) We claim that the system has two minimal proper siphons: $\{p_0\}$ and $\{p_2, p_3\}$.

Let us show the claim. By inspecting $^\bullet p$ and $p^\bullet$ for every place p, we find a single siphon of size one: $\{p_0\}$. Moreover, we have $^\bullet\{p_2, p_3\} = \{t_2, t_3, t_4\} = \{p_2, p_3\}^\bullet$. Now, note that $t_0 \in {}^\bullet p_1$ and $^\bullet t_0 = \{p_0\}$. Therefore, any siphon containing p_1 must also contain p_0. Similarly, any siphon containing p_4 must also contain p_0. Thus, no minimal siphon contains p_1 or p_4, and we are done. $\square$

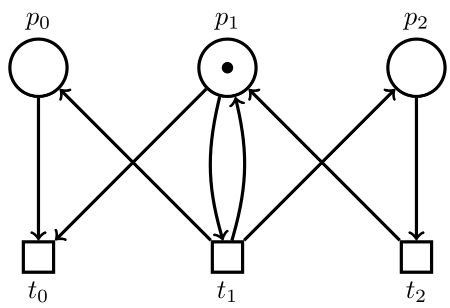
(b) The system is not live. By Commoner's Theorem, the system is live if and only if every minimal proper siphon contains a trap marked at M_0. The minimal siphon $\{p_2, p_3\}$ is also a trap and it is marked at M_0. However, the minimal siphon $\{p_0\}$ is not a trap and hence it does not contain a marked trap.

Solution 7.2

(a) The following system is not live since t_3 is dead. The only proper siphon of the system is $Q = \{p_1, p_2\}$. The set Q is also a trap which is marked at M_0.



(b) The following system is live, however the siphon $\{p_0, p_1, p_2\}$ contains no trap marked at M_0 since the net does not contain any proper trap.



(c) Let $(\mathcal{N}, M_0)$ be a live free-choice system. By Commoner's Liveness Theorem, every proper siphon of $\mathcal{N}$ contains a trap marked at M_0. Let $M \geq M_0$. Each trap marked at M_0 is also marked at M. Therefore, every proper siphon of $\mathcal{N}$ contains a trap marked at M. By Commoner's Liveness Theorem, $(\mathcal{N}, M)$ is live. $\square$

Solution 7.3

(a) Since u is not dead at K, there exist $\sigma_1 \in T^*$ and $L_1 \in \mathbb{N}^P$ such that $K \xrightarrow{\sigma_1} L_1$ and u is enabled at L_1. Let $L'_1 \in \mathbb{N}^P$ be such that $L_1 \xrightarrow{u} L'_1$. By maximality of $d(K)$, we have $d(L'_1) \leq d(K)$. Moreover, every transition dead at K is also dead at L'_1, and hence the transitions dead at K and L'_1 are the same. Therefore, u is not dead at L'_1. This implies that there exist $\sigma_2 \in T^*$ and $L_2 \in \mathbb{N}^P$ such that $L'_1 \xrightarrow{\sigma_2} L_2$ and u is enabled at L_2. By repeating this argument, we obtain an infinite firing sequence $\sigma = \sigma_1 u \sigma_2 u \cdots$ enabled at K and containing infinitely many occurrences of u. $\square$

(b) By (a), there exists an infinite firing sequence $\sigma \in T^\omega$ which is enabled at K and such that σ contains infinitely many occurrences of u. Since $v \in (u^\bullet)^\bullet$, there exists $p \in P$ such that $(u, p), (p, v) \in F$. Since $(\mathcal{N}, M_0)$ is bounded and u produces a token in p, σ must contain infinitely many occurrences of a transition w such that $p \in {}^\bullet w$. Indeed, otherwise p would be unbounded. In particular, this implies that w is not dead at K. Since $\mathcal{N}$ is free-choice, v is enabled at the same markings as w. Therefore, v is not dead at K. □

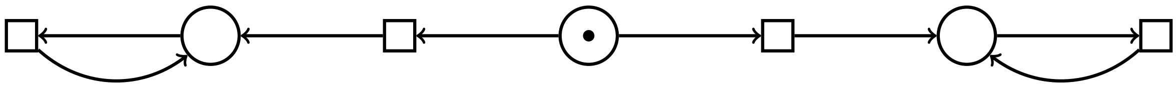
(c) Since $(\mathcal{N}, M_0)$ is deadlock-free and K is reachable from M_0, there exists a transition t enabled at K. Moreover, since $\mathcal{N}$ is strongly connected, there exists a path starting in t that goes through all nodes of $\mathcal{N}$. □

(d) By (c), there exists a path γ that contains all transitions of $\mathcal{N}$ and whose first transition is enabled at K. Let $\gamma = t_1 p_1 t_2 \cdots p_{n-1} t_n$. We claim that, for every $1 \leq i \leq n$, t_i is not dead at K. It follows from this claim that $d(K) = 0$, and hence, by maximality of $d(K)$, that $(\mathcal{N}, M_0)$ is live.

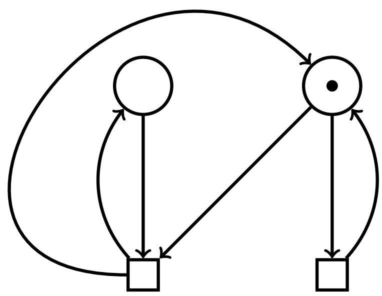
We prove the claim by induction on i. The base case follows immediately. Let $i > 1$ and assume that the claim holds for t_{i-1}. We have $(t_{i-1}, p_{i-1}), (p_{i-1}, t_i) \in F$. Therefore, $t_i \in (t_{i-1}^\bullet)^\bullet$. By induction hypothesis, t_{i-1} is not dead at K and hence, by (b), t_i is also not dead. □

Solution 7.4

(a)



(b)



(c)