Technische Universität München **Summer term 2017**
I7
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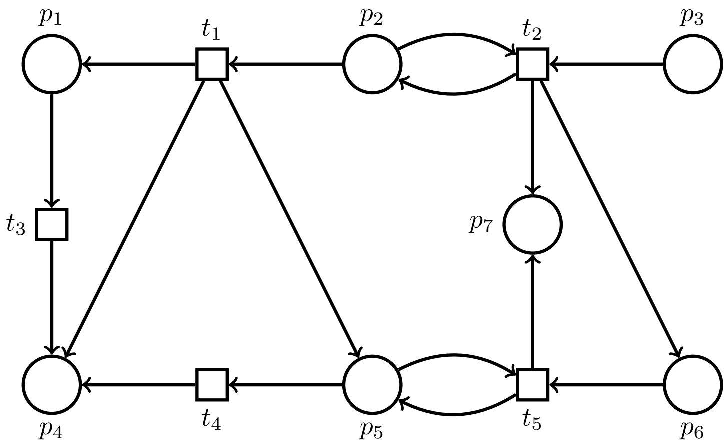
Petri nets — Homework 5

Due 27.06.2017

Exercise 5.1
Consider the following Petri net $\mathcal{N} = (P, T, F)$:



(a) Give a basis of the vector space of T-invariants of $\mathcal{N}$. [Hint:]

(b) Let $M = \{p_1, p_2, p_4, p_4\}$ and $M' = \{p_1, p_3, p_5\}$. We have shown that $\mathcal{N}$ is bounded from any initial marking in #4.3(b). Can you tell whether $(\mathcal{N}, M)$ and $(\mathcal{N}, M')$ are live?

Exercise 5.2

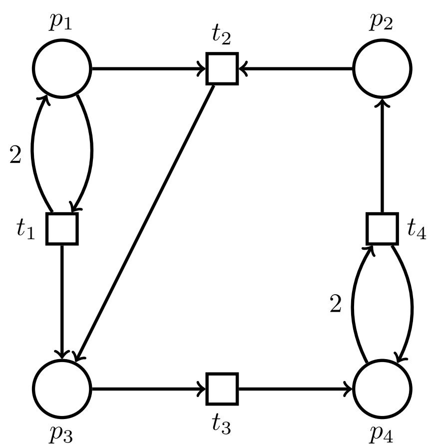
(a) Show that the union of two traps is also a trap.

(b) Give an algorithm that takes as input a Petri net $\mathcal{N} = (P, T, F)$ and a subset $R \subseteq P$, and that returns the largest trap of $\mathcal{N}$ contained in R.

(c) Show that your algorithm is correct and that it always terminates.

Exercise 5.3

Consider the following Petri net $\mathcal{N}$:



(a) Use siphons/traps to prove or disprove that $\mathcal{N}$ is live from $M_0 = \{p_2, 3 \cdot p_4\}$,

(b) Can the marking equation be used to prove or disprove that $\{p_2, p_4\} \xrightarrow{*} \{p_1, 3 \cdot p_2, p_3\}$? Is so, why? If not, can traps or siphons help?

Exercise 5.4

Let $\mathcal{N} = (P, T, F)$ be a Petri net. Let $Q_x = \{p : p \in P, x_p = \texttt{true}\}$ for every $x \in \{\texttt{false}, \texttt{true}\}^P$.

(a) Give a Boolean formula φ over variables $\{x_p : p \in P\}$ such that $\varphi(x)$ holds if and only if Q_x is a siphon of $\mathcal{N}$.

(b) Let $M_0, M \in \mathbb{N}^P$. Modify you previous construction in order to obtain a formula ψ such that $\psi(x)$ holds if and only if Q_x is a siphon of $\mathcal{N}$, $M_0(Q_x) = 0$ and $M(Q_x) > 0$.

(c) Construct ψ for the Petri net of #5.3 with $M_0 = \{p_2, p_4\}$ and $M = \{p_1, 3 \cdot p_2, p_3\}$.

Solution 5.1

(a) Recall that J is a T-invariant if and only if $\sum_{t \in {}^\bullet p} I(t) = \sum_{t \in p^\bullet} I(t)$ for every $p \in P$. This gives rise to the following system of equations:

$$\begin{aligned} J(t_1) &= J(t_3), \\ J(t_2) &= J(t_1) + J(t_2), \\ 0 &= J(t_2), \\ J(t_1) + J(t_3) + J(t_4) &= 0, \\ J(t_1) + J(t_5) &= J(t_4) + J(t_5), \\ J(t_2) &= J(t_5), \\ J(t_2) + J(t_5) &= 0. \end{aligned}$$

This system of equations is equivalent to $J(t_1) = J(t_2) = J(t_3) = J(t_4) = J(t_5) = 0$. Therefore, the vector space of T-invariants of $\mathcal{N}$ is trivial, i.e. it only contains the null vector.

★ This can be verified using `PIPE` by loading the Petri net and clicking on "Invariant Analysis" in the left menu.

(b) Yes. We can actually now show that $\mathcal{N}$ is not live from any initial marking M_0. Assume, $(\mathcal{N}, M_0)$ is live. Since $(\mathcal{N}, M_0)$ is also bounded by #4.3(b), then it is well-formed. We have seen that every well-formed net has a positive T-invariant. This is a contradiction since the only T-invariant of $\mathcal{N}$ is the trivial invariant which is not positive. Therefore, $(\mathcal{N}, M_0)$ is not live. In particular, this implies that both $(\mathcal{N}, M)$ and $(\mathcal{N}, M')$ are not live.

Solution 5.2

(a) Let $U, V \subseteq P$ be traps. We have

$$\begin{aligned} (U \cup V)^\bullet &= U^\bullet \cup V^\bullet && \text{(by def. of post-sets)} \\ &\subseteq {}^\bullet U \cup {}^\bullet V && \text{(since } U \text{ and } V \text{ are traps)} \\ &= {}^\bullet(U \cup V) && \text{(by def. of pre-sets)} \end{aligned}$$

Therefore, $U \cup V$ is a trap. □

(b)

Input: Petri net $\mathcal{N} = (P, T, F)$ and $R \subseteq P$
Output: Largest trap of $\mathcal{N}$ contained in R
$Q \leftarrow R$
while $\exists p \in Q, t \in p^\bullet$ s.t. $t \notin {}^\bullet Q$ **do**
    $Q \leftarrow Q \setminus \{p\}$
return Q

(c) **Termination.** Each iteration of the `while` loop decreases the size of Q. Since no element is ever added to Q after its initialization, the algorithm must terminate.

Correctness. Let $Q_0 = R$ and let Q_i be the snapshot of set Q after the i^{th} iteration of the `while` loop. Let n be the last iteration executed by the algorithm. First note that $Q_i \subseteq R$ for every $0 \leq i \leq n$. Moreover, Q_n does not satisfy the condition of the `while` loop. Therefore, for every $p \in Q_n$ and every $t \in p^\bullet$, we have $t \in {}^\bullet Q_n$. This implies that $Q_n^\bullet \subseteq {}^\bullet Q_n$, and hence that Q_n is a trap.

It remains to show that Q_n is maximal. Let Q' be the maximal trap contained in R. Since traps are closed under union, we have $Q_n \subseteq Q'$. We claim that $Q' \subseteq Q_i$ for every $0 \leq i \leq n$. The validity of the claim completes the proof since it implies that $Q' = Q_n$.

Let us prove the claim by induction on i. First note that $Q' \subseteq R = Q_0$. Let $0 \leq i < n$. Assume that $Q' \subseteq Q_i$. There exists $q \in Q_i \setminus Q_{i+1}$ and $t \in q^\bullet$ such that $t \notin {}^\bullet Q_i$. Since ${}^\bullet Q' \subseteq {}^\bullet Q_i$ and Q' is a trap, this implies that $q \notin Q'$. Therefore, $Q' \subseteq Q_i \setminus \{q\} = Q_{i+1}$. □

Solution 5.3

(a) Let $Q = \{p_1, p_3\}$. We have

$$\begin{aligned} {}^\bullet Q &= \{t_1, t_2\} \\ &\subseteq \{t_1, t_2, t_3\} \\ &= Q^\bullet. \end{aligned}$$

Therefore Q is a siphon. Since Q is not marked by M_0, we conclude that $(\mathcal{N}, M_0)$ is not live. □

(b) The incidence matrix of $\mathcal{N}$ is:

$$\boldsymbol{N} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The marking equation is:

$$\begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \boldsymbol{x}.$$

The unique solution of the marking equation is

$$\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Since it is non negative, we cannot conclude whether the marking is reachable or not.

Let us consider the trap $Q = \{p_4\}$ which is marked by the initial marking. We can conclude that the target marking is not reachable since it does not mark Q.

Solution 5.4

(a) We have

$$\begin{aligned} Q_x \text{ is a siphon} &\iff {}^\bullet Q_x \subseteq Q_x^\bullet \\ &\iff \forall t \in T\ [(t \in {}^\bullet Q_x) \implies (t \in Q_x^\bullet)] \\ &\iff \forall t \in T\ [(\exists q \in t^\bullet\ q \in Q_x) \implies (\exists q \in {}^\bullet t\ q \in Q_x)] \\ &\iff \forall t \in T\ [(\exists q \in t^\bullet\ x_q) \implies (\exists q \in {}^\bullet t\ x_q)]. \end{aligned}$$

Therefore, we can build the following formula:

$$\varphi(x) = \bigwedge_{t \in T} \left[\left(\bigvee_{q \in t^\bullet} x_q \right) \to \left(\bigvee_{q \in {}^\bullet t} x_q \right) \right].$$

(b) We have

$$\begin{aligned} M_0(Q_x) = 0 \wedge M(Q_x) > 0 &\iff (\forall q \in Q_x\ M_0(p) = 0) \wedge (\exists q \in Q_x\ M(q) > 0) \\ &\iff (\forall q \in Q\ (q \in Q_x \implies M_0(p) = 0)) \wedge (\exists q \in Q\ (q \in Q_x \wedge M(q) > 0)) \\ &\iff (\forall q \in Q\ (x_q \implies M_0(q) = 0)) \wedge (\exists q \in Q\ (x_q \wedge M(q) > 0)) \\ &\iff (\forall q \in Q\ (M_0(q) > 0 \implies \neg x_q)) \wedge (\exists q \in Q\ (x_q \wedge M(q) > 0)). \end{aligned}$$

Therefore, we can add the following conjunct to φ:

$$\varphi(x) = \left(\bigwedge_{\substack{q \in Q \\ M_0(q) > 0}} \neg x_q \right) \wedge \left(\bigvee_{\substack{q \in Q \\ M(q) > 0}} x_q \right).$$

(c) ψ is the conjunction of the following constraints:

$$\begin{aligned} x_{p_1} \vee x_{p_3} &\implies x_{p_1} \\ x_{p_3} &\implies x_{p_1} \vee x_{p_2} \\ x_{p_4} &\implies x_{p_3} \\ x_{p_2} \vee x_{p_4} &\implies x_{p_4} \end{aligned}$$

and

$$(\neg x_{p_2} \wedge \neg x_{p_4}) \wedge (x_{p_1} \vee x_{p_2} \vee x_{p_3}).$$

★ It can be solved using `Z3` at `http://rise4fun.com/Z3/` with:

```
(declare-const x1 Bool)
(declare-const x2 Bool)
(declare-const x3 Bool)
(declare-const x4 Bool)

(assert (implies (or x1 x3) x1))
(assert (implies x3 (or x1 x2)))
(assert (implies x4 x3))
(assert (implies (or x2 x4) x4))

(assert (and (and (not x2) (not x4)) (or x1 x2 x3)))

(check-sat)
(get-model)
```

The formula has two solutions: $x_1 = \texttt{true}, x_2 = \texttt{false}, x_3 \in \{\texttt{true}, \texttt{false}\}, x_4 = \texttt{false}$.