Petri nets — Homework 4

Due 14.06.2017

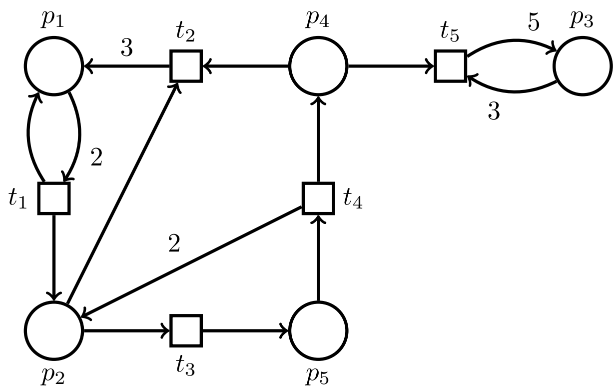
Exercise 4.1

(a) Show that
$$X = \{(x_1, x_2, x_3) \in \mathbb{N}^3 : (x_1 + 3 \leq x_2 \leq x_3 + 1) \vee (x_2 = 2x_1 + x_3 + 5)\}$$
is semilinear by exhibiting its representation as a finite set of roots and periods.

(b) Give a Petri net whose reachability set equals X up to a projection. More precisely, give a Petri net (with weights) $\mathcal{N} = (P, T, W)$ such that $\{p_{\text{init}}, p_1, p_2, p_3\} \subseteq P$ and
$$\{p_{\text{init}}\} \xrightarrow{*} M \text{ and } M(p_{\text{init}}) = 0 \iff (M(p_1), M(p_2), M(p_3)) \in X.$$

Exercise 4.2

Consider the following Petri net (with weights) $\mathcal{N}$:



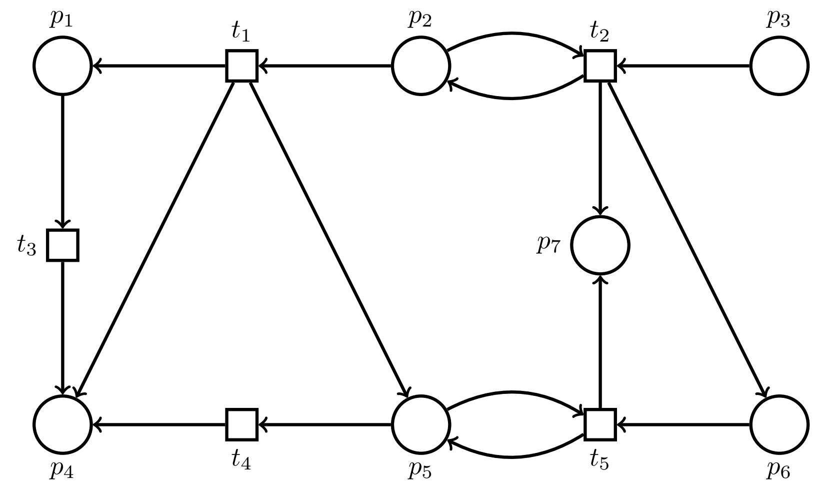
(a) Build the incidence matrix of $\mathcal{N}$.

(b) Let $M_0 = \{p_1, p_1\}$. Try to determine whether
$$M_0 \xrightarrow{*} \{p_1, p_1, p_1, p_4\},$$
$$M_0 \xrightarrow{*} \{p_1, p_1, p_1, p_1, p_2\},$$
$$M_0 \xrightarrow{*} \{p_1, p_2, p_5\},$$
by solving the marking equation.

(c) Does $\{p_1, p_5\} \xrightarrow{*} \{p_2, p_2, p_2, p_4\}$? Prove your answer.

Exercise 4.3

Consider the following Petri net $\mathcal{N} = (P, T, F)$:



(a) Give a basis of the vector space of S-invariants of $\mathcal{N}$. [Hint:]

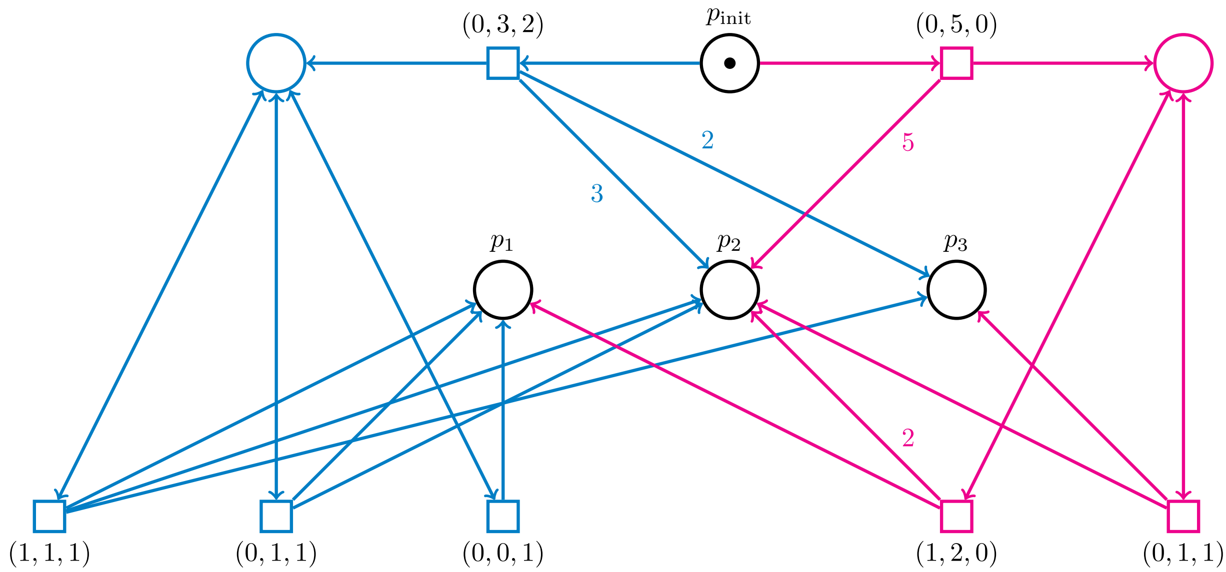
(b) Let $M = \{p_1, p_2, p_4, p_4\}$ and $M' = \{p_1, p_3, p_5\}$. Using (a), can you tell whether $(\mathcal{N}, M)$ and $(\mathcal{N}, M')$ are bounded? live?

Solution 4.1

(a)

$$X = (0,3,2) + \mathbb{N}\cdot(1,1,1) + \mathbb{N}\cdot(0,1,1) + \mathbb{N}\cdot(0,0,1) \cup$$
$$(0,5,0) + \mathbb{N}\cdot(1,2,0) + \mathbb{N}\cdot(0,1,1)$$

(b)



Solution 4.2

(a)

$$\boldsymbol{N} = \begin{array}{c|rrrrr} & t_1 & t_2 & t_3 & t_4 & t_5 \\ \hline p_1 & -1 & 3 & 0 & 0 & 0 \\ p_2 & 1 & -1 & -1 & 2 & 0 \\ p_3 & 0 & 0 & 0 & 0 & 2 \\ p_4 & 0 & -1 & 0 & 1 & -1 \\ p_5 & 0 & 0 & 1 & -1 & 0 \end{array}$$

★ This can be verified using `PIPE` by loading the Petri net, clicking on "Incidence & Marking" in the left menu, and comparing with the "Combined incidence matrix".

(b) Let us first write the markings as vectors:

$$M_0 = \begin{pmatrix}2\\0\\0\\0\\0\end{pmatrix}, \quad M_1 = \begin{pmatrix}3\\0\\0\\1\\0\end{pmatrix}, \quad M_2 = \begin{pmatrix}4\\1\\0\\0\\0\end{pmatrix}, \quad M_3 = \begin{pmatrix}1\\1\\0\\0\\1\end{pmatrix}.$$

We need to solve $M_i = M_0 + \boldsymbol{N}\cdot X$, for each $i \in \{1,2,3\}$, which is equivalent to solving $\boldsymbol{N}\cdot X = M_i - M_0$. All three systems of equations can be solved simultaneously by using Gaussian elimination:

$$\left(\begin{array}{rrrrr|rrr} -1 & 3 & 0 & 0 & 0 & 1 & 2 & -1 \\ 1 & -1 & -1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrrrr|rrr} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 4/3 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1/3 \end{array}\right)$$

Markings M_1 and M_3 are not reachable from M_0 since their associated solutions contain respectively negative and non integer values. Since the marking equation for M_2 has a non negative integer solution, we cannot conclude whether M_2 can be reached or not. In fact, a closer look at the Petri net shows that it is reachable since $M_0 \xrightarrow{t_1 t_3 t_4 t_2} M_2$.

(c) Let us first write the markings as vectors:

$$M_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad M = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Let us solve the marking equation $\boldsymbol{N} \cdot X = M - M_0$:

$$\left(\begin{array}{ccccc|c} -1 & 3 & 0 & 0 & 0 & -1 \\ 1 & -1 & -1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

As there exists a non negative integer solution, we cannot conclude anything immediately. Let us analyze the solution more carefully. Transitions t_1 and t_4 must be fired exactly once, and all other transitions must not be fired. Transition t_1 is disabled in M_0, so t_4 must be fired first, leading to the marking $\{p_1, p_2, p_2, p_4\}$. Transition t_1 is still disabled in thich marking, which implies that M is not reachable.

Solution 4.3

(a) Recall that I is an S-invariant if and only if $\sum_{p \in {}^\bullet t} I(p) = \sum_{p \in t^\bullet} I(p)$ for every $t \in T$. This gives rise to the following system of equations:

$$\begin{aligned} I(p_2) &= I(p_1) + I(p_4) + I(p_5), \\ I(p_2) + I(p_3) &= I(p_2) + I(p_6) + I(p_7), \\ I(p_1) &= I(p_4), \\ I(p_5) &= I(p_4), \\ I(p_5) + I(p_6) &= I(p_5) + I(p_7), \end{aligned}$$

which is equivalent to:

$$\begin{aligned} I(p_2) &= 3 \cdot I(p_1), \\ I(p_3) &= 2 \cdot I(p_6), \\ I(p_4) &= I(p_1), \\ I(p_5) &= I(p_1), \\ I(p_7) &= I(p_6). \end{aligned}$$

Therefore, each S-invariant I is fully determined by $I(p_1)$ and $I(p_6)$, and hence the vector space of S-invariants is given by:

$$x \cdot \begin{pmatrix} 1 & 3 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{for } x, y \in \mathbb{Q}.$$

★ This can be verified using `PIPE` by loading the Petri net and clicking on "Invariant Analysis" in the left menu.

(b) If a Petri net has a positive S-invariant, then it is bounded from any initial marking. By (a), taking $x, y > 0$ yields a positive S-invariants, e.g. $\begin{pmatrix} 1 & 3 & 2 & 1 & 1 & 1 & 1 \end{pmatrix}$ obtained by taking $x = y = 1$. Therefore, $\mathcal{N}$ is bounded both from M and M'.

Assume that $(\mathcal{N}, M)$ is live, then $I \cdot M > 0$ for every semi-positive S-invariant I. By (a), semi-positive S-invariants of $\mathcal{N}$ are obtained by taking $x, y \geq 0$ and $x + y > 0$. Therefore, we have

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ 3x \\ 2y \\ x \\ x \\ y \\ y \end{pmatrix} = 5x$$

When $x = 0$, we have $5x = 0$ which contradicts the fact that $(\mathcal{N}, M)$ is live. Therefore, it is not live.

Let us do the same calculations for M':

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ 3x \\ 2y \\ x \\ x \\ y \\ y \end{pmatrix} = 2x + 2y$$

Since $x + y > 0$, we have $2x + 2y = 2(x + y) > 0$. This implies that $I \cdot M' > 0$ for every semi-positive S-invariant I. Therefore, we cannot conclude whether $(\mathcal{N}, M')$ is live or not.