Petri nets — Homework 2

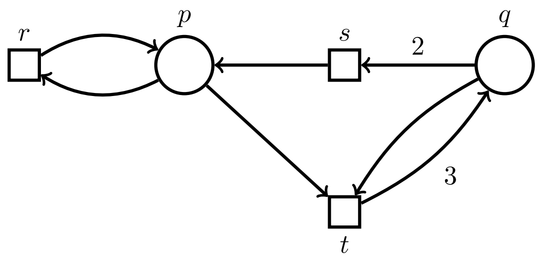
Due 17.05.2017

Exercise 2.1

(a) Give a Petri net $\mathcal{N}$ and two markings M and M' such that $M \leq M'$, $(\mathcal{N}, M)$ is bounded, and $(\mathcal{N}, M')$ is *not* bounded.

(b) Give a Petri net $\mathcal{N}$ and two markings M and M' such that $M \leq M'$, $(\mathcal{N}, M)$ is deadlock-free, and $(\mathcal{N}, M')$ is *not* deadlock-free.

(c) Consider the following Petri net $\mathcal{N}$ (with weighted arcs):



Give markings M and M' such that $M \leq M'$, $(\mathcal{N}, M)$ is live, and $(\mathcal{N}, M')$ is deadlock-free but *not* live.

Exercise 2.2
Let $\mathcal{N} = (P, T, W)$ be a Petri net with weighted arcs. Let $M, M' \in \mathbb{N}^P$, $\sigma, \sigma' \in T^*$ and $t \in T$ be such that $M \xrightarrow{\sigma t \sigma'} M'$. Prove or disprove the following statements:

(a) if t does not consume any token, i.e $W(p,t) = 0$ for every $p \in P$, then $M \xrightarrow{t\sigma\sigma'} M'$.

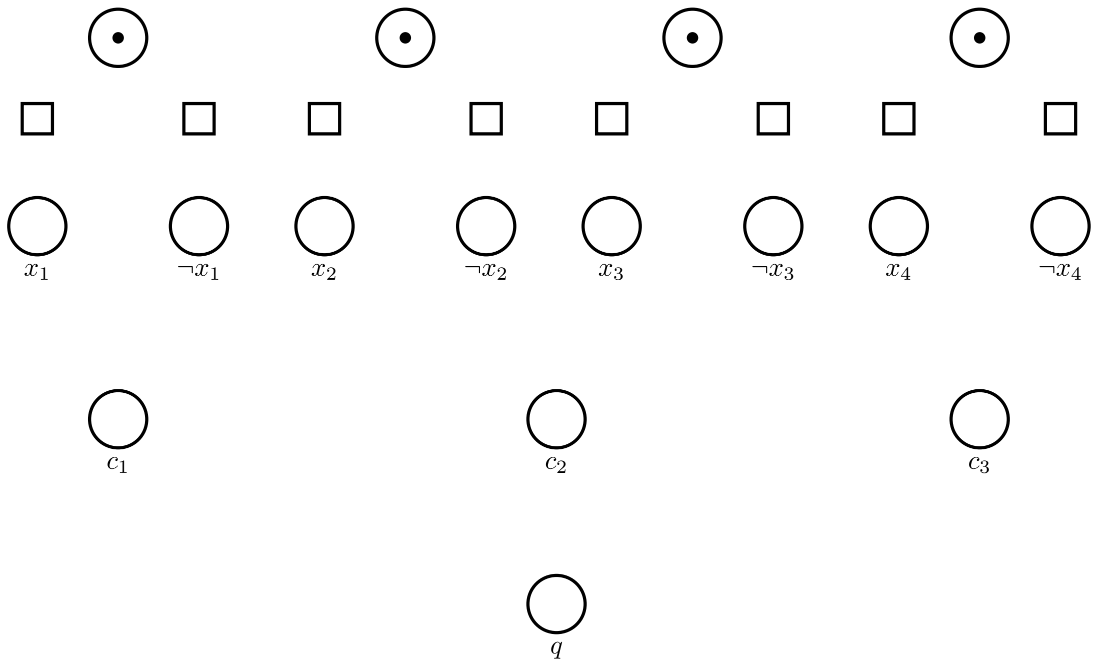
(b) if t consumes no more tokens than it produces, i.e $W(p,t) \leq W(t,p)$ for every $p \in P$, then $M \xrightarrow{t\sigma\sigma'} M'$.

(c) if t does not produce any token, i.e. $W(t,p) = 0$ for every $p \in P$, then $M \xrightarrow{\sigma\sigma' t} M'$.

(d) if t produces no more tokens than it consumes, i.e. $W(t,p) \leq W(p,t)$ for every $p \in P$, then $M \xrightarrow{\sigma\sigma' t} M'$.

Exercise 2.3

(a) Recall that 3-SAT is the problem of determining the satisfiabillity of a Boolean formula, in conjunctive normal form, whose clauses have at most three literals. It is well-known that 3-SAT is NP-complete. Give a polynomial time reduction from 3-SAT to Petri net coverability. You can simply illustrate your reduction for the formula $\varphi(x_1, x_2, x_3, x_4) = (x_1 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_4)$ by extending the following partial Petri net in such a way that φ is satisfiable if and only if $\{q\}$ is coverable:

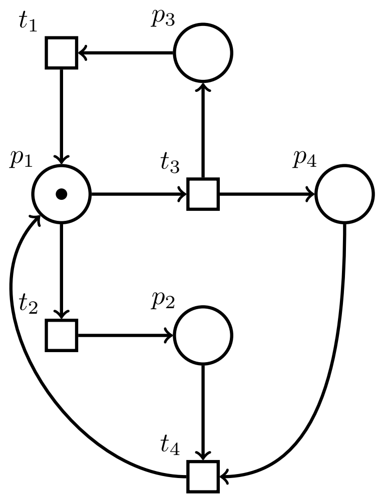


(b) Adapt your previous reduction to boundedness instead of coverability.

(c) ★ Give a polynomial time reduction from coverability to reachability. [Hint:]

(d) ★ Prove that the reduction you gave in (c) is correct. [Hint:]

Exercise 2.4
Consider the following Petri net $\mathcal{N} = (P, T, F)$:



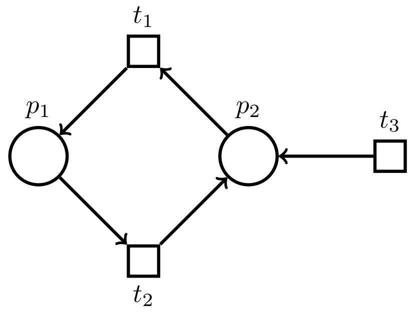
(a) Draw a coverability graph for $(\mathcal{N}, \{p_1\})$.

(b) Is $(\mathcal{N}, \{p_1\})$ bounded? If so, why? If not, which places are bounded?

(c) Describe the set of markings coverable from $\{p_1\}$.

Exercise 2.5

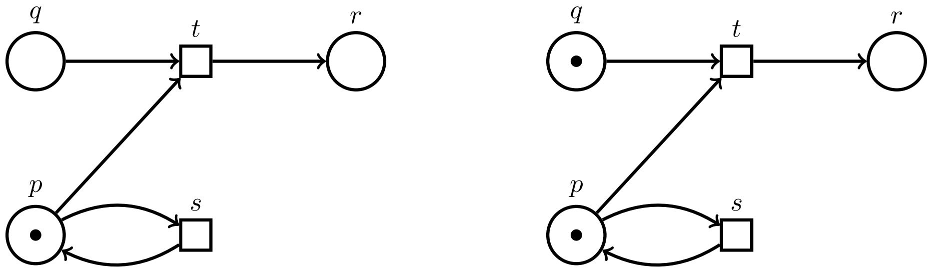
The algorithm Coverability-Graph does not specify how the coverability graph should be traversed during its construction. Show that different traversal strategies can lead to different coverability graphs. More precisely, exhibit a marking M and two different coverability graphs for $(\mathcal{N}, M)$, where $\mathcal{N}$ is the following Petri net:



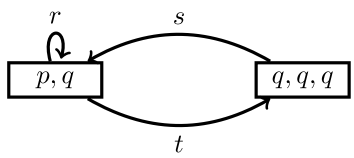
Solution 2.1

(a) The following Petri net is bounded from the empty marking since its reachability set is empty. However, it is not bounded from $\{p\}$ since repetitively firing t increases the number of tokens in q.

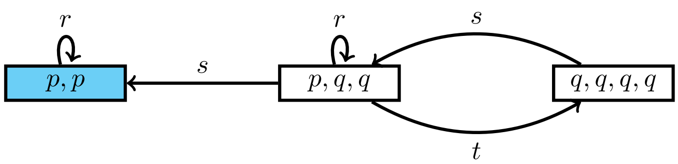
(b) The following Petri net is deadlock-free from $\{p\}$ since s is always enabled. However, it is not deadlock-free from $\{p, q\}$ since $\{p, q\} \xrightarrow{t} \{r\}$ and $\{r\}$ is dead.



(c) $\mathcal{N}$ is live from $M = \{q, q, q\}$, since the reachability graph of $(\mathcal{N}, M)$ is strongly connected and it enables all transitions:



Let us build the reachability graph of $(\mathcal{N}, M')$ where $M' = \{q, q, q, q\}$:



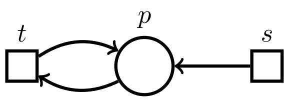
$\mathcal{N}$ is deadlock-free from M', since each marking of the reachability graph enables a transition. However, $\mathcal{N}$ is not live from M', since the bottom strongly connected component colored in blue only enables r.

Solution 2.2

(a) True. Let $A, A' \in \mathbb{N}^P$ be such that $M \xrightarrow{\sigma} A \xrightarrow{t} A' \xrightarrow{\sigma'} M'$. Since $W(p, t) = 0$ for every $p \in P$, t is enabled at any marking. In particular, $A' - A \geq \mathbf{0}$. Thus, $M \xrightarrow{t} M + (A' - A)$ and, by monotonicity, $M + (A' - A) \xrightarrow{\sigma} A + (A' - A)$. Therefore,

$$M \xrightarrow{t} M + (A' - A) \xrightarrow{\sigma} A' \xrightarrow{\sigma'} M'.$$

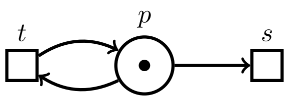
(b) False. Consider the following Petri net:



We have $0 \xrightarrow{st} 1$ and $W(p, t) = W(t, p)$, yet ts cannot be fired from 0.

(c) True. The proof is symmetric to (a).

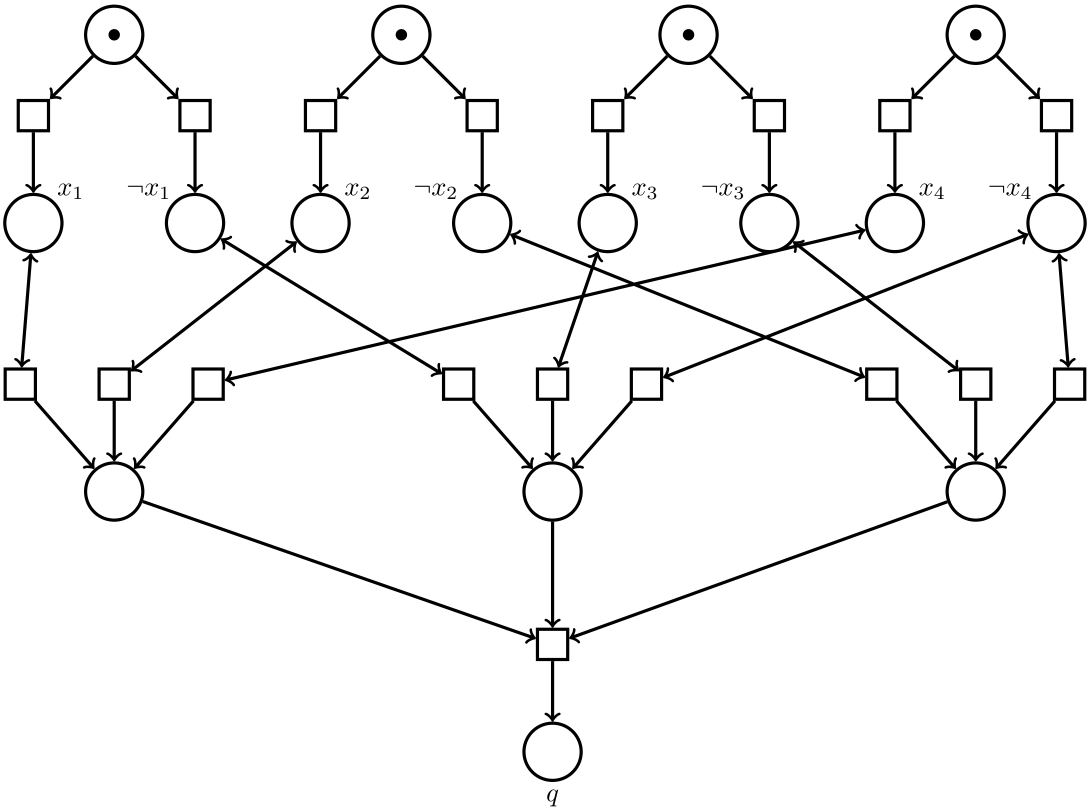
(d) False. Consider the following Petri net:



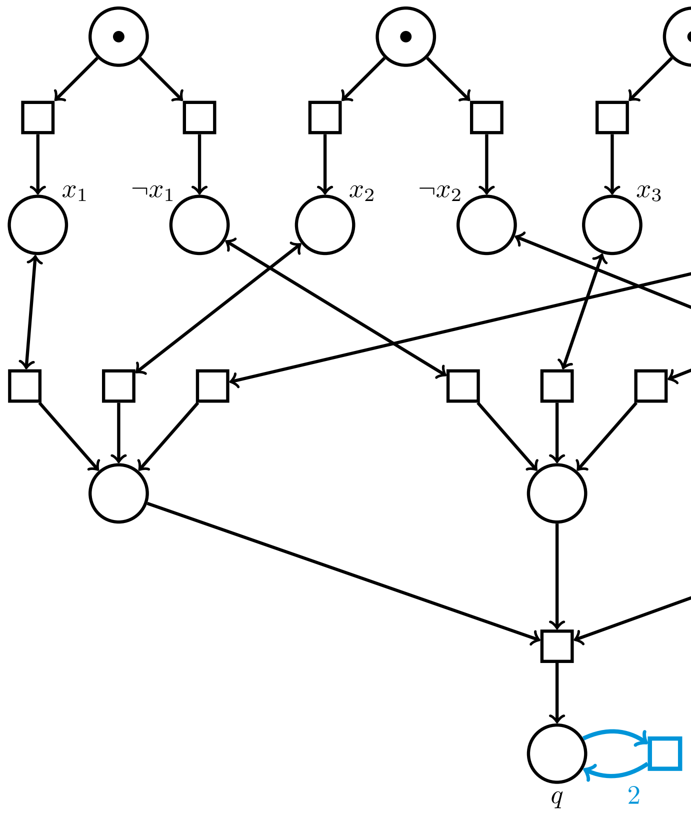
We have $1 \xrightarrow{ts} 0$ and $W(t,p) = W(p,t)$, yet ts cannot be fired from 1.

Solution 2.3

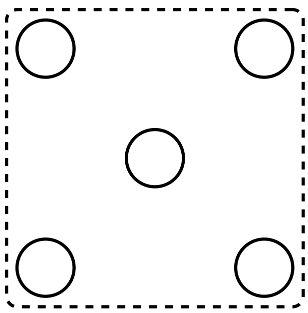
(a)



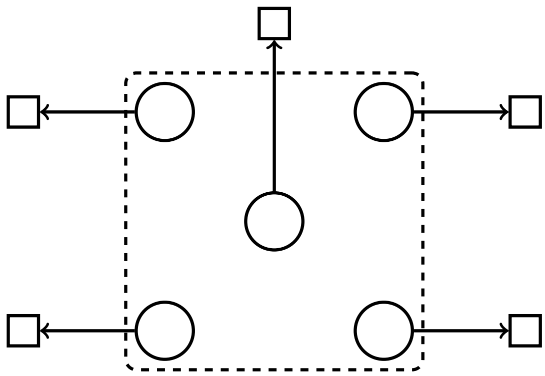
(b)



(c) ★ Given a Petri net, we make it *lossy* by adding, for each place p, a transition s_p that consumes a token from p:



original Petri net



transformed Petri net

More formally, given a Petri net with weighted arcs $\mathcal{N} = (P, T, W)$, we build the Petri net $\mathcal{N}' = (P, T', W')$ where

$$T' = T \cup \{s_p : p \in P\},$$

$$W'(t, p) = \begin{cases} W(t, p) & \text{if } t \in T, \\ 0 & \text{otherwise.} \end{cases}$$

$$W'(p, t) = \begin{cases} W(p, t) & \text{if } t \in T, \\ -1 & \text{if } t = s_p, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for every $M, M' \in \mathbb{N}^P$, M' is coverable in $(\mathcal{N}, M)$ if and only if M' is reachable in $(\mathcal{N}', M)$.

(d) ★ We prove the above claim. Let $M, M' \in \mathbb{N}^P$.

$\Rightarrow$) Suppose that M' is coverable in $(\mathcal{N}, M)$. There exist $M'' \in \mathbb{N}^P$ and $\sigma \in T^*$ such that $M'' \geq M'$ and $M \xrightarrow{\sigma} M''$ in $\mathcal{N}$. This implies that $M \xrightarrow{\sigma} M''$ in $\mathcal{N}'$. Since $M'' \geq M'$, we have $M'' \xrightarrow{*} M'$ in $\mathcal{N}'$ by decreasing the number of tokens accordingly. Therefore, $M \xrightarrow{*} M'' \xrightarrow{*} M'$ in $\mathcal{N}'$.

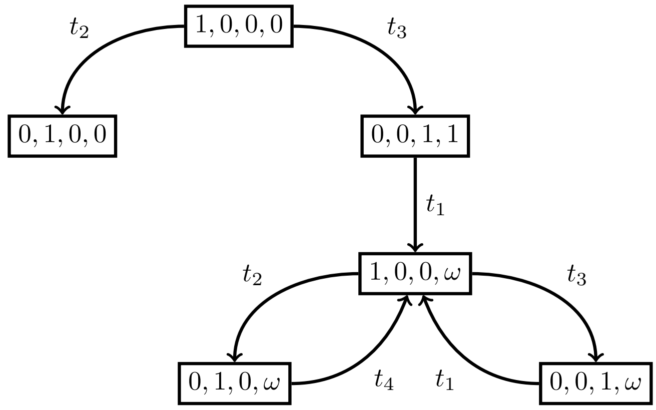
$\Leftarrow$) Suppose that M' is reachable in $(\mathcal{N}', M)$. There exists $\sigma \in (T')^*$ such that $M \xrightarrow{\sigma} M'$. By definition of $\mathcal{N}'$, for every $t \in T' \setminus T$ and $p \in P$, we have $W(t, p) = 0$. Thus, by #2.1(c), all transitions of $T' \setminus T$ occurring in σ can be moved to the end. More formally, there exists $\pi \in T^*$, $\pi' \in (T' \setminus T)^*$ and $M'' \in \mathbb{N}^P$ such that $\sigma = \pi\pi'$ and

$$M \xrightarrow{\pi} M'' \xrightarrow{\pi'} M'.$$

Since π' does not produce any token, we have $M'' \geq M'$. Moreover, $M \xrightarrow{\pi} M''$ is a firing sequence of $\mathcal{N}$ since $\pi \in T^*$. Therefore, M'' is coverable in $(\mathcal{N}, M)$. ☐

Solution 2.4

(a) The following is a coverability graph where nodes are labeled with respect to the total order $p_1 < p_2 < p_3 < p_4$:



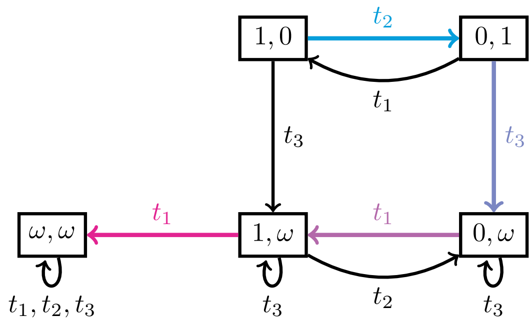
(b) It is not bounded since some markings of the graph contain ω. Places p_1, p_2 and p_3 are bounded because no marking of the graph contains an ω in the three first components.

★ This can also be tested with `LoLA` as follows:

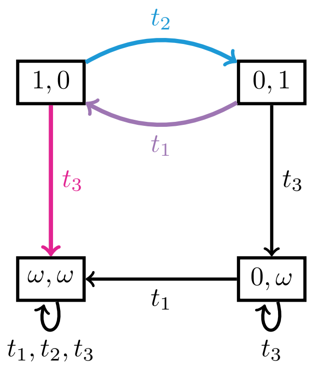
```
> lola pn_2-4.lola -f "AG (p1 < oo)" --search=cover
lola:  result:  yes
lola:  The net satisfies the given formula.


> lola pn_2-4.lola -f "AG (p2 < oo)" --search=cover
lola:  result:  yes
lola:  The net satisfies the given formula.


> lola pn_2-4.lola -f "AG (p3 < oo)" --search=cover
lola:  result:  yes
lola:  The net satisfies the given formula.


> lola pn_2-4.lola -f "AG (p4 < oo)" --search=cover
lola:  result:  no
lola:  The net does not satisfy the given formula.
```

(c) $\{M \in \mathbb{N}^P : M(p_1) + M(p_2) + M(p_3) = 1\}$.

Solution 2.5
Let $M = \{p_1\}$. We exhibit two coverability graphs for $(\mathcal{N}, M)$, where nodes are labeled with respect to the total order $p_1 < p_2$. We construct the first coverability graph by first exploring the path $t_2 t_3 t_1 t_1$:



For the second coverability graph, we first explore the path $t_2 t_1 t_3$:



Note that the subprocedure AddOmegas generates (ω, ω) after exploring $t_2 t_1 t_3$ because, at this point, both $(1, 0)$ and $(0, 1)$ are "ancestors" of the current node labeled by $(1, 1)$.