Solution

Petri nets – Homework 6

Discussed on Thursday 14th July, 2016.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

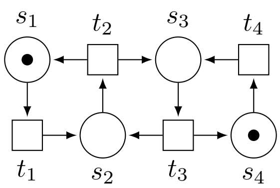
Exercise 6.1 Minimal traps and siphons in free-choice nets

A trap (resp. siphon) is *minimal* if it is proper (not empty) and contains no other proper trap (resp. siphon).

(a) Add arcs to the Petri net below such that it becomes a live and bounded free-choice system.

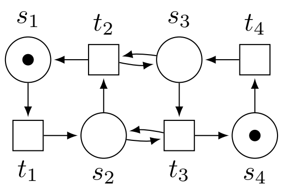
(b) Find all minimal traps and all minimal siphons of the resulting free-choice net.

(c) Does every minimal siphon contain a proper trap? Does every minimal trap contain a proper siphon?



Solution:

(a) The net is already free-choice, however it is not bounded with the given initial marking. We can add arcs from s_3 to t_2 and from s_2 to t_3 to make the Petri net bounded, while remaining live and free-choice.



(b) The minimal traps of the net are $R_1 = \{s_1, s_2\}$, $R_2 = \{s_3, s_4\}$ and $R_3 = \{s_2, s_3\}$. R_1 and R_2 are also the minimal siphons of the net. Note that R_1 and R_2 are initially marked, however R_3 is not.

(c) The minimal siphons R_1 and R_2 are traps themself and therefore contain proper traps. However, the trap R_3 does not contain any non-empty siphon.

As the free-choice system is live, this also follows from Commoner's Liveness Theorem, which states that every minimal siphon needs to contain an initially marked trap.

Exercise 6.2 Characterization of minimal siphons

(a) Let N be a net, R a minimal siphon of N, and N_R the subnet generated by $(R, {}^\bullet R)$. Show: N_R is strongly connected.

Hint: For an arc (x, y) in N_R, with $Q = \{s \in R \mid \text{there exists a path from } s \text{ to } x \text{ in } N_R\}$, show that Q is a proper siphon, and therefore there exists a path from y into Q to X.

(b) Exhibit a strongly connected net in which not every place belongs to a minimal siphon.

Hint: Two places and two transitions suffice.

Solution:

(a) Observe first that N_R is connected, otherwise, N_R has two different connected components, and the set of places of each of them is a proper siphon included in R.

Let (x, y) be an arbitrary arc of N_R. We prove in four steps that N_R contains a path from y to x. Define

$$Q = \{s \in R \mid \text{there exists a path from } s \text{ to } x \text{ in } N_R\}.$$

i) $Q \neq \emptyset$.

Since x is a node of N_R, $x \in R \cup {}^\bullet R$.

If $x \in R$, then $x \in Q$ by the definition of Q, and hence $Q \neq \emptyset$.

If $x \in {}^\bullet R$, then $x \in R^\bullet$ since R is a siphon. So $x \in s^\bullet$ for some place $s \in R$. By the definition of Q, $s \in Q$ and hence $Q \neq \emptyset$

ii) Q is a siphon.

Let t be a transition of ${}^\bullet Q$. We show $t \in Q^\bullet$. Since $t \in {}^\bullet Q$, we have $t \in {}^\bullet s$ for some place $s \in Q$. By the definition of Q, the subnet N_R contains a path π leading from s to x. Since $Q \subseteq R$, and R is a siphon, t is an output transition of some place $s' \in R$. So the path $\pi' = s' t \pi$ leads from s' to x. Since $t \in {}^\bullet Q \subseteq {}^\bullet R$, the path π' only contains elements of $R \cup {}^\bullet R$, and is therefore a path of N_R. Then $s' \in Q$ by the definition of Q. Since $t \in s'^\bullet$, we get $t \in Q^\bullet$.

iii) By i) and ii), Q is a proper siphon. Since Q is included in the minimal siphon Q, we have $Q = R$.

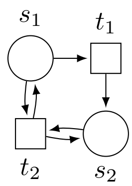
iv) N_R contains a path from y to x.

Since y is a node of N_R, we have $y \in R \cup {}^\bullet R$.

Assume $y \in R$. Then $y \in Q$ by iii) and, by the definition of Q, N_R contains a path from y to x.

Assume $y \in {}^\bullet R$. Then $y \in Q$ by iii). So $y \in {}^\bullet s$ for some $s \in Q$. By the definition of Q, there exists a path π of N_R from s to x. Since $s \in Q \subseteq R$, the path $y\pi$ is contained in N_R, and leads from y to x.

(b) The following net is strongly connected, however s_2 does not belong to any minimal siphon, as $\{s_2\}$ is not a siphon by itself, but $\{s_1\}$ is a minimal siphon. All places together form a siphon as $\{s_1, s_2\}$, which is however not minimal.



Exercise 6.3 Liveness and boundedness in free-choice systems

The result from the previous exercise can be used to show the following proposition (full proof given in Proposition 5.4 of "Free Choice Petri Nets" by J. Desel and J. Esparza):

Proposition 6.3.1. Let N be a well-formed free-choice net and R be a minimial siphon of N. Then

(1) R is a trap of N.

(2) The subnet generated by $(R, {}^\bullet R)$ is an S-component of N.

Using the above proposition, as well as Commoner's Liveness Theorem and Hack's Boundedness Theorem, prove or disprove the following:

(a) A bounded free-choice system (N, M_0) is live iff every minimal siphon of N is a trap marked at M_0.

(b) A live free-choice system (N, M_0) is bounded iff every minimal siphon of N is a trap marked at M_0.

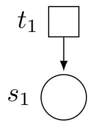
Solution:

(a) ($\Rightarrow$) Let (N, M_0) be a live and bounded free-choice system. Then N is well-formed, and by Proposition 6.3.1, every minimal siphon of N is a trap. By Commoner's Liveness Theorem, every minimal siphon contains a trap marked at M_0, therefore every minimal siphon of N is also a trap marked at M_0.

($\Leftarrow$) Let (N, M_0) be a bounded free-choice system where every minimal siphon of N is a trap marked at M_0. Then every minimal siphon contains a trap marked at M_0, and by Commoner's Liveness Theorem, the system is live.

(b) The ($\Rightarrow$) direction holds as in (a), however the other direction does not. Even though we can infer with Proposition 6.3.1 that every minimal siphon generates an S-component of N, we can not show that every place belongs to a minimal siphon and therefore to an S-component, which would be necessary for Hack's Boundedness Theorem.

The following live free-choice system is a counterexample for this conjecture. It has no minimal siphons, therefore every minimal siphon is a trap marked at M_0, however it is unbounded.



Exercise 6.4 **Reducing SAT to reachability in free-choice systems**

Reduce the satisfiability problem for boolean formulas in conjunctive normal form to the reachability problem in free-choice systems.

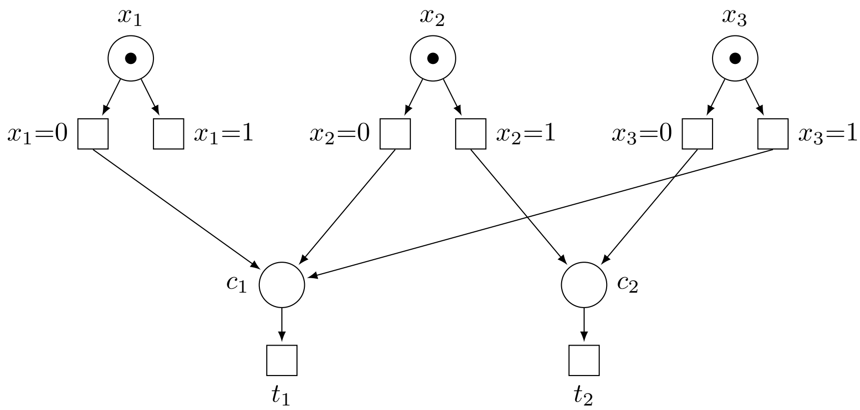
For that, give a polynomial time translation that, for a given formula φ, produces a free-choice system (N, M_0) and a marking M such that φ is satisfiable iff M is reachable in (N, M_0). Describe your reduction informally and give the resulting Petri net when applying it to the formula below.

$$\varphi = (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

Solution:

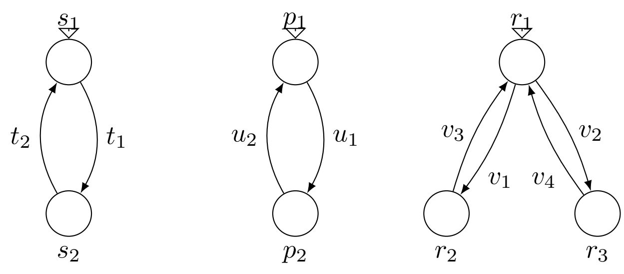
We use places x_i for each variable and transitions to choose their assignment. Places c_j for each clause get marked if the assignment makes that clause true. A transition for each clause can remove additional tokens from the clause places. The target marking M is given by $M(c_j) = 1$ for each clause place c_j and $M(s) = 0$ for all other places s. M is reachable if and only if all clauses can be made true by an assignment of the variables, i.e., φ is satisfiable.

For the given formula, we get the following free-choice system and the target marking with one token in c_1 and c_2 each and no tokens elsewhere.



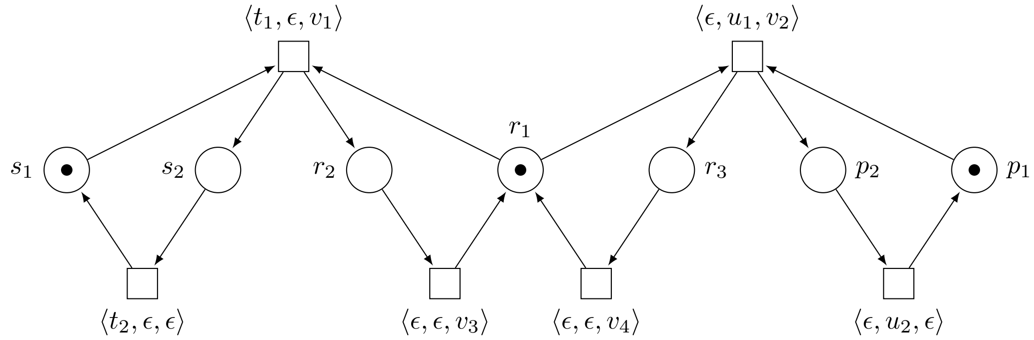
Exercise 6.5 **Unfoldings**

Consider the transition systems below, with the synchronization constraint $\mathbf{T}$:



$$\mathbf{T} = \{\langle t_1, \epsilon, v_1\rangle, \langle t_2, \epsilon, \epsilon\rangle, \langle \epsilon, u_1, v_2\rangle, \langle \epsilon, u_2, \epsilon\rangle, \langle \epsilon, \epsilon, v_3\rangle, \langle \epsilon, \epsilon, v_4\rangle\}$$

The following Petri net represents the product of the transition systems:



Using the search strategy $[w] \prec [w'] \Leftrightarrow |w| < |w'|$ for Mazurkiewicz traces w, w', compute the finite and complete prefix of the unfolding of this product.

Solution:

The following is the unfolding of the net with above search strategy. Each event is labeled with its associated transition and in blue the global state reached by firing the local configuration of the event. Terminal events are also colored blue.