Technische Universität München (I7)
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Solution

Petri nets – Homework 5

Discussed on Wednesday 29th June, 2016.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 5.1 Boundedness and liveness in S/T-systems

Show the following:

(a) An S-system (N, M_0) is bounded for any M_0.

(b) If (N, M_0) is a live S-system and $M_0' \geq M_0$, then (N, M_0') is also live.

(c) If (N, M_0) is a live and bounded T-system, then (N, M_0') is also bounded for any M_0'.

(d) If (N, M_0) is a live T-system and $M_0' \geq M_0$, then (N, M_0') is also live.

Exhibit Petri nets for the following:

(e) Give a bounded T-system (N, M_0) and a marking $M_0' \geq M_0$ such that (N, M_0') is not bounded.

(f) Give a 1-bounded S-system (N, M_0) where $M_0(S) > 1$.

(g) Give a live and 1-bounded T-system (N, M_0) with a circuit γ where $M_0(\gamma) > 1$.

Solution:

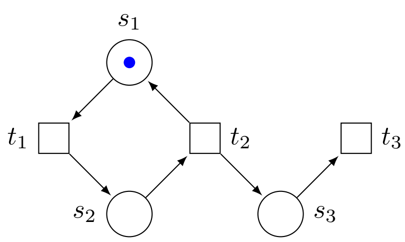
(a) By the fundamental property of S-systems (Proposition 5.1.1), for every reachable marking, we have $M(S) = M_0(S)$ and therefore $M(s) \leq M_0(S)$ for all $s \in S$.

(b) By the liveness theorem for S-systems (Theorem 5.1.3), (N, M_0) is live iff N is strongly connected and $M_0(S) > 0$, and as $M_0'(S) \geq M_0(S) > 0$, (N, M_0') is also live.

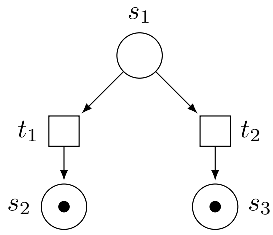
(c) A live T-system (N, M_0) is bounded iff every place s of N belongs to some circuit γ (Theorem 5.2.4). By the fundamental property of T-systems (Proposition 5.2.2), we have $M(\gamma) = M_0'(\gamma')$ for any marking M_0' and marking M reachable from M_0'. Therefore s, belonging to γ, is always bounded by $M_0'(\gamma)$ and thus the system is bounded for any initial marking.

(d) By the liveness theorem for T-systems, (N, M_0) is live iff $M_0(\gamma) > 0$ for every circuit γ, and as $M_0'(\gamma) \geq M_0(\gamma) > 0$, (N, M_0') is also live.

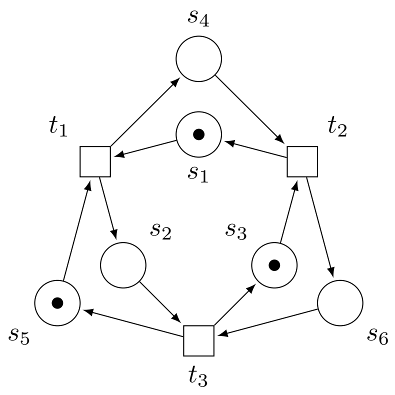
(e) Due to (c), the system needs to be non-live. The following Petri net without any tokens is a non-live, bounded T-system. By adding the blue token to s_1, the net becomes unbounded.



(f) Due to the boundedness theorem for S-systems, the system needs to be non-live. The following Petri net is a non-live, 1-bounded S-system with $M_0(S) > 1$:



(g) In the following live T-system, the inner circuit $s_1 s_2 s_3$ contains 2 tokens, however each place is 1-bounded due to the outer circuits.



Exercise 5.2 Marking equation in S-systems

In the lecture, it was shown that for an S-system (N, M_0), a marking M of N is reachable from M_0 iff the marking equation $M = M_0 + \mathbf{N} \cdot X$ has a nonnegative integer solution, i.e. $X : T \to \mathbb{N}$.

Show the following: For an S-system (N, M_0), a marking M of N is reachable from M_0 iff the marking equation $M = M_0 + \mathbf{N} \cdot X$ has a nonnegative rational solution, i.e. $X : T \to \mathbb{Q}$ with $X \geq 0$.

Note: We have not found a simple, constructive proof, so finding one is probably not that easy, though you should try to see the rationale for why this works. If you find an easy proof, please send it to us.

Solution:

Proof based on graph theory and maximal flow (see L.R. Ford and D.R Fulkerson, "Maximal flow through a network", 1956.):

An S-net (S, T, F) is basically a directed graph (V, E), with the places S as nodes V and an edge $(s, s') \in E$ if there is a $t \in T$ with ${}^\bullet t = \{s\}$ and $t^\bullet = \{s'\}$. A rational solution X to the marking equation $M = M_0 + \mathbf{N} \cdot X$ with $X \geq 0$ then assigns each edge in the graph a nonnegative rational value. For each $s \in S$, we have

$$\sum_{t \in {}^\bullet s} X(t) - \sum_{t \in s^\bullet} X(t) = M(s) - M_0(s).$$

If we think of every place with $M_0(s) > M(s)$ as an input place and every place with $M(s) > M_0(s)$ as an output place, the solution X gives us a flow of tokens from the input places to the output places. As $I = (1, \ldots, 1)$ is an S-invariant of any S-net, we have $M(S) = I \cdot M = I \cdot M_0 + \underbrace{I \cdot \mathbf{N}}_{=0} \cdot X = I \cdot M_0 = M_0(S)$ and therefore the input flow is equal to the output flow.

On the graph (V, E), we add an source vertex v_s and a sink vertex v_t, with an edge from v_s to every $s \in S$ with $M_0(s) > M(s)$ and capacity $M_0(s) - M(s)$. Similarly we add an edge from every $s \in S$ with $M(s) > M_0(s)$ to v_t with capacity $M(s) - M_0(s)$. The solution X then gives us a maximal flow from v_s to v_t, if we assign maximal flow to all edges from v_s and v_t.

The integral flow theorem (following from the correctness of the Ford-Fulkerson algorithm) then states that if all edge capacities are integral, then there is a maximum flw in which all flows are integers. This maximal flow needs to have the same flow as X on the source and edge vertices. If we consider this flow as a Parikh vector Y, it moves the same amount of tokens from each input place to each output place, and therefore we have $M = M_0 + \mathbf{N} \cdot Y$ and $Y \geq 0$, so it is a nonnegative integer solution to the marking equation and thus, there is an occurence sequence leading from M_0 to M.

Note that this integral flow may be found with the Ford-Fulkerson algorithm, or also in polynomial time with the Edmonds-Karp algorithm.

Proof based on linear and integer programming (see A. Schijver, "Theory of Linear and Integer Programming", p. 269, Theorem 19.3 and p. 301, Theorem 21.5, for detailed proofs of the used theorems):

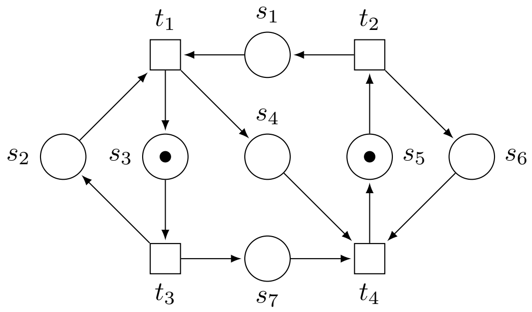
For an S-net, the incidence matrix $\mathbf{N}$ contains exactly one $+1$ and one -1 in every column, and the remaining entries are 0 (as $|{}^\bullet t| = 1 = |t^\bullet|$ for every $t \in T$). Therefore the incidence matrix is totally unimodular.

As $\mathbf{N}$ is totally unimodular and M and M_0 are integral, the polyhedron $\{X \mid X \geq 0,\ \mathbf{N} \cdot X \leq M - M_0\}$ is also integral, i.e. all vertices of the polygon are integral. A rational solution X to the marking equation $M = M_0 + \mathbf{N} \cdot X$ with $X \geq 0$ then satisfies $\mathbf{N} \cdot X \leq M - M_0$. Therefore X is contained in the polyhedron and it is not empty. Further, as X achieves equality, it is part of some face of the polyhedron and we can then find some vertex Y of the polyhedron contained in this face. This Y is then integral and satisfies $\mathbf{N} \cdot Y = M - M_0$, so it is a nonnegative integer solution to the marking equation and therefore M is reachable from M_0. Note that this Y may be found e.g. by a linear program in polynomial time.

Exercise 5.3 Polynomial time algorithm for deciding liveness of a T-system

(a) Give a polynomial time algorithm to check if a T-system is live (note that a T-net may have an exponential number of circuits, so simply enumerating all circuits is infeasible).

(b) Apply your algorithm to the T-system below to decide if it is live.



Solution:

(a) A T-system is live iff $M_0(\gamma) > 0$ for every circuit γ of N (Theorem 5.2.3). While a net may have an exponential number of circuits, we can check the existance of an unmarked circuit in polynomial time by a simple depth-first search restricted to all places unmarked at M_0, with the transitions acting as edges between places. If we visit a place currently on the stack, we have found an unmarked circuit and therefore the net is not live. Otherwise, if we do not find such a circuit, the net is live.

Input: A T-net $N = (S, T, F)$ and a marking M_0.
Output: YES if the T-system (N, M_0) is live, otherwise NO.
Initialization: $visited(s) :=$ *false* and $onstack(s) :=$ *false* for all $s \in S$, $emptycircuit =$ *false*.

begin
  while there are $s \in S$ with $visited(s) =$ *false* and $M_0(s) = 0$ **do**
    dfs(s)
  endwhile
  if $emptycircuit =$ *true* **then**
    return NO
  else
    return YES
  endif
end
function dfs(s) **begin**
  $onstack(s) :=$ *true*
  $visited(s) :=$ *true*
  for $s' \in (s^\bullet)^\bullet$ with $M_0(s) = 0$ **do**
    if $onstack(s') =$ *true* **then**
      $emptycircuit =$ *true*
    else if $visited(s') =$ *false* **then**
      dfs(s')
    endif
  endfor
  $onstack(s) :=$ *false*
end

(b) By starting at all unmarked places and following transitions to unmarked places, we quickly see that there is no unmarked circuit, so the Petri net is live.

In contrast, by enumerating all circuits, we need to look at all these circuits:

$$\gamma_1 = s_1 t_1 s_3 t_3 s_7 t_4 s_5 t_2$$
$$\gamma_2 = s_2 t_1 s_3 t_3$$
$$\gamma_3 = s_5 t_2 s_6 t_4$$
$$\gamma_4 = s_1 t_1 s_4 t_4 s_5 t_2$$

Exercise 5.4 Strong Connectedness Theorem

Let (N, M_0) be a live and bounded Petri net. Show that N is strongly connected.

Hint: To show that the net is strongly connected, you need to show that for every arc $(x, y) \in F$, there is a path from y to x. Use liveness to construct a firing sequence containing the transition of the arc often enough and then use boundedness on the place of the arc to show that there needs to be a path back. You may also use the following lemma, proven in exercise 1.6:

Lemma 5.4.1 (Exchange Lemma)**.** Let u and v be transitions of a net satisfying ${}^\bullet u \cap v^\bullet = \emptyset$. If $M \xrightarrow{vu} M'$ then $M \xrightarrow{uv} M'$.

Solution: Let $(x, y) \in F$. We distinguish between two cases:

Case 1: $x \in S$ and $y \in T$. Let V be the set of all transitions $v \in T$ for which there is a path from y to v and let $U = T \setminus V$. For $u \in U$ and $v \in V$ we have ${}^\bullet u \cap v^\bullet = \emptyset$.

Let b be the bound of x. Liveness implies that there exists a finite firing sequence $M_0 \xrightarrow{\sigma} M$ with $b + 1$ occurrences of y in σ. By Lemma 5.4.1, transitions of σ can repeatedly be swapped, resulting in firing sequences $M_0 \xrightarrow{\sigma_1} M' \xrightarrow{\sigma_2} M$ such that σ_1 contains only transitions in U and σ_2 contains only transitions in V.

Transition y is in the set V, so y occurs $b + 1$ times in σ_2. Since $M'(x) \leq b$ and $y \in x^\bullet$, some transition $v \in {}^\bullet x$ occurs in σ_2. Since σ_2 contains only transitions of V, we have $v \in V$. By definition of V, there is a path from y to v and by extension also from y to x.

Case 2: $x \in T$ and $y \in S$. Let U be the set of all transitions $u \in T$ for which there is a path from u to x and let $V = T \setminus U$. For $u \in U$ and $v \in V$ we have ${}^\bullet u \cap v^\bullet = \emptyset$.

Let b be the bound of y. Liveness implies that there exists a finite firing sequence $M_0 \xrightarrow{\sigma} M$ with $b + 1$ occurrences of x in σ. By Lemma 5.4.1, transitions of σ can repeatedly be swapped, resulting in firing sequences $M_0 \xrightarrow{\sigma_1} M' \xrightarrow{\sigma_2} M$ such that σ_1 contains only transitions in U and σ_2 contains only transitions in V.

Transition x is in the set U, so x occurs $b + 1$ times in σ_1. Since $M'(y) \leq b$ and $x \in {}^\bullet y$, some transition $u \in y^\bullet$ occurs in σ_1. Since σ_1 contains only transitions of U, we have $u \in U$. By definition of U, there is a path from u to x and by extension also from y to x.