Solution

Petri nets – Homework 4

Discussed on Thursday 16th June, 2016.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 4.1 S-invariants and T-invariants

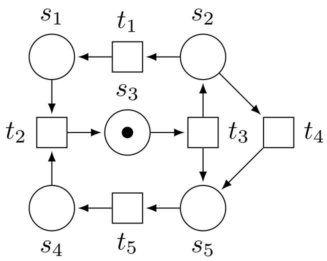
Four each of the following four invariants, check if the net below has such an invariant. If yes, give one such invariant. Can you make any statements about the liveness and boundedness of the net based on the existance of these invariants?

(a) a semi-positive S-invariant

(b) a semi-positive T-invariant

(c) a positive S-invariant

(d) a positive T-invariant



Solution:

A vector I is an S-invariant if $I \cdot \mathbf{N} = 0$ or if $\forall t \in T : \sum_{s \in {}^\bullet t} I(s) = \sum_{s \in t^\bullet} I(s)$. By the second definition, we obtain the following equations for an S-invariant $I = (s_1, s_2, s_3, s_4, s_5)$:

$$\begin{aligned} s_2 &= s_1 \\ s_1 + s_2 &= s_3 \\ s_3 &= s_2 + s_5 \\ s_2 &= s_5 \\ s_5 &= s_4 \end{aligned}$$

These can be simplified and reduced to following equivalent set of equations:

$$\begin{aligned} s_1 &= s_2 = s_4 = s_5 \\ s_3 &= 2s_1 \end{aligned}$$

By specifying s_1, the S-invariant is completely defined. With $s_1 = 1$, we obtain the following S-invariant:

$$I = (1, 1, 2, 1, 1)$$

I is positive and therefore also semi-positive. As the net N has a positive S-invariant, is bounded for all initial markings M_0.

A vector J is a T-invariant if $\mathbf{N} \cdot J = 0$ or if $\forall s \in S : \sum_{t \in {}^\bullet s} J(t) = \sum_{t \in s^\bullet} J(t)$ or $J = \vec{\sigma}$ for some occurrence sequence σ and marking M with $M \xrightarrow{\sigma} M$ (fundamental property of T-invariants). By the second definition, we obtain the following equations for a T-invariant $J = (t_1, t_2, t_3, t_4, t_5)$:

$$t_1 = t_2$$
$$t_3 = t_1 + t_4$$
$$t_2 = t_3$$
$$t_5 = t_2$$
$$t_3 + t_4 = t_5$$

These can be simplified and reduced to following equivalent set of equations:

$$t_1 = t_2 = t_3 = t_5$$
$$t_4 = 0$$

By specifying t_1 the S-invariant is completely defined. By setting $t_1 = 1$, we obtain the following semi-positive T-invariant:

$$I = (1, 1, 1, 1, 0)$$

As $t_4 = 0$ for all T-invariants, there can be no positive T-invariant. As any well-founded net (live and bounded for some initial marking) has a positive T-invariant, the net N is not live for any initial marking M_0.

Exercise 4.2 Properties of invariants

Exhibit counterexamples that disprove the following conjectures:

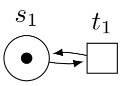
(a) For a Petri net (N, M_0), an S-invariant I of N and a marking M, if $I \cdot M_0 = I \cdot M$, then M is reachable from M_0.

(b) For a Petri net (N, M_0) and a place s of N, if s is bounded, then there is a place invariant I of N with $I(s) > 0$.

(c) For a net N, if N has a positive transition invariant J, then it is well-formed (there is a marking M_0 such that (N, M_0) is live and bounded).

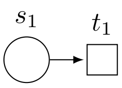
Solution:

(a) In the following net, we have $I = (0)$ as an S-invariant (the zero vector is an S-invariant for any net), and therefore we have $I \cdot M_0 = 0 = I \cdot M$ for any markings M_0 and M. However the marking $M = (2)$ is not reachable from $M_0 = (1)$.

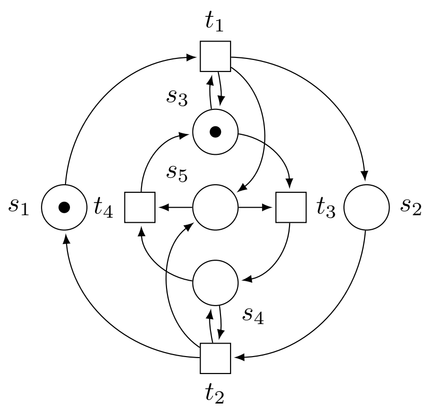


Another example is the net in exercise 4.3. It has $I = (2, 1, 1, 1, 1, 1, 1)$ as a positive S-invariant, and the Petri net is bounded and live. However, the marking M is not reachable from M_0, as shown in the next exercise.

(b) In the following net, the place s_1 is bounded. We have $\mathbf{N} = (-1)$ and therefore $I(s1) = 0$ for any S-invariant.



As a more interesting example, take the following Petri net, known from exercise 2.5(c), which is live and bounded.



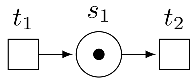
Any S-invariant I would need to satisfy

$$I(s_1) = I(s_2) + I(s_5)$$
$$I(s_2) = I(s_1) + I(s_5)$$
$$I(s_3) = I(s_4) + I(s_5)$$
$$I(s_4) = I(s_3) + I(s_5)$$

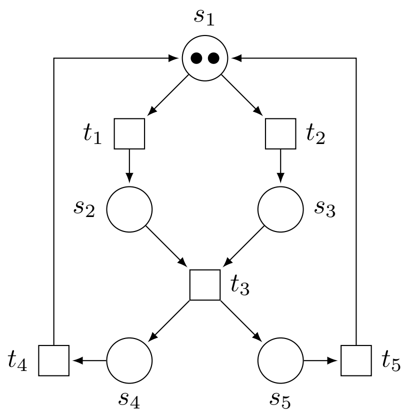
and therefore $I(s_5) = 0$, so there is no positive S-invariant.

(c) For a net N, if N has a positive transition invariant J, then it is well-formed (there is a marking M_0 such that (N, M_0) is live and bounded).

(d) In the following net, we have $J = (1, 1)$ as a positive T-invariant, however it is not bounded for any initial marking.



As another example, in the net below, we have $J = (1, 1, 1, 1, 1)$ as a positive T-invariant, however it is not live for any initial marking.



Exercise 4.3 **Encoding traps into SAT**

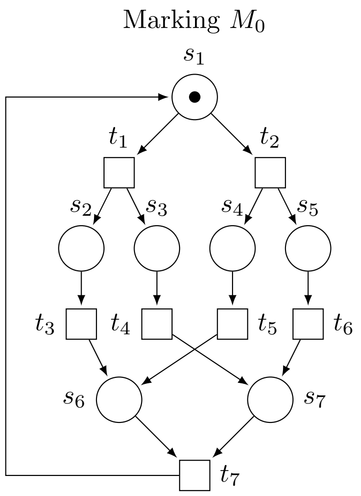
(a) Give a procedure that, given a net N, constructs a boolean formula φ satisfying the following properties:
- The formula contains variables r_s for each place $s \in S$,
- if φ is satisfiable, then N has a trap,
- and if φ is not satisfiable, then N has no trap.
- Additionally, if A is a model of φ, then the set given by $R = \{s \mid A(r_s)\}$ is a trap of N.

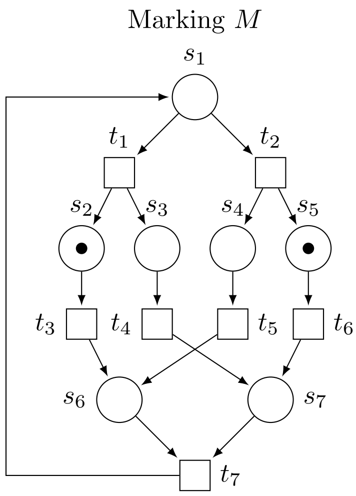
(b) Apply your procedure to the Petri net on the left below and give the resulting constraints.

(c) Adapt your procedure such that, given two marking M_0 and M, it adds additional constraints to ensure that any trap R obtained as a solution by the constraints is marked at M_0 and unmarked at M. The constraints should be satisfiable iff a trap marked at M_0 and unmarked at M exists.

(d) Construct the constraints for the Petri net below with the markings M_0 and M.

(e) Use your constraints and the trap property to show that M is not reachable from M_0 in the net below.





Solution:

(a) Any trap R satisfiest $R^\bullet \subseteq {}^\bullet R$ and therefore $\forall t \in T : \exists s \in {}^\bullet t : s \in R \implies \exists s' \in t^\bullet : s' \in R$. This can be encoded with the following formula, which can be unrolled for a given net N:

$$\bigwedge_{t \in T} \left(\left(\bigvee_{s \in {}^\bullet t} r_s \right) \implies \left(\bigvee_{s' \in t^\bullet} r'_s \right) \right)$$

Any assignment satisfying the formula gives rise to a set R which satisfies the trap condition and is therefore a trap.

(b) The constraints are as follows:

$$\begin{aligned} r_{s_1} &\implies r_{s_2} \vee r_{s_3} \\ r_{s_1} &\implies r_{s_4} \vee r_{s_5} \\ r_{s_2} &\implies r_{s_6} \\ r_{s_3} &\implies r_{s_7} \\ r_{s_4} &\implies r_{s_6} \\ r_{s_5} &\implies r_{s_7} \\ r_{s_6} \vee r_{s_7} &\implies r_{s_1} \end{aligned}$$

(c) To ensure that the trap is marked at M_0 and unmarked at M, we can add the following constraint:

$$\left(\bigvee_{s \in S : M_0(s) > 0} r_s \right) \wedge \left(\bigwedge_{s \in S : M(s) > 0} \neg r_s \right)$$

(d) The additonal constraints are:

$$r_{s_1} \wedge (\neg r_{s_2} \wedge \neg r_{s_5})$$

(e) We obtain a satisying assignment A for the constraints by setting $A(r_{s_1}) = A(r_{s_3}) = A(r_{s_4}) = A(r_{s_6}) = A(r_{s_7}) = 1$ and $A(r_{s_2}) = A(r_{s_5}) = 0$. The trap obtained from these constraints is $R = \{s_1, s_3, s_4, s_6, s_7\}$. As the trap is marked at M_0, it needs to stay marked in any reachable marking, therefore the marking M is not reachable.

Exercise 4.4 Algorithm for the largest siphon

Recall the following algorithm for computing the largest siphon Q contained in a given set R of places:

Input: A net $N = (S, T, F)$ and $R \subseteq S$.
Output: The largest siphon $Q \subseteq R$.
Initialization: $Q := R$.

begin
  while there are $s \in Q$ and $t \in {}^\bullet s$ such that $t \notin Q^\bullet$ **do**
    $Q: = Q \setminus \{s\}$
  endwhile
end

Show that the algorithm is correct by showing

(a) that the algorithm terminates, and

(b) that after termination, Q is the largest siphon contained in R.

Solution:

(a) In every iteration of the while loop, a place s is removed from Q. Q contains only finitely many places initially, therefore the while loop and the algorithm terminates.

(b) Let Q' be the largest siphon contained in R. First we show that $Q \subseteq Q'$. Let $s \in Q$. Then for all $t \in {}^\bullet s$, we have $t \in Q^\bullet$, therefore Q is a siphon. As Q' contains all siphons in R, $Q \subseteq Q'$.

Now let $Q_0, Q_1, \ldots, Q_n$ be the intermediate sets in the algorithm, with $Q_0 = R$ and $Q_n = Q$. We show that in each step i, we have $Q' \subseteq Q_i$.

Initially, with $i = 0$, we have $Q' \subseteq R = Q_0$. Now assume that $Q' \subseteq Q_i$ and we execute the body of the while loop in step i. Then there is $s \in Q_i$ and $t \in {}^\bullet s$ such that $t \notin {Q_i}^\bullet$. As ${Q'}^\bullet \subseteq {Q_i}^\bullet$, we also have $t \notin {Q'}^\bullet$ and therefore $s \notin Q'$. Thus $Q' \subseteq Q_{i+1} = Q_i \setminus \{s\}$.

Exercise 4.5 **S-invariants and traps**

Prove: Let N be a net and I a semi-positive S-invariant of N. The set $R = \{s \mid I(s) > 0\}$ of places is a trap of N.

Solution:

Let $R = \{s \mid I(s) > 0\}$. For $t \in R^\bullet$, there is an $s \in R$ with $s \in {}^\bullet t$. As $I(s) > 0$, $I(s') \geq 0$ for all $s' \in R$ and $\sum_{s \in {}^\bullet t} I(s) = \sum_{s' \in t^\bullet} I(s')$, there is an $s' \in t^\bullet$ with $I(s') > 0$. Therefore $s' \in R$ and $t \in {}^\bullet R$, so $R^\bullet \subseteq {}^\bullet R$ and R is a trap.