Solution

Petri nets – Homework 4

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Exercise 4.1 S-invariants and T-invariants

Four each of the following four invariants, check if the net below has such an invariant. If yes, give one such invariant. Can you make any statements about the liveness and boundedness of the net based on the existance of these invariants?

(a) a semi-positive S-invariant
(b) a semi-positive T-invariant
(c) a positive S-invariant
(d) a positive T-invariant

\[ s_2 = s_1 \\
\]
\[ s_1 + s_2 = s_3 \\
\]
\[ s_3 = s_2 + s_5 \\
\]
\[ s_2 = s_5 \\
\]
\[ s_5 = s_4 \\
\]

These can be simplified and reduced to following equivalent set of equations:

\[ s_1 = s_2 = s_4 = s_5 \\
\]
\[ s_3 = 2s_1 \]

By specifying \( s_1 \), the S-invariant is completely defined. With \( s_1 = 1 \), we obtain the following S-invariant:

\[ I = (1, 1, 2, 1, 1) \]

I is positive and therefore also semi-positive. As the net \( N \) has a positive S-invariant, is bounded for all initial markings \( M_0 \).

A vector \( J \) is a T-invariant if \( N \cdot J = 0 \) or if \( \forall s \in S : \sum_{t \in \cdot s} J(t) = \sum_{t \in \cdot s} J(t) \) or \( J = \vec{\sigma} \) for some occurrence sequence \( \sigma \) and marking \( M \) with \( M \xrightarrow{\vec{\sigma}} M \) (fundamental property of T-invariants). By the second definition, we obtain the following equations for a T-invariant \( J = (t_1, t_2, t_3, t_4, t_5) \):
\[
\begin{align*}
t_1 &= t_2 \\
t_3 &= t_1 + t_4 \\
t_2 &= t_3 \\
t_5 &= t_2 \\
t_3 + t_4 &= t_5
\end{align*}
\]

These can be simplified and reduced to following equivalent set of equations:
\[
\begin{align*}
t_1 &= t_2 = t_3 = t_5 \\
t_4 &= 0
\end{align*}
\]
By specifying \( t_1 \) the S-invariant is completely defined. By setting \( t_1 = 1 \), we obtain the following semi-positive T-invariant:
\[
I = (1, 1, 1, 1, 0)
\]
As \( t_4 = 0 \) for all T-invariants, there can be no positive T-invariant. As any well-founded net (live and bounded for some initial marking) has a positive T-invariant, the net \( N \) is not live for any initial marking \( M_0 \).

**Exercise 4.2 Properties of invariants**

Exhibit counterexamples that disprove the following conjectures:

(a) For a Petri net \( (N, M_0) \), an S-invariant \( I \) of \( N \) and a marking \( M \), if \( I \cdot M_0 = I \cdot M \), then \( M \) is reachable from \( M_0 \).

(b) For a Petri net \( (N, M_0) \) and a place \( s \) of \( N \), if \( s \) is bounded, then there is a place invariant \( I \) of \( N \) with \( I(s) > 0 \).

(c) For a net \( N \), if \( N \) has a positive transition invariant \( J \), then it is well-formed (there is a marking \( M_0 \) such that \( (N, M_0) \) is live and bounded).

**Solution:**

(a) In the following net, we have \( I = (0) \) as an S-invariant (the zero vector is an S-invariant for any net), and therefore we have \( I \cdot M_0 = 0 = I \cdot M \) for any markings \( M_0 \) and \( M \). However the marking \( M = (2) \) is not reachable from \( M_0 = (1) \).

\[
\begin{align*}
\text{s}_1 & \quad \text{t}_1 \\
&
\end{align*}
\]

Another example is the net in exercise 4.3. It has \( I = (2, 1, 1, 1, 1, 1) \) as a positive S-invariant, and the Petri net is bounded and live. However, the marking \( M \) is not reachable from \( M_0 \), as shown in the next exercise.

(b) In the following net, the place \( s_1 \) is bounded. We have \( N = (-1) \) and therefore \( I(s_1) = 0 \) for any S-invariant.

\[
\begin{align*}
\text{s}_1 & \quad \text{t}_1 \\
&
\end{align*}
\]

As a more interesting example, take the following Petri net, known from exercise 2.5(c), which is live and bounded.

\[
\begin{align*}
I(s_1) &= I(s_2) + I(s_5) \\
I(s_2) &= I(s_1) + I(s_5) \\
I(s_3) &= I(s_4) + I(s_5) \\
I(s_4) &= I(s_3) + I(s_5)
\end{align*}
\]
and therefore \( I(s_5) = 0 \), so there is no positive S-invariant.

(c) For a net \( N \), if \( N \) has a positive transition invariant \( J \), then it is well-formed (there is a marking \( M_0 \) such that \((N,M_0)\) is live and bounded).

(d) In the following net, we have \( J = (1,1) \) as a positive T-invariant, however it is not bounded for any initial marking.

\[
\begin{array}{c}
t_1 \\
\bullet \\
t_2
\end{array}
\]

As another example, in the net below, we have \( J = (1,1,1,1,1) \) as a positive T-invariant, however it is not live for any initial marking.

\[
\begin{array}{c}
s_1 \\
t_1 \\
\bullet \\
t_2 \\
s_2 \\
t_3 \\
s_3 \\
t_4 \\
s_4 \\
t_5 \\
s_5 \\
t_6 \\
t_7
\end{array}
\]

**Exercise 4.3 Encoding traps into SAT**

(a) Give a procedure that, given a net \( N \), constructs a boolean formula \( \varphi \) satisfying the following properties:

- The formula contains variables \( r_s \) for each place \( s \in S \),
- if \( \varphi \) is satisfiable, then \( N \) has a trap,
- and if \( \varphi \) is not satisfiable, then \( N \) has no trap.
- Additionally, if \( A \) is a model of \( \varphi \), then the set given by \( R = \{ s \mid A(r_s) \} \) is a trap of \( N \).

(b) Apply your procedure to the Petri net on the left below and give the resulting constraints.

(c) Adapt your procedure such that, given two marking \( M_0 \) and \( M \), it adds additional constraints to ensure that any trap \( R \) obtained as a solution by the constraints is marked at \( M_0 \) and unmarked at \( M \). The constraints should be satisfiable iff a trap marked at \( M_0 \) and unmarked at \( M \) exists.

(d) Construct the constraints for the Petri net below with the markings \( M_0 \) and \( M \).

(e) Use your constraints and the trap property to show that \( M \) is not reachable from \( M_0 \) in the net below.

**Solution:**
(a) Any trap \( R \) satisfies \( R^* \subseteq \bullet R \) and therefore \( \forall t \in T: \exists s \in \bullet t: s \in R \implies \exists s' \in \bullet t: s' \in R \). This can be encoded with the following formula, which can be unrolled for a given net \( N \):

\[
\bigwedge_{t \in T} \left( \left( \bigvee_{s \in \bullet t} r_s \right) \implies \left( \bigvee_{s' \in \bullet t} r'_s \right) \right)
\]

Any assignment satisfying the formula gives rise to a set \( R \) which satisfies the trap condition and is therefore a trap.

(b) The constraints are as follows:

\[
\begin{align*}
    r_{s_1} & \implies r_{s_2} \lor r_{s_3} \\
    r_{s_1} & \implies r_{s_4} \lor r_{s_5} \\
    r_{s_2} & \implies r_{s_6} \\
    r_{s_3} & \implies r_{s_7} \\
    r_{s_4} & \implies r_{s_6} \\
    r_{s_5} & \implies r_{s_7} \\
    r_{s_6} \lor r_{s_7} & \implies r_{s_1}
\end{align*}
\]

(c) To ensure that the trap is marked at \( M_0 \) and unmarked at \( M \), we can add the following constraint:

\[
\left( \bigvee_{s \in S:M_0(s) > 0} r_s \right) \land \left( \bigwedge_{s \in S:M(s) > 0} \neg r_s \right)
\]

(d) The additional constraints are:

\[
r_{s_1} \land (\neg r_{s_2} \land \neg r_{s_5})
\]

(e) We obtain a satisfying assignment \( A \) for the constraints by setting \( A(r_{s_1}) = A(r_{s_2}) = A(r_{s_3}) = A(r_{s_4}) = A(r_{s_5}) = A(r_{s_6}) = 1 \) and \( A(r_{s_7}) = A(r_{s_8}) = 0 \). The trap obtained from these constraints is \( R = \{s_1, s_2, s_4, s_5, s_6, s_7\} \). As the trap is marked at \( M_0 \), it needs to stay marked in any reachable marking, therefore the marking \( M \) is not reachable.

Exercise 4.4 Algorithm for the largest siphon

Recall the following algorithm for computing the largest siphon \( Q \) contained in a given set \( R \) of places:

**Input:** A net \( N = (S,T,F) \) and \( R \subseteq S \).

**Output:** The largest siphon \( Q \subseteq R \).

**Initialization:** \( Q := R \).

begin
    while there are \( s \in Q \) and \( t \in \bullet s \) such that \( t \notin Q^* \) do
        \( Q := Q \setminus \{s\} \)
    endwhile
end

Show that the algorithm is correct by showing

(a) that the algorithm terminates, and

(b) that after termination, \( Q \) is the largest siphon contained in \( R \).

Solution:

(a) In every iteration of the while loop, a place \( s \) is removed from \( Q \). \( Q \) contains only finitely many places initially, therefore the while loop and the algorithm terminates.

(b) Let \( Q' \) be the largest siphon contained in \( R \). First we show that \( Q \subseteq Q' \). Let \( s \in Q \). Then for all \( t \in \bullet s \), we have \( t \in Q^* \), therefore \( Q \) is a siphon. As \( Q' \) contains all siphons in \( R \), \( Q \subseteq Q' \).

Now let \( Q_0, Q_1, \ldots, Q_n \) be the intermediate sets in the algorithm, with \( Q_0 = R \) and \( Q_n = Q \). We show that in each step \( i \), we have \( Q' \subseteq Q_i \).

Initially, with \( i = 0 \), we have \( Q' \subseteq R = Q_0 \). Now assume that \( Q' \subseteq Q_i \) and we execute the body of the while loop in step \( i \). Then there is \( s \in Q_i \) and \( t \in \bullet s \) such that \( t \notin Q_i^* \). As \( Q'^* \subseteq Q_i^* \), we also have \( t \notin Q'^* \) and therefore \( s \notin Q' \). Thus \( Q' \subseteq Q_{i+1} = Q_i \setminus \{s\} \).
Exercise 4.5  S-invariants and traps

Prove: Let $N$ be a net and $I$ a semi-positive S-invariant of $N$. The set $R = \{ s \mid I(s) > 0 \}$ of places is a trap of $N$.

Solution:

Let $R = \{ s \mid I(s) > 0 \}$. For $t \in R^*$, there is an $s \in R$ with $s \in \cdot t$. As $I(s) > 0$, $I(s') \geq 0$ for all $s' \in R$ and $\sum_{s' \in \cdot t} I(s) = \sum_{s' \in \cdot t} I(s')$, there is an $s' \in \cdot t$ with $I(s') > 0$. Therefore $s' \in R$ and $t \in \cdot R$, so $R^* \subseteq \cdot R$ and $R$ is a trap.