Solution

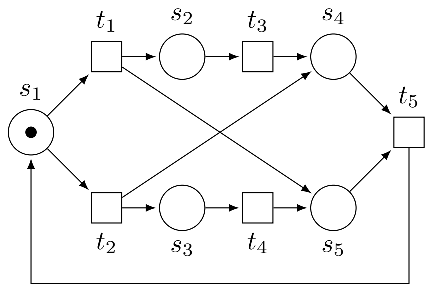
Petri nets – Homework 3

Discussed on Thursday 2nd June, 2016.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 3.1 Marking equation

(a) Construct the incidence matrix $\mathbf{N}$ of the following Petri net:



(b) For the marking M marking $\{s_2, s_3\}$, solve the marking equation $M = M_0 + \mathbf{N} \cdot X$ for X. Note that $M_0 = (1, 0, 0, 0, 0)$ and $M = (0, 1, 1, 0, 0)$ as vectors with the ordering $(s_1, s_2, s_3, s_4, s_5)$ for the places. Does the equation have a solution over the integers? Does it have a non-negative integer solution? If yes, give such a solution.

(c) Can we use the result from (b) to decide if the marking M is reachable?

Solution:

(a) The following is the incidence matrix of the Petri net:

$$\mathbf{N} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

(b) To solve the marking equation, we need to solve the matrix equation $\mathbf{N} \cdot X = M - M_0$ for X. This gives the following matrix, together with its reduced row-echolon form (obtained e.g. by Gauss elimination):

$$\left(\begin{array}{ccccc|c} -1 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The equation has no unique solution. The solution space is given by $X = (0, 1, -1, 0, 0) + \lambda(-1, 1, -1, 1, 0) + \mu(1, 0, 1, 0, 1)$ for $\lambda, \mu \in \mathbb{Q}$. If we choose $\lambda, \nu \in \mathbb{Z}$, we obtain integer solutions, e.g. $X = (0, 1, -1, 0, 0)$. If we additionally choose $\mu > \lambda \geq 0$, we obtain non-negative integer solutions, e.g. $X = (1, 1, 0, 0, 1)$.

(c) No, as even for an unreachable marking, the marking equation can have a non-negative integer solution. In fact, in the net above, M is unreachable, which however can not be concluded by using marking equation alone.

Exercise 3.2 Marking equation in acyclic nets

Show the following: If a net N is structurally acyclic (there is no directed cycle with regard to the flow relation), then a marking M is reachable from an initial marking M_0 *iff* there exists a nonnegative integer solution X satisfying the marking equation $M = M_0 + \mathbf{N} \cdot X$.

Solution: Necessity follows directly from the marking equation lemma, only sufficiency remains to be shown.

For a given acyclic net N, we show: For any marking M and initial marking M_0 and vector $X : T \to \mathbb{N}$, if $M = M_0 + \mathbf{N} \cdot X$, then M is reachable from M_0. We show this by induction on $n := \sum_{t \in T} X(t)$.

Induction base: $n = 0$. Then $X = 0$ and $M = M_0$.

Induction hypothesis: Let $n > 0$ and assume that for all M', M_0' and $X' : T \to \mathbb{N}$ with $\sum_{t \in T} X'(t) < n$ and $M' = M_0' + \mathbf{N} \cdot X'$, M' is reachable from M_0'.

As $n > 0$, there is a t with $X(t) > 0$. The net is acyclic, so from all t with $X(t) > 0$, let this t be one which is minimal with regard to the topological order between them, i.e., there is no t' with $X(t') > 0$ such that there is a path from t' to t. Define $Y : T \to \mathbb{N}$ with $Y(t) := X(t) - 1$ and $Y(u) := X(u)$ if $u \neq t$. We have $X = Y + \vec{t}$ and so

$$M = M_0 + \mathbf{N} \cdot X = M_0 + \mathbf{N} \cdot \left(Y + \vec{t}\right) = M_0 + \mathbf{N} \cdot Y + \mathbf{N} \cdot \vec{t} \geq 0$$

For $s \in {}^\bullet t$ we have $s \notin t^\bullet$ due to acyclicity, so $\left(\mathbf{N} \cdot \vec{t}\right)(s) = \mathbf{N}(s,t) = -1$. None of the transitions in $\langle Y \rangle \subseteq \langle X \rangle$ put tokens in s, so $(\mathbf{N} \cdot Y)(s) \leq 0$. With that we get $M_0(s) \geq 1$, so t is enabled at M_0. With $M_0 \xrightarrow{t} M_1$ we have $M_1 = M_0 + \mathbf{N} \cdot \vec{t}$ and $M = M_1 + \mathbf{N} \cdot Y$. We can apply the induction hypothesis to M, M_1 and Y to obtain that M is reachable from M_1. By extension, M is also reachable from M_0.

Exercise 3.3 Transition liveness levels

For a Petri net (N, M_0) and a transition t of N, we define liveness levels in the following way:

- t is L_0-live (or dead) if t occurs in no firing sequence σ of N enabled at M_0.
- t is L_1-live if t occurs in some firing sequence σ of N enabled at M_0.
- t is L_2-live if for any $k \in \mathbb{N}$, t occurs at least k times in some firing sequence σ of N enabled at M_0.
- t is L_3-live if t occurs infinitely often in some infinite firing sequence σ of N enabled at M_0.
- t is L_4-live if for any reachable marking $M \in [M_0\rangle$, t occurs in some firing sequence σ of N enabled at M, i.e. t can always fire again. *Note*: If this holds for all transitions, this coincides with our standard definition of liveness for Petri nets.

(a) For each $i \in \{0, 1, 2, 3\}$, exhibit a Petri net (N, M_0) and a transition t of N such that t is L_i-live, but not L_{i+i}-live.

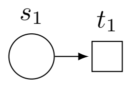
(b) For each $i \in \{0, 1, 2\}$, sketch an algorithm to decide the following problem:

Given a Petri net (N, M_0) and a transition t, is t L_i-live?

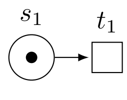
Note: You may also try to find a decision procedure for L_3-liveness, however this is non-trivial, so don't spend too much time on it.

Solution:

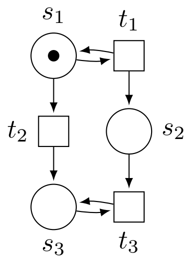
(a)
- In the following net, t_1 is L_0-live (dead). As t_1 is dead, it cannot be L_i-live for $i \geq 1$.



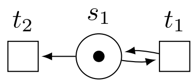
- In the following net, t_1 is L_1-live, as it is enabled at M_0, but not L_1-live, as it can fire at most once.



- In the following net, t_3 is L_2-live, as for any $k \in \mathbb{N}$, it occurs k times in the firing sequence $\sigma = t_1^k t_2 t_3^k$. However, it is not L_3-live, as it needs to be enabled by the occurrence of t_2, after which t_1 cannot occur anymore, so the number of times t_3 can occur is limited by the number of tokens in s_2 after the occurrence of t_2.



- In the following net, t_3 is L_3-live, as it occurs infinitely often in $\sigma = t_1 t_1 t_1 \dots$. However, it is not L_4-live, as it is disabled by the occurrence of t_2.



(b)

- A transition t is L_0-live (dead) iff it is not L_1 live. To decide these two properties, check if the marking M putting one token in each $s \in {}^\bullet t$ and no tokens elsewhere is coverable. If yes, t can be enabled, so it is L_1-live. Otherwise, t is L_0-live (dead).
- To decide L_2-liveness, first add a new place s^+ to the Petri net as an output place of t. Then check if s^+ is unbounded, e.g. by constructing the coverability graph, and checking if an ω-marking M with $M(s^+) = \omega$ exists in the graph. If yes, we can put an arbitrary number of tokens in s^+, especially k tokens for any $k \in \mathbb{N}$, so t can occur k times and it is L_2-live. Otherwise, the number of times t can occur is also bounded, so it is not L_2-live.
- For L_3-liveness, if t is L_3-live, then there is an occurrence sequence σ enabled at M_0 where t occurs infinitely often. This σ can be split up as $\sigma = \sigma_1 t \sigma_2 t \sigma_3 t \sigma_4 \dots$. Therefore $M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{t} M_1' \xrightarrow{\sigma_2} M_2 \xrightarrow{t} M_2' \xrightarrow{\sigma_3} M_3 \xrightarrow{t} M_3' \xrightarrow{\sigma_4} \dots$. By Dickson's lemma, there are i, j with $j > i$ and $M_j \geq M_i$.

  Then if t is L_3-live, there exist occurrence sequences τ, τ' and markings M, M' with $M_0 \xrightarrow{\tau} M \xrightarrow{\tau'} M'$ such that $M' \geq M$ and t occurs in τ' (choose $M = M_i$ and $M' = M_j'$). On the other hand, if such occurence sequences exist, then t is L_3-live, as we can repeat τ' infinitely often.

  Sketch to decide the existance of sequences $M_0 \xrightarrow{\tau} M \xrightarrow{\tau'} M'$ with $M' \geq M$ and t occuring in τ':

  We create a net that in the first phase simulates τ in both the original net and a copy the net. Then control is transferred nondeterministically to the second phase, in which τ' is simulated and only the copy of the net is affected, while the tokens in the original net are frozen. Firing t in the second phase produces an additional token in a new place s^+. Finally, in a third phase, tokens are removed either both from a place in the original net and its copy, only from places in the copy, or from s^+.

  We have that if $M_0 \xrightarrow{\tau} M \xrightarrow{\tau'} M'$, then our net can simulate τ and τ' such that first, we have the marking M in the original net and M' in the copy of the net. If t occurs in τ', we also have s^+ marked. We can then check $M' \geq M$ by removing tokens simultaneously from M and M' or only from M', until we have $M = M' = \mathbf{0}$. We can remove tokens from s^+ until only one token is left. Then, the marking where there is one token in s^+, no tokens in the original net and its copy and control is in the third phase is reachable iff if there are sequences τ, τ' with above property iff t is L_3-live.

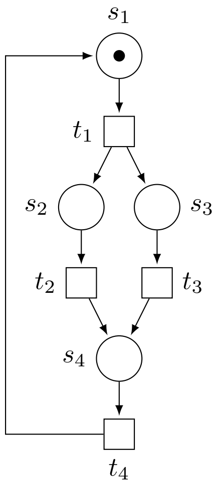
  Note: This problem is also in EXPSPACE, as it can be shown that the length of the shortest occurrence sequence $\tau\tau'$ is bounded by an exponential function, e.g. by adapting the proof of Rackoff's theorem.

<u>Exercise 3.4</u> Number of tokens in bounded nets

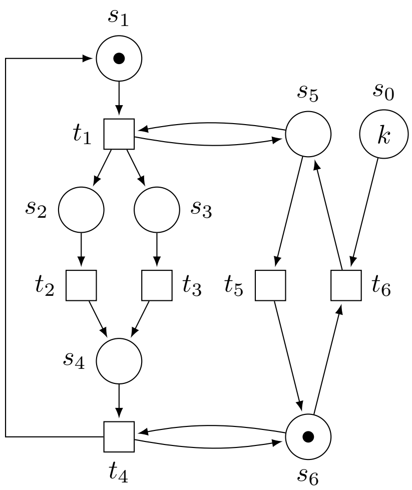
Give a family of bounded Petri nets $\{N_k\}_{k \in \mathbb{N}}$ such that the size of N_k is bounded by $O(k)$ (that is, there is a $c \in \mathbb{N}$ such that for all $N_k = (S, T, F, M_0)$, we have $|S| + |T| + |F| \leq ck$ and $\forall s \in S : M_0(s) \leq ck$), but each N_k has a reachable marking M and a place s with $M(s) \geq 2^{2^k}$.

Hint: Construct a net that doubles the number of tokens in a place. Modify it so that one occurrence sequence for doubling removes exactly one token from a certain place. Use this construct again or the construct from the lecture to put 2^k tokens into that place.

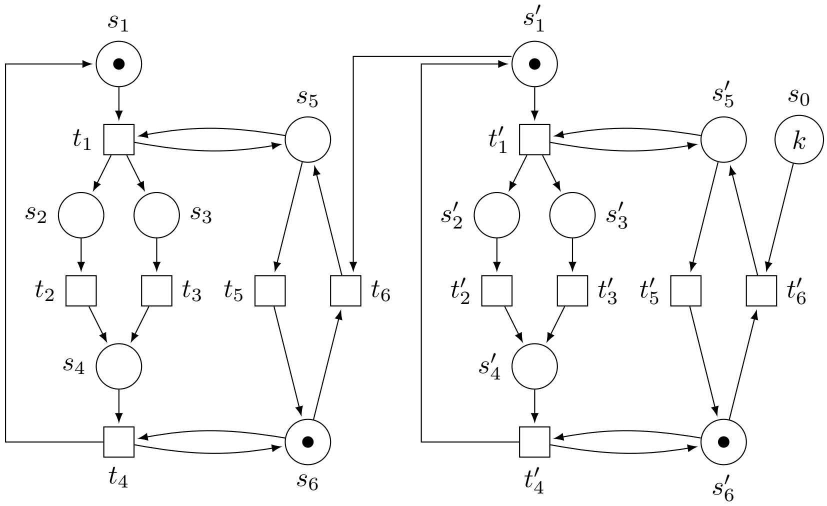
Solution: In the following net, we can fire $t_1 t_2 t_3 t_4 t_4$ to duplicate a token in s_1. If there are n tokens in s_1, the firing sequence $t_1^n t_2^n t_3^n t_4^{2n}$ doubles the number of tokens in s_1.



By modifying the net as follows, we ensure that to fire $t_1^n t_2^n t_3^n t_4^{2n}$, we need to move the token from s_6 to s_5 and back and remove one token from s_0. Now the net is bounded, and with k tokens in s_0, we can put up to 2^k tokens in s_1.



We can duplicate the net and use the output place s_1 as the input place s_0 for the other net. In the following net, we can fire the transitions in the right net to put 2^k tokens in s_1', and then fire the transitions in the left net to put 2^{2^k} tokens in s_1. The net has a constant size, and we have $M(s) \leq k$ for all places s.



The construction could even be repeated, to obtain a family of bounded Petri nets of size $O(k)$ with a reachable marking with $\underbrace{2^{2^{2^{\cdots^{2}}}}}_{k \text{ times}}$ tokens, i.e. an exponential tower of height k.