SOLUTION

Petri nets – Endterm

Last name: ______________________________

First name: ______________________________

Student ID no.: ______________________________

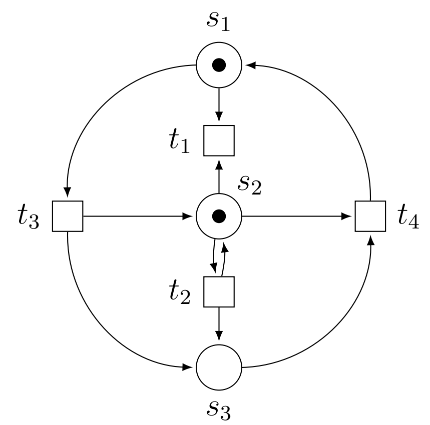
Signature: ______________________________

- If you feel ill, let us know immediately.
- Please, **do not write** until told so.
- You will be given **90 minutes** to fill in all the required information and write down your solutions.
- Don't forget to **sign**.
- Write with a non-erasable **pen**, do not use red or green color.
- You are not allowed to use **auxiliary means** other than your pen.
- You may answer in **English or German**.
- Please turn off your **cell phone**.
- Should you require additional **scrap paper**, please tell us.
- You can obtain **40 points** in the exam. You need **17 points** in total to pass (grade 4.0).
- Don't fill in the table below.
- Good luck!

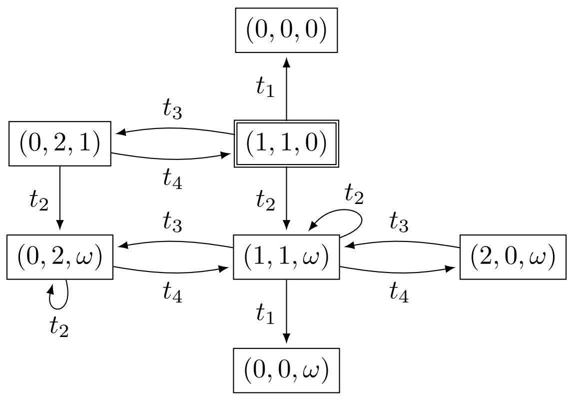
Ex 1	Ex 2	Ex 3	Ex 4	Ex 5	$\sum$

Exercise 1 **5P**

Construct the coverability graph of the Petri net below.



Solution:



Exercise 2 **6P**

Reduce the coverability problem to the reachability problem.

For that, describe an algorithm that, given a Petri net (N, M_0) and a marking M, constructs a Petri net (N', M'_0) and a marking M' such that M' is reachable in N' from M'_0 **if and only if** M is coverable in N from M_0. The algorithm should run in polynomial time. You don't have to describe N' formally.

Give a brief argument showing that your construction is correct, i.e. show that if M is coverable in N from M_0, then M' is reachable in N' from M'_0, and if M' is reachable in N' from M'_0, then M is coverable in N from M_0.

Solution:

Informal answer (sufficient for full points):

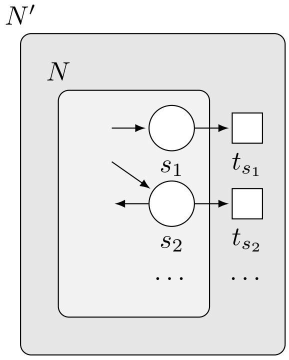
Let N' be a copy of N and for each place of N, add a transition to N' with that place as its only input place and no output places. Let the initial marking and target marking for N' be the same as for N, i.e. $M'_0 = M_0$ and $M' = M$.

If M is coverable in N by some marking $M_1 \geq M$, then we can also reach M_1 in N', and fire the additional transitions to reduce tokens until we reach $M = M'$ in N'.

On the other hand, if M' is reachable in N', then we can execute the sequence to reach M' without firing the additional transitions. That sequence is also enabled in N at M_0 and leads to a marking $M_1 \geq M' = M$, so M is coverable in N.

Formal answer (given for clarity):

Define the net $N' = (S', T', F')$ with $S' = S$, $T' = T \uplus \{t_s \mid s \in S\}$ and $F' = F \cup \{(s, t_s) \mid s \in S\}$ and the markings $M'_0 = M_0$ and $M' = M$. Below is a sketch of the construction:



If M is coverable in N from M_0, then there is a marking M_1 and an occurrence sequence σ with $M_0 \xrightarrow{\sigma} M_1$ in N and $M_1 \geq M$. Then also $M'_0 \xrightarrow{\sigma} M_1$ in N'. From M_1, for each $s \in S$, we can fire t_s exactly $M_1(s) - M(s)$ times. This yields our target marking $M = M'$, so M' is reachable in N' from M'_0.

On the other hand, if M' is reachable in N' from M'_0, then there is an occurrence sequence σ with $M'_0 \xrightarrow{\sigma} M'$ in N'. Let τ be the occurrence sequence obtained from σ by removing all occurrences of t_s for $s \in S$. As every t_s only removes tokens in N', by the monotonicity property of Petri nets, τ is also enabled at M'_0 in N' and as τ only contains transitions from T, it is also enabled at M_0 in N. This yields $M_0 \xrightarrow{\tau} M_1$ in N for some marking M_1 with $M_1 \geq M' = M$, so M is coverable in N from M_0.

<u>Exercise 3</u> **9P=3+3+3**

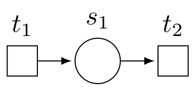
(a) Exhibit a net having a positive T-invariant but no positive S-invariant.

(b) Exhibit a net having a positive S-invariant but no positive T-invariant.

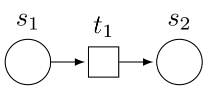
(c) Exhibit a net with a minimal siphon containing two input places of the same transition.

Solution:

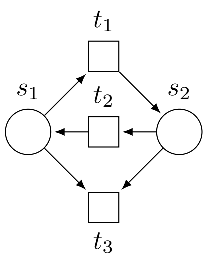
(a) In the following net, $J = (1, 1)$ is a positive T-invariant, but any S-invariant I has to satisfy $I(s_1) = 0$, therefore there is no positive S-invariant.



(b) In the following net, $I = (1, 1)$ is a positive S-invariant, but any T-invariant J has to satisfy $J(t_1) = 0$, therefore there is no positive T-invariant.

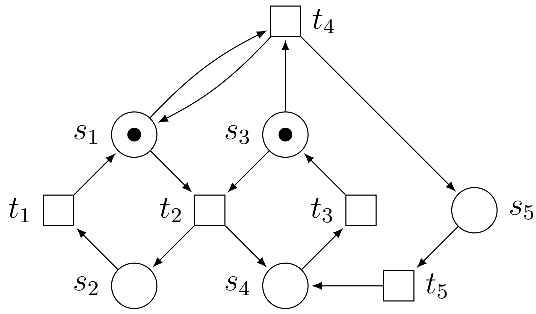


(c) In the following net, $R = \{s_1, s_2\}$ is a minimal siphon with $|R \cap {}^\bullet t_3| = 2$.



<u>Exercise 4</u> **10P=3+3+4**

Consider the following Petri net:



(a) Give a basis of the space of S-invariants of the net.

(b) Find all three minimal traps of the net.

(c) Use (a) and (b) to show that s_2 and s_5 are mutually exclusive, i.e. there is no reachable marking M with $M(s_2) \geq 1$ and $M(s_5) \geq 1$.

Solution:

(a) Any S-invariant I needs to satisfy:

$$\begin{aligned}
t_1: &\quad I(s_2) = I(s_1) \\
t_2: &\quad I(s_1) + I(s_3) = I(s_2) + I(s_4) \\
t_3: &\quad I(s_3) = I(s_4) \\
t_4: &\quad I(s_1) + I(s_3) = I(s_1) + I(s_5) \\
t_5: &\quad I(s_5) = I(s_4)
\end{aligned}$$

From these constraints, we can derive $I(s_1) = I(s_2)$ and $I(s_3) = I(s_4) = I(s_5)$. Any S-invariant is defined by specifying $I(s_1)$ and $I(s_3)$, giving us two invariants for the basis, for instance $I_1 = (1, 1, 0, 0, 0)$ and $I_2 = (0, 0, 1, 1, 1)$.

(b) For any semi-positive S-invariant I, the set of places s with $I(s) > 0$ form a trap. From I_1 and I_2, we obtain the traps $R_1 = \{s_1, s_2\}$ and $R_2 = \{s_3, s_4, s_5\}$, which are already minimal.

The third trap can be found with the trap constraints, as any trap R needs to satisfy:

$$\begin{aligned}
t_1: &\quad s_2 \in R \implies s_1 \in R \\
t_2: &\quad s_1 \in R \lor s_3 \in R \implies s_2 \in R \lor s_4 \in R \\
t_3: &\quad s_4 \in R \implies s_3 \in R \\
t_4: &\quad s_1 \in R \lor s_3 \in R \implies s_1 \in R \lor s_5 \in R \\
t_5: &\quad s_5 \in R \implies s_4 \in R
\end{aligned}$$

By the implications, we see that if s_2 or s_5 are in the trap, then we obtain a superset of R_1 or R_2. Therefore, if we look for a trap without s_2 and s_5, then the only satisfying assignment for a proper trap is $R_3 = \{s_1, s_3, s_4\}$, which is the last minimal trap.

(c) Let M be a reachable marking. From I_1 and I_2, we obtain the positive S-invariant $I_3 = I_1 + I_2 = (1, 1, 1, 1, 1)$, and as $M \cdot I_3 = M_0 \cdot I_3$, we get $M(s_1) + M(s_2) + M(s_3) + M(s_4) + M(s_5) = 2$. From R_3, as $M_0(R_3) \geq 1$, we get $M(R) \geq 1$ and therefore $M(s_1) + M(s_3) + M(s_4) \geq 1$. In combination, we get $M(s_2) + M(s_5) \leq 1$, which shows mutual exclusion.

Exercise 5 — 10P=3+3+4

(a) Prove: If (N, M_0) is a live and bounded Petri net, then N has a positive T-invariant.

(b) Prove: If (N, M_0) is a live and bounded free-choice system and $M_0' \geq M_0$, then (N, M_0') is also live and bounded.

(c) Prove: Let N be a net and M a marking of N. The equation $M = \mathbf{N} \cdot X$ has a nonnegative integer solution $X : T \to \mathbb{N}$ **iff** there is marking M_0 of N such that $M + M_0$ is reachable from M_0 in N.

Note: $M + M_0$ is defined as the marking with $(M + M_0)(s) = M(s) + M_0(s)$.

Solution:

(a) Let (N, M_0) be a live and bounded Petri net. By liveness there is an infinite occurrence sequence $\sigma_1\sigma_2\sigma_3\dots$ such that every σ_i is a finite occurrence sequence containing all transitions of N. We have

$$M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \xrightarrow{\sigma_3} \cdots .$$

By boundedness there are indices $i < j$ such that $M_i = M_j$. So the sequence $\sigma_{i+1}\dots\sigma_j$ satisfies

$$M_i \xrightarrow{\sigma_{i+1}\dots\sigma_j} M_i$$

and so $J = \vec{\sigma}_{i+1} + \cdots + \vec{\sigma}_j$ is a T-invariant of N. Further, J is positive because every transition occurs at least once in $\sigma_{i+1}\dots\sigma_j$.

(b) Let (N, M_0) be a live and bounded free-choice system and $M_0' \geq M_0$.

By Commoner's Liveness Theorem, every proper siphon of N contains a trap marked at M_0. As $M_0' \geq M_0$, every such trap is also marked at M_0', therefore the system (N, M_0') is also live.

By Hack's Boundedness Theorem, every place of N belongs to an S-component, therefore (N, M_0') is also bounded.

(c) ($\Rightarrow$): Let X be a nonnegative integer solution of $M = \mathbf{N} \cdot X$. Then let M_0 be a marking sufficiently large to consecutively enable all transitions $t \in T$ exactly $X(t)$ times in some order. Clearly, such a marking exists. Let σ be a corresponding occurrence sequence enabled at M_0 with $\vec{\sigma} = X$. We then have $M_0 \xrightarrow{\sigma} M_1$ for some marking M_1 and with the marking equation, we have $M_1 = M_0 + \mathbf{N} \cdot \vec{\sigma} = M_0 + \mathbf{N} \cdot X = M_0 + M$, so $M_0 + M$ is reachable from M_0.

($\Leftarrow$): Let M_0 be a marking of N such that $M + M_0$ is reachable from M_0 in N. Then there is an occurence sequence σ with $M_0 \xrightarrow{\sigma} M + M_0$. With the marking equation, we have $M + M_0 = M_0 + \mathbf{N} \cdot \vec{\sigma}$ and, by subtracting M_0 from both sides, $M = \mathbf{N} \cdot \vec{\sigma}$. So $X := \vec{\sigma}$ is a nonnegative integer solution of $M = \mathbf{N} \cdot X$.