SOLUTION

Petri nets – Endterm

Last name: ______________________________________

First name: ______________________________________

Student ID no.: ___________________________________

Signature: ________________________________________

- If you feel ill, let us know immediately.
- Please, do not write until told so.
- You will be given 90 minutes to fill in all the required information and write down your solutions.
- Don’t forget to sign.
- Write with a non-erasable pen, do not use red or green color.
- You are not allowed to use auxiliary means other than your pen.
- You may answer in English or German.
- Please turn off your cell phone.
- Should you require additional scrap paper, please tell us.
- You can obtain 40 points in the exam. You need 17 points in total to pass (grade 4.0).
- Don’t fill in the table below.
- Good luck!

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Exercise 1

Construct the coverability graph of the Petri net below.

![Petri net diagram](image)

Solution:

![Coverability graph](image)

Exercise 2

Reduce the coverability problem to the reachability problem.

For that, describe an algorithm that, given a Petri net \((N, M_0)\) and a marking \(M\), constructs a Petri net \((N', M'_0)\) and a marking \(M'\) such that \(M'\) is reachable in \(N'\) from \(M'_0\) \textbf{if and only if} \(M\) is coverable in \(N\) from \(M_0\). The algorithm should run in polynomial time. You don’t have to describe \(N'\) formally.

Give a brief argument showing that your construction is correct, i.e. show that if \(M\) is coverable in \(N\) by some marking \(M_1 \geq M\), then we can also reach \(M_1\) in \(N'\), and if \(M'\) is reachable in \(N'\) from \(M'_0\), then \(M\) is coverable in \(N\) from \(M_0\).

Solution:

\textbf{Informal answer} (sufficient for full points):

Let \(N'\) be a copy of \(N\) and for each place of \(N\), add a transition to \(N'\) with that place as its only input place and no output places. Let the initial marking and target marking for \(N'\) be the same as for \(N\), i.e. \(M'_0 = M_0\) and \(M' = M\).

If \(M\) is coverable in \(N\) by some marking \(M_1 \geq M\), then we can also reach \(M_1\) in \(N'\), and fire the additional transitions to reduce tokens until we reach \(M = M'\) in \(N'\).

On the other hand, if \(M'\) is reachable in \(N'\), then we can execute the sequence to reach \(M'\) without firing the additional transitions. That sequence is also enabled in \(N\) at \(M_0\) and leads to a marking \(M_1 \geq M' = M\), so \(M\) is coverable in \(N\).

\textbf{Formal answer} (given for clarity):

Define the net \(N' = (S', T', F')\) with \(S' = S\), \(T' = T \cup t_s \mid s \in S\) and \(F' = F \cup \{(s, t_s) \mid s \in S\}\) and the markings \(M'_0 = M_0\) and \(M' = M\). Below is a sketch of the construction:
If $M$ is coverable in $N$ from $M_0$, then there is a marking $M_1$ and an occurrence sequence $\sigma$ with $M_0 \overset{\sigma}{\rightarrow} M_1$ in $N$ and $M_1 \geq M$. Then also $M'_0 \overset{\sigma}{\rightarrow} M_1$ in $N'$. From $M_1$, for each $s \in S$, we can fire $t_s$ exactly $M_1(s) - M(s)$ times. This yields our target marking $M = M'$, so $M'$ is reachable in $N'$ from $M'_0$.

On the other hand, if $M'$ is reachable in $N'$ from $M'_0$, then there is an occurrence sequence $\sigma$ with $M'_0 \overset{\sigma}{\rightarrow} M'$ in $N'$. Let $\tau$ be the occurrence sequence obtained from $\sigma$ by removing all occurrences of $t_s$ for $s \in S$. As every $t_s$ only removes tokens in $N'$, by the monotonicity property of Petri nets, $\tau$ is also enabled at $M'_0$ in $N'$ and as $\tau$ only contains transitions from $T$, it is also enabled at $M_0$ in $N$. This yields $M_0 \overset{\tau}{\rightarrow} M_1$ in $N$ for some marking $M_1$ with $M_1 \geq M' = M$, so $M$ is coverable in $N$ from $M_0$.

**Exercise 3**

(a) Exhibit a net having a positive T-invariant but no positive S-invariant.
(b) Exhibit a net having a positive S-invariant but no positive T-invariant.
(c) Exhibit a net with a minimal siphon containing two input places of the same transition.

**Solution:**

(a) In the following net, $J = (1, 1)$ is a positive T-invariant, but any S-invariant $I$ has to satisfy $I(s_1) = 0$, therefore there is no positive S-invariant.

```
  \[
  \begin{array}{c}
  t_1 \\
  \hline
  s_1 \\
  \hline
  t_2 \\
  \end{array}
  \]
```

(b) In the following net, $I = (1, 1)$ is a positive S-invariant, but any T-invariant $J$ has to satisfy $J(t_1) = 0$, therefore there is no positive T-invariant.

```
  \[
  \begin{array}{c}
  \hline
  s_1 \\
  \hline
  t_1 \\
  \hline
  s_2 \\
  \end{array}
  \]
```

(c) In the following net, $R = \{s_1, s_2\}$ is a minimal siphon with $|R \cap \ast t_3| = 2$.

```
  \[
  \begin{array}{c}
  \hline
  t_1 \\
  \hline
  s_1 \\
  \hline
  t_2 \\
  \hline
  s_2 \\
  \hline
  t_3 \\
  \end{array}
  \]
```

**Exercise 4**

Consider the following Petri net:

```
\[
\begin{array}{c}
N' \\
\hline
N \\
\hline
\end{array}
\]
```

```
\[
\begin{array}{c}
\hline
s_1 \\
\hline
t_{s_1} \\
\hline
s_2 \\
\hline
t_{s_2} \\
\hline
\ldots \\
\hline
\end{array}
\]
```
(a) Give a basis of the space of S-invariants of the net.

(b) Find all three minimal traps of the net.

(c) Use (a) and (b) to show that \( s_2 \) and \( s_5 \) are mutually exclusive, i.e. there is no reachable marking \( M \) with \( M(s_2) \geq 1 \) and \( M(s_5) \geq 1 \).

**Solution:**

(a) Any S-invariant \( I \) needs to satisfy:

\[
\begin{align*}
    t_1 : & \quad I(s_2) = I(s_1) \\
    t_2 : & \quad I(s_1) + I(s_3) = I(s_2) + I(s_4) \\
    t_3 : & \quad I(s_3) = I(s_4) \\
    t_4 : & \quad I(s_1) + I(s_3) = I(s_1) + I(s_5) \\
    t_5 : & \quad I(s_5) = I(s_4)
\end{align*}
\]

From these constraints, we can derive \( I(s_1) = I(s_2) \) and \( I(s_3) = I(s_4) = I(s_5) \). Any S-invariant is defined by specifying \( I(s_1) \) and \( I(s_3) \), giving us two invariants for the basis, for instance \( I_1 = (1, 1, 0, 0, 0) \) and \( I_2 = (0, 0, 1, 1, 1) \).

(b) For any semi-positive S-invariant \( I \), the set of places \( s \) with \( I(s) > 0 \) form a trap. From \( I_1 \) and \( I_2 \), we obtain the traps \( R_1 = \{s_1, s_2\} \) and \( R_2 = \{s_3, s_4, s_5\} \), which are already minimal.

The third trap can be found with the trap constraints, as any trap \( R \) needs to satisfy:

\[
\begin{align*}
    t_1 : & \quad s_2 \in R \implies s_1 \in R \\
    t_2 : & \quad s_1 \in R \lor s_3 \in R \implies s_2 \in R \lor s_4 \in R \\
    t_3 : & \quad s_4 \in R \implies s_3 \in R \\
    t_4 : & \quad s_1 \in R \lor s_3 \in R \implies s_1 \in R \lor s_5 \in R \\
    t_5 : & \quad s_5 \in R \implies s_4 \in R
\end{align*}
\]

By the implications, we see that if \( s_2 \) or \( s_5 \) are in the trap, then we obtain a superset of \( R_1 \) or \( R_2 \). Therefore, if we look for a trap without \( s_2 \) and \( s_5 \), then the only satisfying assignment for a proper trap is \( R_3 = \{s_1, s_3, s_4\} \), which is the last minimal trap.

(c) Let \( M \) be a reachable marking. From \( I_1 \) and \( I_2 \), we obtain the positive S-invariant \( I'_1 = I_1 + I_2 = (1, 1, 1, 1, 1) \), and as \( M \cdot I'_3 = M_0 \cdot I_1 \), we get \( M(s_1) + M(s_2) + M(s_3) + M(s_4) + M(s_5) = 2 \). From \( R_3 \), as \( M_0(R_3) \geq 1 \), we get \( M(R) \geq 1 \) and therefore \( M(s_1) + M(s_3) + M(s_4) \geq 1 \). In combination, we get \( M(s_2) + M(s_5) \leq 1 \), which shows mutual exclusion.

**Exercise 5**

\( \text{10P}=3+3+4 \)

(a) Prove: If \((N, M_0)\) is a live and bounded Petri net, then \( N \) has a positive T-invariant.

(b) Prove: If \((N, M_0)\) is a live and bounded free-choice system and \(M'_0 \geq M_0\), then \((N, M'_0)\) is also live and bounded.

(c) Prove: Let \( N \) be a net and \( M \) a marking of \( N \). The equation \( M = N \cdot X \) has a nonnegative integer solution \( X : T \to \mathbb{N} \) iff there is marking \( M_0 \) of \( N \) such that \( M + M_0 \) is reachable from \( M_0 \) in \( N \).

\textbf{Note:} \( M + M_0 \) is defined as the marking with \( (M + M_0)(s) = M(s) + M_0(s) \).
Solution:

(a) Let $(N, M_0)$ be a live and bounded Petri net. By liveness there is an infinite occurrence sequence $\sigma_1 \sigma_2 \sigma_3 \ldots$ such that every $\sigma_i$ is a finite occurrence sequence containing all transitions of $N$. We have

$$M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \xrightarrow{\sigma_3} \ldots.$$ 

By boundedness there are indices $i < j$ such that $M_i = M_j$. So the sequence $\sigma_{i+1} \ldots \sigma_j$ satisfies

$$M_i \xrightarrow{\sigma_{i+1} \ldots \sigma_j} M_i$$

and so $J = \sigma_{i+1} + \ldots + \sigma_j$ is a T-invariant of $N$. Further, $J$ is positive because every transition occurs at least once in $\sigma_{i+1} \ldots \sigma_j$.

(b) Let $(N, M_0)$ be a live and bounded free-choice system and $M'_0 \geq M_0$.

By Commoner's Liveness Theorem, every proper siphon of $N$ contains a trap marked at $M_0$. As $M'_0 \geq M_0$, every such trap is also marked at $M'_0$, therefore the system $(N, M'_0)$ is also live.

By Hack's Boundedness Theorem, every place of $N$ belongs to an S-component, therefore $(N, M'_0)$ is also bounded.

(c) $(\Rightarrow)$: Let $X$ be a nonnegative integer solution of $M = N \cdot X$. Then let $M_0$ be a marking sufficiently large to consecutively enable all transitions $t \in T$ exactly $X(t)$ times in some order. Clearly, such a marking exists. Let $\sigma$ be a corresponding occurrence sequence enabled at $M_0$ with $\bar{\sigma} = X$. We then have $M_0 \xrightarrow{\sigma} M_1$ for some marking $M_1$ and with the marking equation, we have $M_1 = M_0 + N \cdot \bar{\sigma} = M_0 + N \cdot X = M_0 + M$, so $M_0 + M$ is reachable from $M_0$.

$(\Leftarrow)$: Let $M_0$ be a marking of $N$ such that $M + M_0$ is reachable from $M_0$ in $N$. Then there is an occurrence sequence $\sigma$ with $M_0 \xrightarrow{\sigma} M + M_0$. With the marking equation, we have $M + M_0 = M_0 + N \cdot \bar{\sigma}$ and, by subtracting $M_0$ from both sides, $M = N \cdot \bar{\sigma}$. So $X := \bar{\sigma}$ is a nonnegative integer solution of $M = N \cdot X$.\