Solution

Petri nets – Homework 7

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For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 7.1  Boundedness and liveness in S/T-systems

Show the following:
(a) An S-system \((N, M_0)\) is bounded for any \(M_0\).
(b) If \((N, M_0)\) is a live S-system and \(M_0' \geq M_0\), then \((N, M_0')\) is also live.
(c) If \((N, M_0)\) is a live and bounded T-system, then \((N, M_0')\) is also bounded for any \(M_0'\).
(d) If \((N, M_0)\) is a live T-system and \(M_0' \geq M_0\), then \((N, M_0')\) is also live.

Exhibit Petri nets for the following:
(e) Give a bounded T-system \((N, M_0)\) and a marking \(M_0' \geq M_0\) such that \((N, M_0')\) is not bounded.
(f) Give a 1-bounded S-system \((N, M_0)\) where \(M_0(S) > 1\).
(g) Give a live and 1-bounded T-system \((N, M_0)\) with a circuit \(\gamma\) where \(M_0(\gamma) > 1\).

Solution:
(a) By the fundamental property of S-systems, for every reachable marking, we have \(M(S) = M_0(S)\) and therefore \(M(s) \leq M_0(S)\) for all \(s \in S\).
(b) By the liveness theorem for S-systems, \((N, M_0)\) is live iff \(N\) is strongly connected and \(M_0(S) > 0\), and as \(M_0'(S) \geq M_0(S) > 0\), \((N, M_0')\) is also live.
(c) A live T-system is bounded iff \(N\) is strongly connected, therefore if \((N, M_0)\) is bounded, then \((N, M_0')\) is also bounded.
(d) By the liveness theorem for T-systems, \((N, M_0)\) is live iff \(M_0(\gamma) > 0\) for every circuit \(\gamma\), and as \(M_0'(\gamma) \geq M_0(\gamma) > 0\), \((N, M_0')\) is also live.
(e) Due to (c), the system needs to be non-live. The following Petri net without any tokens is a non-live, bounded T-system. By adding the blue token to \(s_1\), the net becomes unbounded.

(f) Due to the boundedness theorem for S-systems, the system needs to be non-live. The following Petri net is a non-live, 1-bounded S-system with \(M_0(S) > 1\):
In the following live T-system, the inner circuit $s_1s_2s_3$ contains 2 tokens, however each place is 1-bounded due to the outer circuits.

**Exercise 7.2  Circuits in T-systems**

Consider the T-system $(N, M_0)$ below.

(a) Find all circuits of the net.
(b) Use the circuits to decide if the system is live.
(c) For each place $s$, determine the bound of $s$ by analyzing the circuits containing $s$.
(d) Apply the construction from the proof of Genrich’s Theorem (Theorem 5.2.9) to find a marking $M'_0$ such that $(N, M'_0)$ is live and 1-bounded.

**Solution:**

(a) The circuits of the net are $\gamma_1 = s_1s_5$, $\gamma_2 = s_2s_6s_4$ and $\gamma_3 = s_2s_6s_7s_3$.
(b) We have $M_0(\gamma_1) = 1$, $M_0(\gamma_2) = 1$ and $M_0(\gamma_3) = 2$. Each circuit is initially marked, so the system is live.
(c) The system is bounded, as each place belongs to some circuit. The bound of each place $s$ is equal to the minimal number of tokens among the circuits where $s$ is contained. With that, we get the following bounds:

- $s_1$ : contained in $\gamma_1$ with $M_0(\gamma_1) = 1$, so the bound is 1.
- $s_2, s_4, s_6$ : contained in $\gamma_2$ with $M_0(\gamma_2) = 1$ and in $\gamma_3$ with $M_0(\gamma_3) = 2$, so the bound is $\min(1, 2) = 1$.
- $s_3, s_7$ : contained in $\gamma_3$ with $M_0(\gamma_3) = 2$, so the bound is 2.

(d) The system is not 1-bounded, as the bound of $s_3$ and $s_7$ is 2. By firing $M_0 \xrightarrow{t_5} M$, we obtain a marking $M$ with $M(s_7) = 2$. Define the marking $L$ by $L(s_7) = 1$ and $L(s) = M(s)$ for all other $s$. By construction, $(N, L)$ is still live and bounded. Now for the circuit $\gamma_3$, we have $L(\gamma_3) = 1$, so $(N, L)$ is 1-bounded. We have the marking $M'_0 = L = (1, 0, 0, 1, 0, 1)$.

**Exercise 7.3  Paths and transitions in T-systems**

For a live T-system $(N, M_0)$, with the reachability theorem (Theorem 5.2.7), we have that a marking $M$ is reachable iff $M_0 \sim M$. From the proof, we can easily infer the following corollary:

**Corollary 7.3.1.** Let $(N, M_0)$ be a live T-system, $M$ a marking of $N$ and $X : T \rightarrow N$ a vector such that $M = M_0 + N \cdot X$. There is an occurrence sequence $M_0 \xrightarrow{\sigma} M$ such that $\sigma = X$.

For a live T-system $(N, M_0)$, use this corollary to show the following. Remember that $J = (1, \ldots, 1)$ is a T-invariant of any T-system.

(a) There is an occurrence sequence $\sigma$ with $M_0 \xrightarrow{\sigma} M_0$ such that $\sigma$ contains every transition of $N$ exactly once.
(b) For a reachable marking $M$, there exists an occurrence sequence $\sigma$ with $M_0 \xrightarrow{\sigma} M$ such that $\sigma$ does not contain all transitions of $N$.

Solution:

(a) For the T-invariant $J = (1, \ldots, 1)$, we have $M_0 + N \cdot J = M_0$. With the corollary, there is an occurrence sequence $M_0 \xrightarrow{\sigma} M_0$ with $\bar{\sigma} = J$, so $\sigma$ contains every transition exactly once. Note that this result implies that each transitions can be fired within a sequence of length at most $|T|$.

(b) Let $\sigma$ be a minimal occurrence sequence with $M_0 \xrightarrow{\sigma} M$. Assume that $\sigma$ contains every transition of $N$. Then with $J = (1, \ldots, 1)$, we have $\bar{\sigma} - J \geq 0$ and

$$M_0 + N \cdot (\bar{\sigma} - J) = M_0 + N \cdot \bar{\sigma} - N \cdot J \equiv 0 \equiv M_0 + N \cdot \bar{\sigma} = M,$$

so there is a transition sequence $\tau$ with $\bar{\tau} = \bar{\sigma} - J$ and $M_0 \xrightarrow{\tau} M$, which contradicts the minimality of $\sigma$.

Exercise 7.4 Path length in T-systems

For each $n \in \mathbb{N}$, give a 1-bounded T-system $(N, M_0)$ with $n$ transitions and a reachable marking $M$ such that the minimal occurrence sequence $\sigma$ with $M_0 \xrightarrow{\sigma} M$ has a length of $n(n-1)/2$.

Hint: First try find a Petri net and a marking for $n = 3$, where the minimal sequence has length 3. For this a net with 4 places suffices. Then try to generalize your solution.

Solution:

For $n = 3$, we can take the following net with the marking $M = (0, 0, 1, 1)$. To reach this marking, we need to fire $t_1$ and $t_2$ to mark $s_3$ and $s_4$. However, firing $t_2$ undoes the effect of $t_1$ on $s_3$, so we need to fire $t_1$ twice. The minimal sequence is then $\sigma = t_1 t_2 t_1$ of length 3.

This construction can be repeated for arbitrary $n$, as shown in the following sketch of a Petri net. To reach the marking $M$ with $M(s_{i,1}) = 0$ and $M(s_{i,2}) = 1$ for all $1 \leq i \leq n - 1$ with a minimal sequence, we need to fire $\sigma = t_1 t_2 \ldots t_{n-1} t_1 t_2 \ldots t_{n-2} \ldots t_1$, which has a length of $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$.