Solution

Petri nets – Homework 7

Discussed on Thursday 25th June, 2015.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 7.1 Boundedness and liveness in S/T-systems

Show the following:

(a) An S-system (N, M_0) is bounded for any M_0.

(b) If (N, M_0) is a live S-system and $M_0' \geq M_0$, then (N, M_0') is also live.

(c) If (N, M_0) is a live and bounded T-system, then (N, M_0') is also bounded for any M_0'.

(d) If (N, M_0) is a live T-system and $M_0' \geq M_0$, then (N, M_0') is also live.

Exhibit Petri nets for the following:

(e) Give a bounded T-system (N, M_0) and a marking $M_0' \geq M_0$ such that (N, M_0') is not bounded.

(f) Give a 1-bounded S-system (N, M_0) where $M_0(S) > 1$.

(g) Give a live and 1-bounded T-system (N, M_0) with a circuit γ where $M_0(\gamma) > 1$.

Solution:

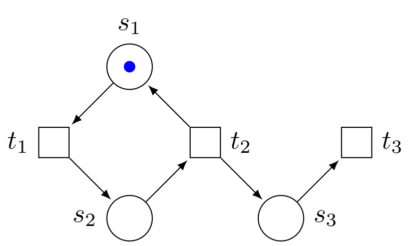
(a) By the fundamental property of S-systems, for every reachable marking, we have $M(S) = M_0(S)$ and therefore $M(s) \leq M_0(S)$ for all $s \in S$.

(b) By the liveness theorem for S-systems, (N, M_0) is live iff N is strongly connected and $M_0(S) > 0$, and as $M_0'(S) \geq M_0(S) > 0$, (N, M_0') is also live.

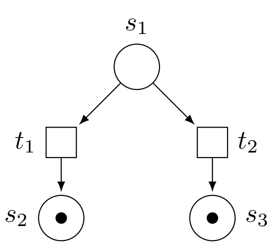
(c) A live T-system is bounded iff N is strongly connected, therefore if (N, M_0) is bounded, then (N, M_0') is also bounded.

(d) By the liveness theorem for T-systems, (N, M_0) is live iff $M_0(\gamma) > 0$ for every circuit γ, and as $M_0'(\gamma) \geq M_0(\gamma) > 0$, (N, M_0') is also live.

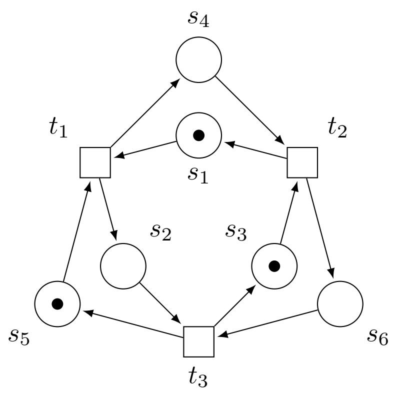
(e) Due to (c), the system needs to be non-live. The following Petri net without any tokens is a non-live, bounded T-system. By adding the blue token to s_1, the net becomes unbounded.



(f) Due to the boundedness theorem for S-systems, the system needs to be non-live. The following Petri net is a non-live, 1-bounded S-system with $M_0(S) > 1$:

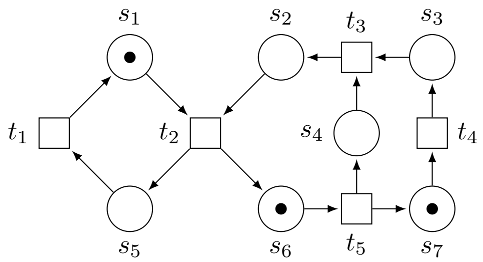


(g) In the following live T-system, the inner circuit $s_1 s_2 s_3$ contains 2 tokens, however each place is 1-bounded due to the outer circuits.



Exercise 7.2 Circuits in T-systems

Consider the T-system (N, M_0) below.



(a) Find all circuits of the net.

(b) Use the circuits to decide if the system is live.

(c) For each place s, determine the bound of s by analyzing the circuits containing s.

(d) Apply the construction from the proof of Genrich's Theorem (Theorem 5.2.9) to find a marking M_0' such that (N, M_0') is live and 1-bounded.

Solution:

(a) The circuits of the net are $\gamma_1 = s_1 s_5$, $\gamma_2 = s_2 s_6 s_4$ and $\gamma_3 = s_2 s_6 s_7 s_3$.

(b) We have $M_0(\gamma_1) = 1$, $M_0(\gamma_2) = 1$ and $M_0(\gamma_3) = 2$. Each circuit is initially marked, so the system is live.

(c) The system is bounded, as each place belongs to some circuit. The bound of each place s is equal to the minimal number of tokens among the circuits where s is contained. With that, we get the following bounds:

$$\begin{aligned} s_1 &: \text{contained in } \gamma_1 \text{ with } M_0(\gamma_1) = 1, \text{ so the bound is 1.} \\ s_2, s_4, s_6 &: \text{contained in } \gamma_2 \text{ with } M_0(\gamma_2) = 1 \text{ and in } \gamma_3 \text{ with } M_0(\gamma_3) = 2, \text{ so the bound is } \min(1,2) = 1. \\ s_3, s_7 &: \text{contained in } \gamma_3 \text{ with } M_0(\gamma_3) = 2, \text{ so the bound is 2.} \end{aligned}$$

(d) The system is not 1-bounded, as the bound of s_3 and s_7 is 2. By firing $M_0 \xrightarrow{t_5} M$, we obtain a marking M with $M(s_7) = 2$. Define the marking L by $L(s_7) = 1$ and $L(s) = M(s)$ for all other s. By construction, (N, L) is still live and bounded. Now for the circuit γ_3, we have $L(\gamma_3) = 1$, so (N, L) is 1-bounded. We have the marking $M_0' = L = (1, 0, 0, 1, 0, 1)$.

Exercise 7.3 Paths and transitions in T-systems

For a live T-system (N, M_0), with the reachability theorem (Theorem 5.2.7), we have that a marking M is reachable iff $M_0 \sim M$. From the proof, we can easily infer the following corollary:

Corollary 7.3.1. Let (N, M_0) be a live T-system, M a marking of N and $X : T \to \mathbb{N}$ a vector such that $M = M_0 + \mathbf{N} \cdot X$. There is an occurrence sequence $M_0 \xrightarrow{\sigma} M$ such that $\vec{\sigma} = X$.

For a live T-system (N, M_0), use this corollary to show the following. Remember that $J = (1, \dots, 1)$ is a T-invariant of any T-system.

(a) There is an occurrence sequence σ with $M_0 \xrightarrow{\sigma} M_0$ such that σ contains every transition of N exactly once.

(b) For a reachable marking M, there exists an occurrence sequence σ with $M_0 \xrightarrow{\sigma} M$ such that σ does not contain all transitions of N.

Solution:

(a) For the T-invariant $J = (1, \ldots, 1)$, we have $M_0 + \mathbf{N} \cdot J = M_0$. With the corollary, there is an occurrence sequence $M_0 \xrightarrow{\sigma} M_0$ with $\vec{\sigma} = J$, so σ contains every transition exactly once. Note that this result implies that each transitions can be fired within a sequence of length at most $|T|$.

(b) Let σ be a minimal occurrence sequence with $M_0 \xrightarrow{\sigma} M$. Assume that σ contains every transition of N. Then with $J = (1, \ldots, 1)$, we have $\vec{\sigma} - J \geq 0$ and

$$M_0 + \mathbf{N} \cdot (\vec{\sigma} - J) = M_0 + \mathbf{N} \cdot \vec{\sigma} - \underbrace{\mathbf{N} \cdot J}_{=0} = M_0 + \mathbf{N} \cdot \vec{\sigma} = M,$$

so there is a transition sequence τ with $\vec{\tau} = \vec{\sigma} - J$ and $M_0 \xrightarrow{\tau} M$, which contradicts the minimality of σ.

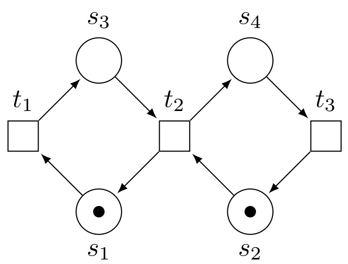
Exercise 7.4 Path length in T-systems

For each $n \in \mathbb{N}$, give a 1-bounded T-system (N, M_0) with n transitions and a reachable marking M such that the minimal occurrence sequence σ with $M_0 \xrightarrow{\sigma} M$ has a length of $\frac{n(n-1)}{2}$.

Hint: First try find a Petri net and a marking for $n = 3$, where the minimal sequence has length 3. For this a net with 4 places suffices. Then try to generalize your solution.

Solution:

For $n = 3$, we can take the following net with the marking $M = (0, 0, 1, 1)$. To reach this marking, we need to fire t_1 and t_2 to mark s_3 and s_4. However, firing t_2 undoes the effect of t_1 on s_3, so we need to fire t_1 twice. The minimal sequence is then $\sigma = t_1 t_2 t_1$ of length 3.



This construction can be repeated for arbitrary n, as shown in the following sketch of a Petri net. To reach the marking M with $M(s_{i,1}) = 0$ and $M(s_{i,2}) = 1$ for all $1 \leq i \leq n-1$ with a minimal sequence, we need to fire $\sigma = t_1 t_2 \ldots t_{n-1}\ t_1 t_2 \ldots t_{n-2}\ \ldots\ t_1$, which has a length of $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$.