Solution

Petri nets – Homework 6

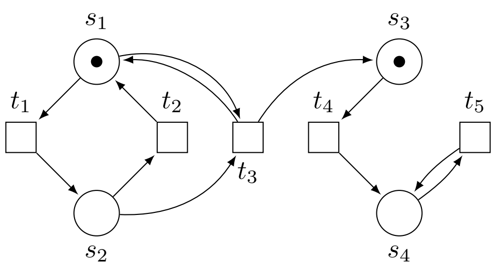
Discussed on Thursday 18th June, 2015.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 6.1 Siphons and traps

(a) Find all the proper siphons and traps in the Petri net below.

(b) Check if each proper siphon contains an initially marked trap.



Solution:

(a) A set R of places is a siphon if ${}^\bullet R \subseteq R^\bullet$. This is equivalent to requiring that every transition that puts a token into R also takes a token from R. This constraint is expressed by requiring that for all transitions t, we have $\left(\bigvee_{s \in t^\bullet} s \in R\right) \implies \left(\bigvee_{s \in {}^\bullet t} s \in R\right)$. For this net, the constraints are as follows, with the added constraint for a proper siphon:

$$\begin{aligned}
t_1 :&\quad s_2 \in R \implies s_1 \in R \\
t_2 :&\quad s_1 \in R \implies s_2 \in R \\
t_3 :&\quad s_1 \in R \vee s_3 \in R \implies s_1 \in R \vee s_2 \in R \\
t_4 :&\quad s_4 \in R \implies s_3 \in R \\
t_5 :&\quad s_4 \in R \implies s_4 \in R \\
&\quad s_1 \in R \vee s_2 \in R \vee s_3 \in R \vee s_4 \in R
\end{aligned}$$

By enumerating the solutions, we obtain the set of siphons $\{\{s_1, s_2\}, \{s_1, s_2, s_3\}, \{s_1, s_2, s_3, s_4\}\}$.

Similarly, a set R of places is a trap if $R^\bullet \subseteq {}^\bullet R$. This is equivalent to requiring that every transition that takes a token from R also puts a token into R. This constraint is expressed by requiring that for all transitions t, we have $\left(\bigvee_{s \in {}^\bullet t} s \in R\right) \implies \left(\bigvee_{s \in t^\bullet} s \in R\right)$. For this net, the constraints are as follows, with the added constraint for a proper trap:

$$\begin{aligned}
t_1 :&\quad s_1 \in R \implies s_2 \in R \\
t_2 :&\quad s_2 \in R \implies s_1 \in R \\
t_3 :&\quad s_1 \in R \vee s_2 \in R \implies s_1 \in R \vee s_3 \in R \\
t_4 :&\quad s_3 \in R \implies s_4 \in R \\
t_5 :&\quad s_4 \in R \implies s_4 \in R \\
&\quad s_1 \in R \vee s_2 \in R \vee s_3 \in R \vee s_4 \in R
\end{aligned}$$

By enumerating the solutions, we obtain the set of traps $\{\{s_1, s_2\}, \{s_4\}, \{s_3, s_4\}, \{s_1, s_2, s_4\}, \{s_1, s_2, s_3, s_4\}\}$.

(b) The trap $\{s_1, s_2\}$ is initially marked and contained in every proper siphon. Therefore the net is deadlock-free.

Exercise 6.2 Algorithm for the largest siphon

Recall the following algorithm for computing the largest siphon Q contained in a given set R of places:

Input: A net $N = (S, T, F)$ and $R \subseteq S$.
Output: The largest siphon $Q \subseteq R$.
Initialization: $Q := R$.

begin
  while there are $s \in Q$ and $t \in {}^\bullet s$ such that $t \notin Q^\bullet$ **do**
    $Q\colon = Q \setminus \{s\}$
  endwhile
end

Show that the algorithm is correct by showing

(a) that the algorithm terminates, and

(b) that after termination, Q is the largest siphon contained in R.

Solution:

(a) In every iteration of the while loop, a place s is removed from Q. Q contains only finitely many places initially, therefore the while loop and the algorithm terminates.

(b) Let Q' be the largest siphon contained in R. First we show that $Q \subseteq Q'$. Let $s \in Q$. Then for all $t \in {}^\bullet s$, we have $t \in Q^\bullet$, therefore Q is a siphon. As Q' contains all siphons in R, $Q \subseteq Q'$.

Now let $Q_0, Q_1, \ldots, Q_n$ be the intermediate sets in the algorithm, with $Q_0 = R$ and $Q_n = Q$. We show that in each step i, we have $Q' \subseteq Q_i$.

Initially, with $i = 0$, we have $Q' \subseteq R = Q_0$. Now assume that $Q' \subseteq Q_i$ and we execute the body of the while loop in step i. Then there is $s \in Q_i$ and $t \in {}^\bullet s$ such that $t \notin {Q_i}^\bullet$. As $Q'^\bullet \subseteq {Q_i}^\bullet$, we also have $t \notin Q'^\bullet$ and therefore $s \notin Q'$. Thus $Q' \subseteq Q_{i+1} = Q_i \setminus \{s\}$.

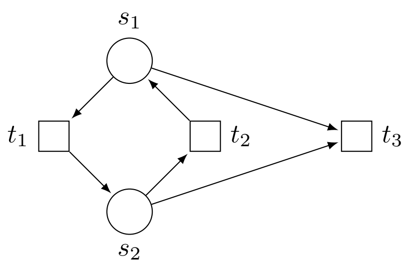
Exercise 6.3 Minimal siphons

(a) Exhibit a net having a minimal siphon R and a transition t such that $|{}^\bullet t \cap R| \geq 2$.

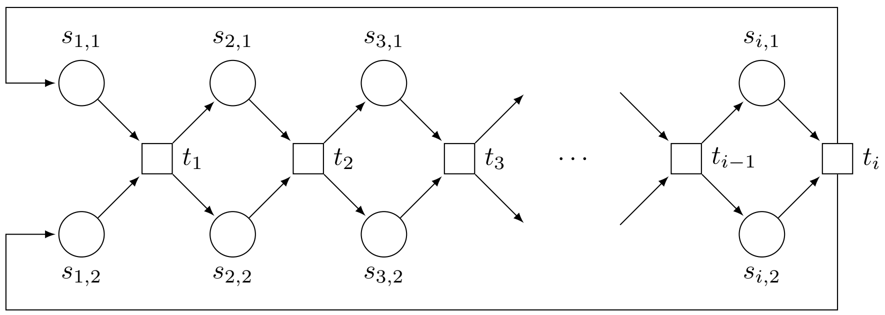
(b) Construct for each $i \in \mathbb{N}$ a net with at most $2i$ places and at least 2^i minimal siphons.

Solution:

(a) In the Petri net below, $R = \{s_1, s_2\}$ is a minimal siphon, as neither $\{s_1\}$ nor $\{s_2\}$ are a siphon on their own, and with $t = t_3$, we have $|{}^\bullet t_3 \cap \{s_1, s_2\}| = 2$.



(b) For a given i, the Petri net is indicated below. It has $2i$ places and for each $k \in \{1,2\}^i$, the set $R_k = \{s_{1,k_1}, s_{2,k_2}, \ldots, s_{i,k_i}\}$ is a siphon, as ${}^\bullet R_k = {R_k}^\bullet = T$, and is minimal, so there are 2^i minimal siphons.



Exercise 6.4 **Characterization of traps**

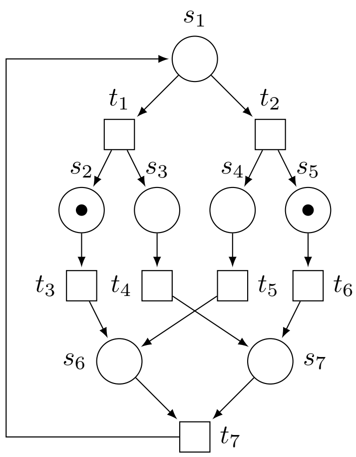
Show the following proposition, a characterization of traps by their fundamental property.

Proposition 6.4.1. Let N be a net and R a set of places of N. R is a trap of N *iff* for all markings M of N, if $M(R) > 0$, then $M'(R) > 0$ for all $M' \in [M\rangle$.

Solution: If R is a trap, the property follows from the fundamental property of traps. If R is not a trap, then there is a $s \in R$ and a $t \in s^\bullet$ such that $t \notin {}^\bullet R$. Define the marking M with $M(s) = 1$ if $s \in {}^\bullet t$ and $M(s) = 0$ otherwise. We have $M(R) > 0$. Further, t is a enabled at M, and we reach $M \xrightarrow{t} M'$ with $M'(R) = 0$, as no $s \in t^\bullet$ is in R. This proves the property.

Exercise 6.5 **Using traps to show non-reachability**

Consider the Petri net below. We want to show that M_0 is not reachable from some reachable marking M (thus showing that M_0 is not a home marking and the net is not cyclic).



(a) Find a trap R not marked at M_0.

(b) Find a marking M reachable from M_0 that marks R.

(c) Use R to construct a constraint over the markings reachable from M and show that M_0 is not reachable from M.

Solution:

(a) The trap $R = \{s_1, s_3, s_4, s_6, s_7\}$ is not marked at M_0.

(b) By firing t_3 at M_0, we reach the marking $M = (0, 0, 0, 0, 1, 1, 0)$ which marks s_6 and therefore R.

(c) As $M(R) > 0$, all markings $M' \in [M\rangle$ need to satisfy $M'(R) > 0$. Initially, we have $M_0(R) = 0$, so M_0 can not be reachable from M.

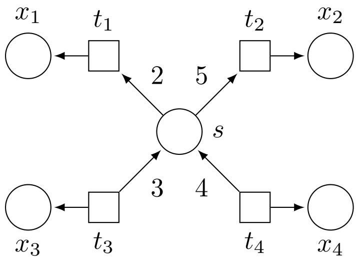
Exercise 6.6 **Linear inequation net**

Consider the following set, defined by a linear inequation.

$$X = \{(x_1, x_2, x_3, x_4) \in \mathbb{N}^4 \mid 2x_1 + 5x_2 \leq 3x_3 + 4x_4\}$$

Give a Petri net (N, M_0) (with or without weighted arcs) containing four designated places x_1, x_2, x_3 and x_4 (and possibly other places) such that $\{(M(x_1), M(x_2), M(x_3), M(x_4)) \mid M \in [M_0\rangle\} = X$, i.e., the reachable markings represent the set X.

Solution: The Petri net below has the places x_1, x_2, x_3 and x_4 for keeping track of the values and the place s, which keeps track of the difference $(3x_3 + 4x_4) - (2x_1 + 5x_2)$ between the right-hand side and the left-hand side of the inequation. The transitions t_1 to t_4 increase each x_i by one and add or remove the appropriate amount to s, depending on whether x_i appears on the left-hand side or right-hand side of the inequation.



x_1
t_1
t_2
x_2
2
5
s
3
4
x_3
t_3
t_4
x_4