Solution

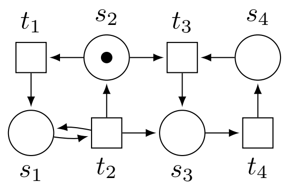
Petri nets – Homework 5

Discussed on Thursday 11th June, 2015.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 5.1 Marking equation

(a) Construct the incidence matrix $\mathbf{N}$ of the following Petri net:



(b) Use the marking equation to decide whether the following markings are not reachable, or may be reachable. Does it make a difference whether the solution space is restricted to the natural numbers or to the rationals?

$$M_1 = (0,0,0,0) \qquad M_2 = (1,1,1,1) \qquad M_3 = (1,4,1,1)$$

Solution:

(a) The following is the incidence matrix of the Petri net:

$$\mathbf{N} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

(b) For each M, we need to solve the marking equation $M = M_0 + \mathbf{N} \cdot X$ for X. We can solve the system of linear equations $\mathbf{N} \cdot X = M - M_0$ simultaneously for M_1, M_2 and M_3:

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right)$$

We obtain the (in this case unique) solutions $X_1 = (0,0,1,1)$, $X_2 = (1,2,1,2)$ and $X_3 = (1,2,-2,-1)$. When looking at solutions over the rationals, we can not make any statement about the reachability of the markings. However, X_3 is no solution over the natural numbers, and there is no other solution, so we can conclude that M_3 is not reachable. M_1 and M_2 may or may not be reachable (in fact, M_1 is not reachable, but M_2 is reachable).

Exercise 5.2 Marking equation in acyclic nets

Show the following: In a Petri net (N, M_0) which is structurally acyclic (there is no directed cycle in the net N), a marking M is reachable from M_0 *iff* there exists a nonnegative integer solution X satisfying the marking equation $M = M_0 + \mathbf{N} \cdot X$

Solution: Necessity follows directly from the marking equation lemma, only sufficiency remains to be shown.

We show: For any two markings M_s and M_d and $X \geq 0$, if $M_d = M_s + \mathbf{N} \cdot X$, then M_d is reachable from M_s. We show this by induction on $n := \sum_{t \in T} X(t)$.

Induction base: $n = 0$. Then $X = 0$ and $M_d = M_s$.

Induction hypothesis: Let $n > 0$ and assume that for all M'_d, M'_s and $X' \geq 0$ with $\sum_{t \in T} X'(t) < n$ and $M'_d = M'_s + \mathbf{N} \cdot X'$, M'_d is reachable from M'_s.

As $n > 0$, there is a t with $X(t) > 0$. The net is acyclic, so from all t with $X(t) > 0$, let this t be one which is minimal with regard to the topological order between them, i.e., there is no t' with $X(t') > 0$ such that there is a path from t' to t. Define $Y \in \mathbb{N}^{|T|}$ with $Y(t) := X(t) - 1$ and $Y(u) := X(u)$ if $u \neq t$. We have $X = Y + \vec{t}$ and so

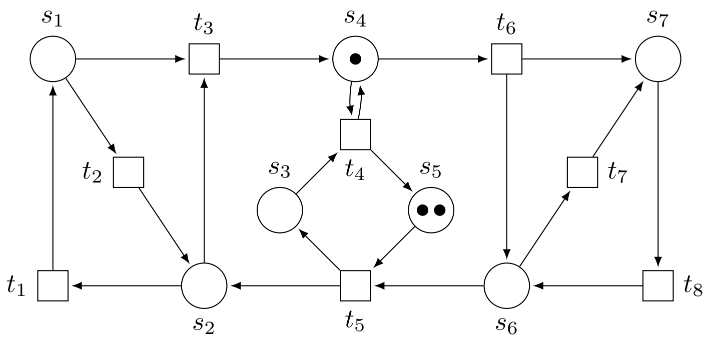
$$M_d = M_s + \mathbf{N} \cdot X = M_s + \mathbf{N} \cdot (Y + \vec{t}) = M_s + \mathbf{N} \cdot Y + \mathbf{N} \cdot \vec{t} \geq 0$$

For $s \in {}^\bullet t$ we have $s \notin t^\bullet$ due to acyclicity, so $(\mathbf{N} \cdot \vec{t})(s) = \mathbf{N}(s,t) = -1$. None of the transitions in $\langle Y \rangle \subseteq \langle X \rangle$ put tokens in s, so $(\mathbf{N} \cdot Y)(s) \leq 0$. With that we get $M_s(s) \geq 1$, so t is enabled at M_s. With $M_s \xrightarrow{t} M'_s$ we have $M'_s = M_s + \mathbf{N} \cdot \vec{t}$ and $M_d = M'_s + \mathbf{N} \cdot Y$. We can apply the induction hypothesis to M_d, M'_s and Y to obtain that M_d is reachable from M'_s. By extension, M_d is also reachable from M_s.

Exercise 5.3 S-invariants and T-invariants

Give a basis of the space of S-invariants and a basis of the space of T-invariants of the following net. Does the net have positive S-invariants and T-invariants? Can you make any statements about the boundedness and liveness of the net based on the invariants?

Hint: Use the alternative definitions for S-invariants and T-invariants to find them more easily.



Solution:

A vector I is an S-invariant if $I \cdot \mathbf{N} = 0$ or if $\forall t \in T : \sum_{s \in {}^\bullet t} I(s) = \sum_{s \in t^\bullet} I(s)$. By the second definition, we obtain the following equations for an S-invariant $I = (s_1, s_2, s_3, s_4, s_5, s_6, s_7)$:

$$\begin{aligned}
s_2 &= s_1 \\
s_1 &= s_2 \\
s_1 + s_2 &= s_4 \\
s_3 + s_4 &= s_4 + s_5 \\
s_5 + s_6 &= s_2 + s_3 \\
s_4 &= s_6 + s_7 \\
s_6 &= s_7 \\
s_7 &= s_6
\end{aligned}$$

These can be simplified and reduced to following equivalent set of equations:

$$\begin{aligned}
s_1 &= s_2 = s_6 = s_7 \\
s_4 &= 2s_1 \\
s_3 &= s_5
\end{aligned}$$

By specifying s_1 and s_3, the S-invariant is completely defined. As a basis, we obtain the following two S-invariants:

$$\begin{aligned}
I_1 &= (1, 1, 0, 2, 0, 1, 1) \\
I_2 &= (0, 0, 1, 0, 1, 0, 0)
\end{aligned}$$

A vector J is a T-invariant if $\mathbf{N} \cdot J = 0$ or if $\forall s \in S : \sum_{t \in {}^\bullet s} J(t) = \sum_{t \in s^\bullet} J(t)$ or $J = \vec{\sigma}$ for some occurrence sequence σ and marking M with $M \xrightarrow{\sigma} M$ (fundamental property of T-invariants). With the third definition, we can identify the minimal sequences which return a marking to itself, which are $\sigma_1 = t_1t_2$, $\sigma_2 = t_7t_8$ and $\sigma_3 = t_1t_3t_4t_4t_5t_5t_6t_8$. This gives us the following three T-invariants $J = (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)$ as a basis:

$$J_1 = (1, 1, 0, 0, 0, 0, 0, 0)$$
$$J_2 = (0, 0, 0, 0, 0, 0, 1, 1)$$
$$J_3 = (1, 0, 1, 2, 2, 1, 0, 1)$$

The S-invariant $I_1 + I_2$ and the T-invariant $J_1 + J_2 + J_3$ are positive invariants. As there is a positive S-invariant, we can conclude that the net is bounded. We can not make a statement about the liveness of the net, as having positive or certain semi-positive invariants are only necessary conditions for liveness. In fact, this net is not live, as firing $t_6t_8t_5t_5t_1t_3t_6$ leads to a marking from which t_5 is never enabled again.

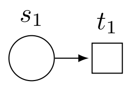
Exercise 5.4 Bounded net with no positive S-invariant

(a) Exhibit a Petri net (N, M_0) which is bounded, but has no positive S-invariant.

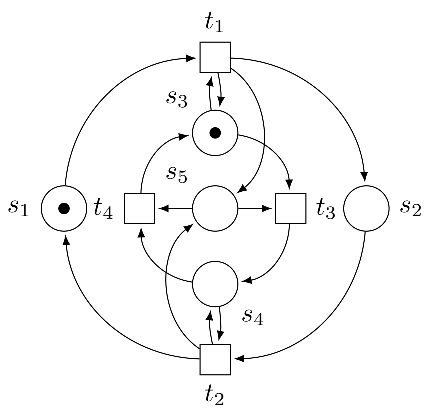
(b) As (a), but (N, M_0) is required to be live and bounded.

Solution:

(a) The following net is bounded, but all S-invariants I need to satisfy $I(s_1) = 0$, so there is no positive S-invariant.



(b) The following net, known from exercise 2.3(c), is live and bounded.



Any S-invariant I would need to satisfy

$$I(s_1) = I(s_2) + I(s_5)$$
$$I(s_2) = I(s_1) + I(s_5)$$
$$I(s_3) = I(s_4) + I(s_5)$$
$$I(s_4) = I(s_3) + I(s_5)$$

and therefore $I(s_5) = 0$, so there is no positive S-invariant.

Exercise 5.5 Reproduction lemma

Let (N, M_0) be a bounded system and let $M_0 \xrightarrow{\sigma}$ be an infinite occurrence sequence. Show the following:

(a) There exists sequences σ_1, σ_2, σ_3 such that $\sigma = \sigma_1\sigma_2\sigma_3$, σ_2 is not the empty sequence and

$$M_0 \xrightarrow{\sigma_1} M \xrightarrow{\sigma_2} M \xrightarrow{\sigma_3}$$

for some marking M.

(b) There exists a semi-positive T-invariant J such that $\langle J \rangle \subseteq \mathcal{A}(\sigma)$, where $\mathcal{A}(\sigma)$ is the set of transitions appearing in σ.

Solution:

(a) Assume $\sigma = t_1 t_2 t_3 \ldots$. Define $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \ldots$. By boundedness, the markings $M_0, M_1, M_2, \ldots$ cannot be pairwise different. Suppose $M = M_i = M_j$ for two indices i, j, $0 \leq i < j$. Define $\sigma_1 = t_1 \ldots t_i$, $\sigma_2 = t_{i+1} \ldots t_j$ and $\sigma_3 = t_{j+1} t_{j+2} \ldots$. The sequence σ_2 is not empty because $i < j$.

(b) Take, with the notions of (a), $J = \vec{\sigma_2}$. The result then follows from the fundamental property of T-invariants because $M \xrightarrow{\sigma_2} M$.