Solution

Petri nets – Homework 4

Discussed on Thursday 28th May, 2015.

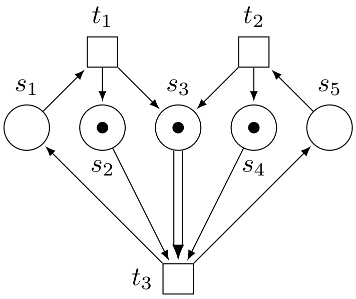
For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 4.1 Coverability with transfer arcs

In exercise 3.4, we defined Petri nets with transfer arcs and showed that coverability is decidable with the backwards reachability algorithm. However, this does not work with the coverability graph algorithm, as spurios ω may be introduced.

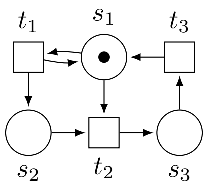
Give a Petri net with transfer arcs where there are occurrence sequences $M_0 \xrightarrow{\sigma_0} M_1 \xrightarrow{\sigma_1} M_2$ such that $M_2 \geq M_1$ and $M_2 \neq M_1$ (thus introducing an ω in the coverability graph), but the net is bounded.

Solution: Take the following net, where the arc from s_3 to t_3 is a transfer arc. In the occurrence sequence $M_0 = (0,1,1,1,0) \xrightarrow{t3} (1,0,0,0,1) \xrightarrow{t1} (0,1,1,0,1) \xrightarrow{t2} (0,1,2,1,0)$, the marking $(0,1,2,1,0)$ strictly covers $(0,1,1,1,0)$, however we cannot further increase the number of tokens in s_3, as firing t_3 removes all tokens in s_3.



Exercise 4.2 Backwards reachability algorithm

Apply the backwards reachability algorithm to the Petri net below to decide if the marking $M = (0,0,2)$ can be covered. Record all intermediate sets of markings with their finite representation of minimal elements.



Solution: We start with $m_0 = \{(0,0,2)\}$ and compute the predecessors for $(0,0,2)$ for each transition t, that is, the minimal marking M such that $M \xrightarrow{t} M'$ with $M' \geq (0,0,2)$.

$$pre((0,0,2),t_1) = (1,0,2)$$
$$pre((0,0,2),t_2) = (1,1,1)$$
$$pre((0,0,2),t_3) = (0,0,3)$$

After adding the new markings to m_0 and eliminating non-minimal markings, our new set is $m_1 = \{(0,0,2),(1,1,1)\}$. For the new marking $(1,1,1)$, we compute the predecessors:

$$pre((1,1,1),t_1) = (1,0,1)$$
$$pre((1,1,1),t_2) = (2,2,0)$$
$$pre((1,1,1),t_3) = (0,1,2)$$

We add the new markings, take the minimal elements and obtain $m_2 = \{(0,0,2),(1,0,1),(2,2,0)\}$. For $(1,0,1)$ and $(1,2,0)$, we compute the predecessors:

$$
\begin{aligned}
pre((1,0,1),t_1) &= (1,0,1)\\
pre((1,0,1),t_2) &= (2,1,0)\\
pre((1,0,1),t_3) &= (0,0,2)\\
pre((2,2,0),t_1) &= (2,1,0)\\
pre((2,2,0),t_2) &= (3,3,0)\\
pre((2,2,0),t_3) &= (1,2,1)
\end{aligned}
$$

The new minimal marking set is now $m_3 = \{(0,0,2),(1,0,1),(2,1,0)\}$. Next we compute the predecessors for $(2,1,0)$:

$$
\begin{aligned}
pre((2,1,0),t_1) &= (2,0,0)\\
pre((2,1,0),t_2) &= (3,2,0)\\
pre((2,1,0),t_3) &= (1,1,1)
\end{aligned}
$$

The new set is $m_4 = \{(0,0,2),(1,0,1),(2,0,0)\}$. Finally we compute the predecessors for $(2,0,0)$:

$$
\begin{aligned}
pre((2,0,0),t_1) &= (2,0,0)\\
pre((2,0,0),t_2) &= (3,1,0)\\
pre((2,0,0),t_3) &= (1,0,1)
\end{aligned}
$$

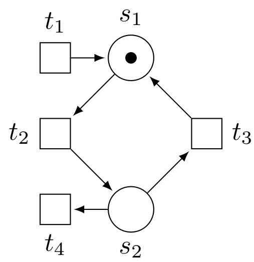
After that, we do not obtain any new minimal markings, so we have reached a fixpoint. As for no marking $M' \in m_4$, we have $M_0 \geq M'$, we can conclude that $M = (0,0,2)$ is not coverable.

Exercise 4.3 Reduction of reachability problems

A variant of the reachability problem is the *zero-reachability problem*:

Definition 4.3.1 (Zero-reachability problem)**.** For a Petri net (N, M_0), is there a marking $M \in [M_0\rangle$ with $M(s) = 0$ for all $s \in S$?

Let $\mathbf{0}$ be the marking M with $M(s) = 0$ for all $s \in S$. A reduction from the zero-reachability problem to the reachability problem is straightforward: simpliy specify the target marking $\mathbf{0}$. For the other way, reduce the reachability problem to the zero-reachability problem. Describe an algorithm that, given a Petri net (N, M_0) and a marking M, constructs in polynomial time a Petri net (N', M_0') such that M is reachable from M_0 in N if and only if $\mathbf{0}$ is reachable from M_0' in N'. Apply the algorithm to the Petri net below with the marking $M = (0,2)$.



Solution:

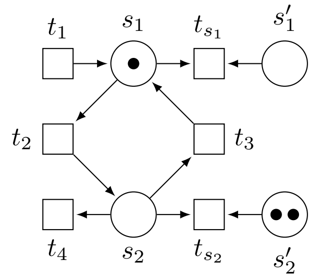
Create a disjunct copy of the places S in the original net. For each place $s \in S$, add a transition t_s with arcs (s, t_s) and (s', t_s), where s' is the copy of s. The initial marking of each place s' in the copy is equal to the marking of the original place in the target marking M. The other places keep their initial marking. Formally, the net (N', M_0') is defined as:

$$
\begin{aligned}
S' &= S \uplus \{s' \mid s \in S\}\\
T' &= T \cup \{t_s \mid s \in S\}\\
F' &= F \cup \{(s, t_s), (s', t_s) \mid s \in S\}\\
M_0'(s) &= M_0(s) \text{ for } s \in S\\
M_0'(s') &= M(s) \ \text{ for } s' \in S' \setminus S
\end{aligned}
$$

Now, if M is reachable in the net N, then, with the same occurrence sequence, we can also reach a marking M' in N' with $M'(s) = M(s) = M'(s')$ for all $s \in S$. Then for each $s \in S$, we can fire t_s exactly $M(s)$ times to remove all the tokens in s and s'.

If, on the other hand, the empty marking is reachable in N' with the occurrence sequence $M'_0 \xrightarrow{\sigma} \mathbf{0}$, then then we had to fire each t_s exactly $M(s)$ times. With the exchange lemma, we can move the t_s to the end of σ and obtain sequences $M'_0 \xrightarrow{\sigma_1} M' \xrightarrow{\sigma_2} \mathbf{0}$ such that all transitions occuring in σ_1 are in T. We have $M'(s) = M(s)$ for all $s \in S$ and we can fire σ_1 in N to reach M.

For the above Petri net, applying the reduction results in the following net:

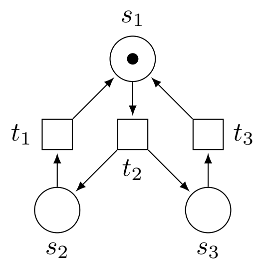


Exercise 4.4 Semilinear sets

(a) Show that the following set is semilinear by giving a finite set of pairs of roots and periods $\{(r_1, P_1), \dots, (r_n, P_n)\}$ representing the linear sets.

$$X = \{(x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_1 \le x_2 + 1 \le x_3\}$$

(b) Show that the set of reachable markings of the following Petri net is semilinear.



(c) Use the representation of the reachable markings of the previous Petri net as a semilinear set to show that the marking $M = (0, 0, 1)$ is not reachable.

Solution:

(a) Take the linear sets Y_1 given by $r_1 = (0, 0, 1)$ and $P_1 = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ and Y_2 given by $r_2 = (1, 0, 1)$ and $P_1 = P_2$ and the semilinear set $Y = Y_1 \cup Y_2$. Clearly, $Y \subseteq X$. Now take $x = (x_1, x_2, x_3) \in X$. If $x_1 \le x_2$, then $x = (0, 0, 1) + x_1(1, 1, 1) + (x_2 - x_1)(0, 1, 1) + (x_3 - x_2 - 1)(0, 0, 1)$, so $x \in Y_1$. If $x_1 = x_2 + 1$, then $x = (1, 0, 1) + (x_1 - 1)(1, 1, 1) + (x_2 - x_1 + 1)(0, 1, 1) + (x_3 - x_2 - 1)(0, 0, 1)$, so $x \in Y_2$. Therefore $X = Y$.

(b) Initially, we can only fire $(1, 0, 0) \xrightarrow{t_2} (0, 1, 1)$. From $(1, 0, 0)$ or $(0, 1, 1)$, we can add any number of tokens to each place., as indicated by the following occurrence sequences:

$$(1,0,0) \xrightarrow{(t_2 t_3 t_1)^{x_1}} (1 + x_1, 0, 0) \qquad (0,1,1) \xrightarrow{(t_3 t_1 t_2)^{x_1}} (x_1, 1, 1)$$
$$(1,0,0) \xrightarrow{(t_2 t_3)^{x_2}} (1, x_2, 0) \qquad (0,1,1) \xrightarrow{(t_3 t_2)^{x_2}} (0, 1 + x_2, 1)$$
$$(1,0,0) \xrightarrow{(t_2 t_1)^{x_3}} (1, 0, x_3) \qquad (0,1,1) \xrightarrow{(t_1 t_2)^{x_3}} (0, 1, 1 + x_3)$$

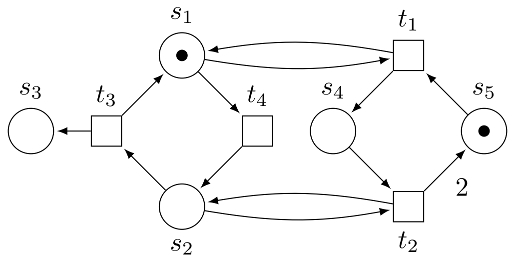
We define the linear sets $\mathcal{M}_1$ by $r_1 = (1, 0, 0)$ and $P_1 = \{(1, 0, 0), (0, 1, 0,), (0, 0, 1)\}$ and $\mathcal{M}_2$ by $r_2 = (0, 1, 1)$ and $P_2 = P_1$ as well as the semilinear set $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. As seen above, if $M \in \mathcal{M}$, then $M \in [M_0\rangle$. Next we show that if $M \in [M_0\rangle$, then $M \in \mathcal{M}$. We see that $M_0 \in \mathcal{M}$. Now assume $M \in \mathcal{M}$ and $M \xrightarrow{t} M'$ for some $t \in T$. We show that $M' \in \mathcal{M}$. If $M = (x_1, x_2, x_3) \xrightarrow{t_2} (x_1 - 1, x_2 + 1, x_3 + 1) = M'$, then $M' \in \mathcal{M}_2$. If $M = (x_1, x_2, x_3) \xrightarrow{t_1} (x_1 + 1, x_2 - 1, x_3) = M'$ or $M = (x_1, x_2, x_3) \xrightarrow{t_3} (x_1 + 1, x_2, x_3 - 1) = M'$, then $M' \in \mathcal{M}_1$. Therefore $\mathcal{M} = [M_0\rangle$.

(c) There is no solution for either of the following equations, therefore $(0, 0, 1)$ is not reachable.

$$(0,0,1) = (1,0,0) + \lambda_1(1,0,0) + \lambda_2(0,1,0) + \lambda_3(0,0,1) \qquad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{N}$$
$$(0,0,1) = (0,1,1) + \mu_1(1,0,0) + \mu_2(0,1,0) + \mu_3(0,0,1) \qquad \mu_1, \mu_2, \mu_3 \in \mathbb{N}$$

Exercise 4.5 **Petri net with a non-semilinear reachability set**

We want to show that the following Petri net with weighted arcs has a non-semilinear reachability set (we can also obtain the result with unweighted arcs by the standard reduction).



Consider the following sets of markings, given as $M = (s_1, s_2, s_3, s_4, s_5)$:

$$\mathcal{M}_1 = \{(1, 0, x_1, x_2, x_3) \mid 0 < x_2 + x_3 \le 2^{x_1}\}$$
$$\mathcal{M}_2 = \{(0, 1, x_1, x_2, x_3) \mid 0 < 2x_2 + x_3 \le 2^{x_1+1}\}$$
$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$$

Clearly, $\mathcal{M}$ is a non-semilinear set. We claim: $\mathcal{M}$ is equal to the set of reachable markings for the above Petri net.

(a) Show that if $M \in [M_0\rangle$, then $M \in \mathcal{M}$. For this, show that $M_0 \in \mathcal{M}$ and if $M \in \mathcal{M}$ and $M \xrightarrow{t} M'$ for some transition t, then also $M' \in \mathcal{M}$.

(b) Show that if $M \in \mathcal{M}$, then $M \in [M_0\rangle$.

Note: This is a rather hard exercise. *Hint*: Do this by induction on $x_1 = M(s_3)$ for $M \in \mathcal{M}$. In the induction step at x_1, do a case distinction between $M \in \mathcal{M}_1$ and $M \in \mathcal{M}_2$. In each case, find an M' for which you can apply the induction hypothesis and from which M is reachable.

Solution:

(a) We have $M_0 \in \mathcal{M}_1$. Now assume $M \in \mathcal{M}$ and $M \xrightarrow{t} M'$ for some transitions t. We show that $M' \in \mathcal{M}$.

- $M \xrightarrow{t_1} M'$: Then $M = (1, 0, x_1, x_2, x_3)$ with $x_3 \ge 1$ and $M' = (1, 0, x_1, x_2+1, x_3-1)$. We have $0 < x_2+1+x_3-1 = x_2 + x_3 \le 2^{x_1}$, therefore $M' \in \mathcal{M}_1$.
- $M \xrightarrow{t_2} M'$: Then $M = (0, 1, x_1, x_2, x_3)$ with $x_2 \ge 1$ and $M' = (0, 1, x_1, x_2-1, x_3+2)$. We have $0 < 2(x_2-1)+x_3+2 = 2x_2 + x_3 \le 2x_2 + x_3 \le 2^{x_1+1}$, therefore $M' \in \mathcal{M}_2$.
- $M \xrightarrow{t_3} M'$: Then $M = (0, 1, x_1, x_2, x_3)$ and $M' = (1, 0, x_1 + 1, x_2, x_3)$. We have $0 < 2x_2 + x_3 \le 2^{x_1+1}$ and so $0 < x_2 + x_3 \le 2^{x_1+1}$, therefore $M' \in \mathcal{M}_1$.
- $M \xrightarrow{t_4} M'$: Then $M = (1, 0, x_1, x_2, x_3)$ and $M' = (0, 1, x_1, x_2, x_3)$. We have $0 < x_2+x_3 \le 2^{x_1}$ and so $0 < 2x_2+x_3 \le 2^{x_1+1}$, therefore $M' \in \mathcal{M}_2$.

(b) We show for all M, if $M \in \mathcal{M}$, then $M \in [M_0\rangle$, by induction on $x_1 = M(s_3)$.

Induction base: $x_1 = 0$. Then M is one of the following and can be reached from M_0:

- $M = (1, 0, 0, 0, 1)$: $M_0 \xrightarrow{\epsilon} M$.
- $M = (1, 0, 0, 1, 0)$: $M_0 \xrightarrow{t_1} M$.
- $M = (0, 1, 0, 0, 1)$: $M_0 \xrightarrow{t_4} M$.
- $M = (0, 1, 0, 1, 0)$: $M_0 \xrightarrow{t_1 t_4} M$.
- $M = (0, 1, 0, 0, 2)$: $M_0 \xrightarrow{t_1 t_4 t_2} M$.

Induction hypothesis: Let $x_1 > 0$ and assume that for all M' with $M'(s_3) < x_1$, if $M' \in \mathcal{M}$, then $M' \in [M_0\rangle$.

- *Case 1*: $M \in \mathcal{M}_1$. Then $M = (1, 0, x_1, x_2, x_3)$ with $0 < x_2 + x_3 \le 2^{x_1}$. With $M' := (0, 1, x_1 - 1, 0, x_2 + x_3)$, we have $M' \in \mathcal{M}_2$, so it is reachable by the induction hypothesis. M is then reachable from M' with $\sigma = t_3 t_1^{x_2}$.
- *Case 2*: $M \in \mathcal{M}_2$. Then $M = (0, 1, x_1, x_2, x_3)$ with $0 < 2x_2+x_3 \le 2^{x_1+1}$. With $M' := (0, 1, x_1-1, 0, x_2+x_3 \text{ div } 2+ x_3 \bmod 2)$, we have $M' \in \mathcal{M}_2$, so it is reachable by the induction hypothesis. M is then reachable from M' with $\sigma = t_3 t_1^{(x_2+x_3 \text{ div } 2)} t_4 t_2^{(x_3 \text{ div } 2)}$, as $x_3 = 2(x_3 \text{ div } 2) + x_3 \bmod 2$.