Solution

Petri nets – Homework 2

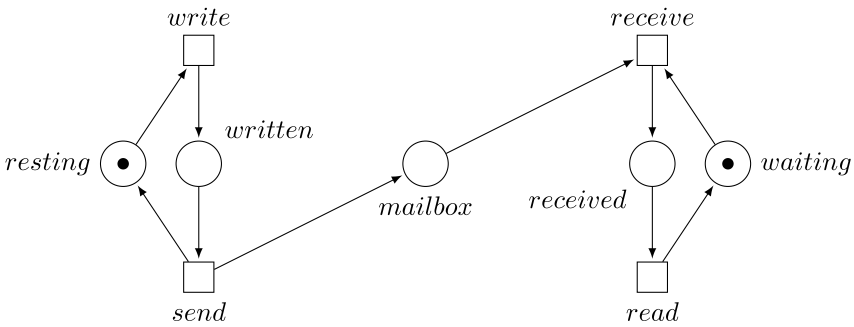
Discussed on Wednesday 13th May, 2015.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 2.1 Bounded mailbox

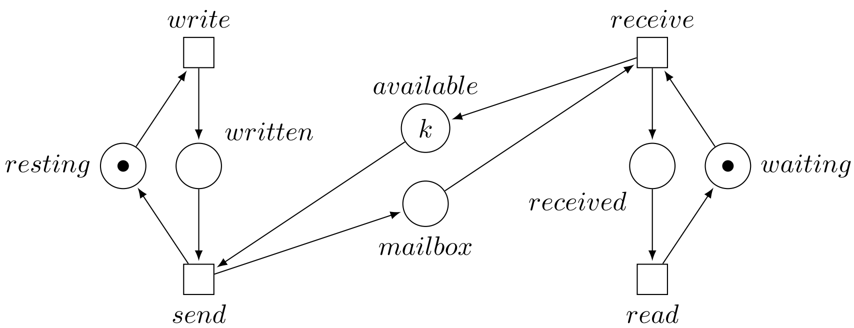
The following Petri net models a communication of two processes by sending mails to a mailbox and receiving them from the mailbox. This is an instance of the producer/consumer problem.

Currently the mailbox is unbounded. Show how, for a given k, the net can be modified to enforce a maximum capacity of k on the mailbox, that is, the place *mailbox* should be k-bounded in the modified net. Try to minimize the modifications.



Solution:

An additional place, *available*, with an initial marking of k tokens, represents the remaining capacity of the mailbox. For the transition $send \in {}^\bullet mailbox \setminus mailbox^\bullet$, add an arc $(available, send)$ and for the transition $receive \in mailbox^\bullet \setminus {}^\bullet mailbox$, add an arc $(receive, available)$. This ensures that the sum of tokens in *available* and *mailbox* is always equal k and there are never more than k tokens in *mailbox*.



Exercise 2.2 A vending machine

We would like to model a simple vending machine with a Petri net. The vending machine accepts 1 euro coins and dispenses chocolate bars for 1 euro each. After inserting 1 euro, the machine should eventually dispense a chocolate bar. A chocolate bar should only be dispensed after inserting one euro, and one euro is only valid for one chocolate bar.

(a) Model the vending machine as a Petri net. It should have transitions for inserting a 1 euro coin and for dispensing a chocolate bar.

(b) Modify the Petri net so that the vending machine has a limited amount of storage for chocolate bars, and initially contains 4 bars. The net should have a transition for refilling the storage, however not above the maximum capacity of 4 bars.

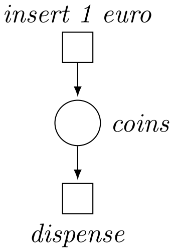
(c) Now the vending machine is upgraded to offer both small chocolate bars for 1 euro and large chocolate bars for 2 euros. Modify the Petri net such that it additionally allows for dispension of large chocolate bars after inserting at least 2 euros. The large chocolate bars are in a separate storage space, which should be limited to 3 bars and also be able to be refilled.

Note: You may use Petri nets with weighted arcs. However, also think about a solution without weights. For this, you may have more than one transition corresponding to the insertion of a coin or to the dispension of a chocolate bar.

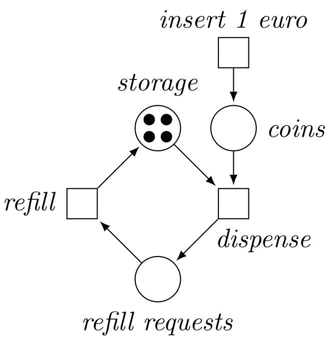
(d) Further modify the Petri net so that the vending machine also accepts 2 euro coins. A 2 euro coin should allow dispension of either two small chocolate bars or one large chocolate bar.

Solution:

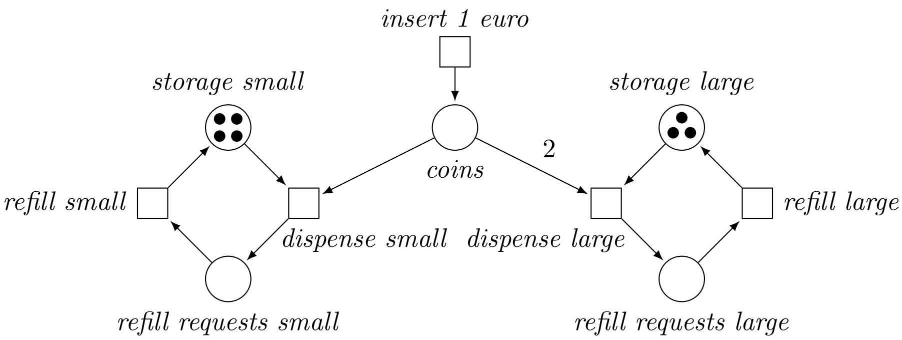
(a) The vending machine can be modeled by a Petri net having a place *coins* for accumulating coins, a transition *insert 1 euro* for inserting a 1 euro coin by putting a token into the place *coins* and a transition *dispense* for dispensing a chocolate bar by removing a token from the place *coins*.



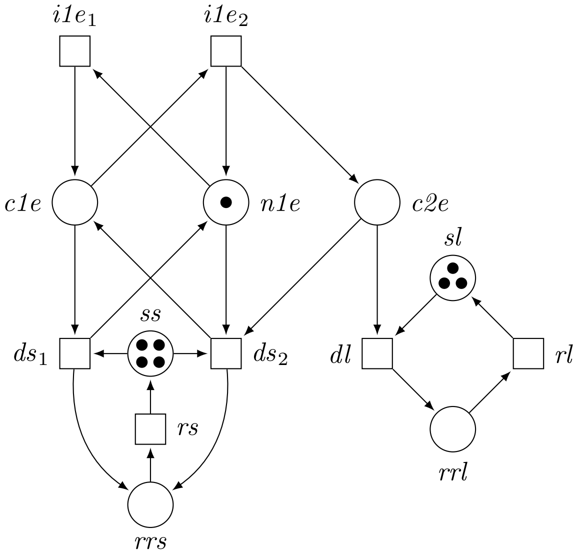
(b) By adding a place *storage* with 4 tokens initially and having *dispense* remove a token from *storage*, at most 4 chocolate bars can be dispensed without refilling. Each dispensed bar creates a token in a place *refill requests*, which allows refilling the storage with the transition *refill*. This ensures that there are never more than 4 tokens in the place *storage*.



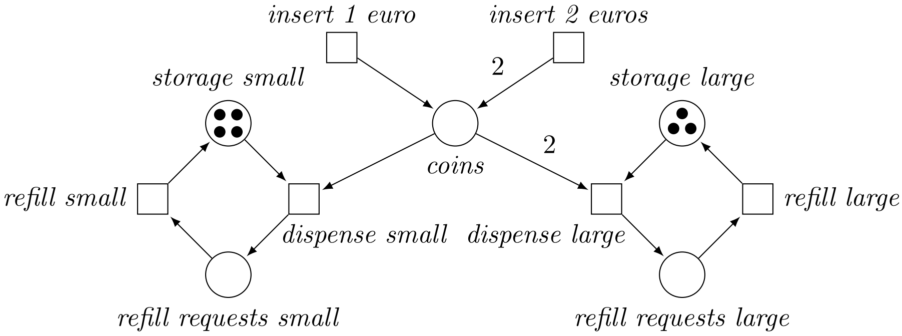
(c) When using arcs with weights, we can duplicate the part for dispensing, storing and refilling small chocolate bars for large chocolate bars, except that the transition *dispense large* for dispensing a large chocolate bar takes two tokens from the place *coins*, corresponding to using 2 euros. The storage for the large bars is limited to 3 bars by having 3 tokens initially in the place *storage large*.



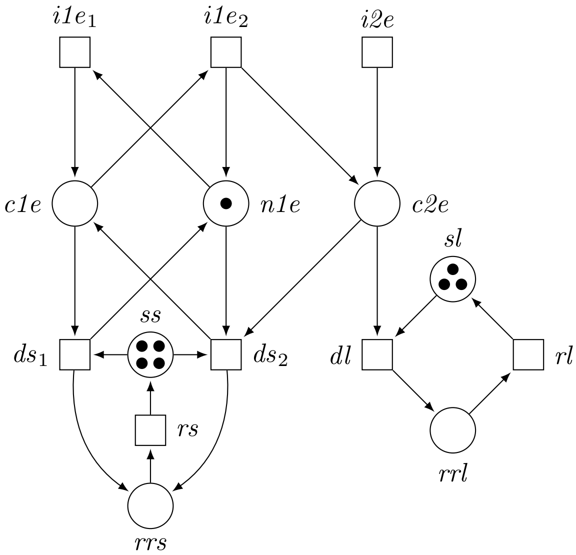
For the alternative solution without weights, we use two places for accumulating the coins. The place *c1e* represents 1 euro coins and the place *c2e* represents 2 euro coins which were "upgraded" from two 1 euro coins. The place *n1e* is marked iff there is no token in the place *c1e*. A 1 euro coin can be inserted with the transitions $i1e_1$ or $i1e_2$ by either adding a token to the place *c1e*, if there is none, or by taking the token from *c1e* and creating a token in *c2e*. There are two transitions for dispensing a small chocolate bar, ds_1 and ds_2, either by taking 1 euro from *c1e* or 2 euros from *c2e* and returning 1 euro in *c1e*. The large chocolate bar can simply be dispensed with *dl* by taking a token from *c2e*. The storage constraints are modelled as in the previous solution.



(d) With weights, we can add a transition *insert 2 euros* that models insertion of a 2 euro coin by adding 2 tokens to the place *coins*.



Without weights, the transition *i2e* models insertion of a 2 euro coin by directly adding a token to the place *c2e*.



Exercise 2.3 Monotonicity of properties

Exhibit counterexamples that disprove the following conjectures:

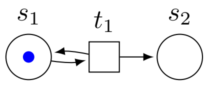
(a) If (N, M_0) is bounded and $M \geq M_0$, then (N, M) is bounded.

(b) If (N, M_0) is live and $M \geq M_0$, then (N, M) is live.

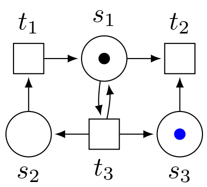
(c) If (N, M_0) is live and bounded and $M \geq M_0$, then (N, M) is bounded.

Solution:

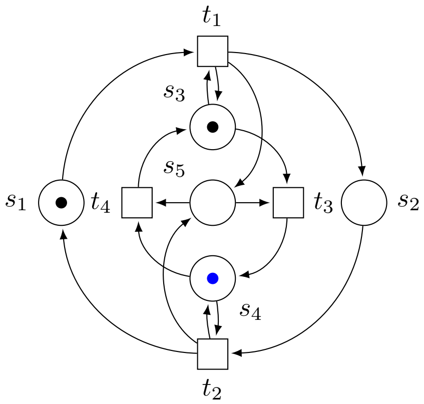
(a) The following Petri net is bounded without tokens, but not bounded with the blue token in s_1, as repeatedly firing t_1 can put an arbitrary number of tokens in s_2.



(b) The following Petri net is live with the black tokens, but not live with the additional blue token in s_3, as firing t_2 leads to a dead marking.



(c) The following Petri net is live and bounded with the black tokens, but not bounded with the additional blue token in s_4, as repeatedly firing t_1t_2 can put an arbitrary number of tokens in s_5.



Exercise 2.4 Home markings

Definition 2.4.1 (Home marking)**.** Let (N, M_0) be a Petri net. A marking M of the net N is a *home marking* of (N, M_0) if it is reachable from every marking of $[M_0\rangle$.

We say that (N, M_0) has a home marking if some reachable marking is a home marking.

(a) Does the Petri net from exercise 1.2 have a home marking?

(b) Find all home markings for the three Petri nets A, B and C from exercise 1.4.

(c) Exhibit a Petri net (N, M_0) which has home markings, but also an infinite occurrence sequence $M_0 \xrightarrow{\sigma}$ such that none of the markings along the occurrence of σ is a home marking.

(d) Show that every reachable marking of a cyclic Petri net is a home marking.

(e) Prove that every bounded Petri net (N, M_0) has a reachable marking M which is a home marking of (N, M).

Solution:

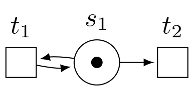
(a) No, as the markings $(0, 2, 0)$ and $(0, 0, 0)$ are both reachable markings in different bottom SCCs, therefore there is no marking reachable from both of these markings.

(b) Petri net A has $(0, 1, 0, 0, 2)$ as its only home marking.

Every reachable marking in Petri net B is a home marking.

Petri net C has $(1, 0, 1, 0)$, $(0, 0, 2, 1)$ and $(0, 1, 2, 0)$ as home markings.

(c) The following Petri net has (0) as its only home marking, but the only marking reached along the infinite sequence $M_0 \xrightarrow{t_1t_1t_1\cdots}$ is $M_0 = (1)$.



(d) Let (N, M_0) be a Petri net and M a reachable marking. We want to show: M is a home marking.

Let M' be a reachable marking. As the Petri net is cyclic, there is a firing sequence $M' \xrightarrow{\sigma_1} M_0$. As M is reachable, there is a firing sequence $M_0 \xrightarrow{\sigma_2} M$. Therefore there is a firing sequence $M' \xrightarrow{\sigma_1\sigma_2} M$.

(e) Let G be the reachability graph of the Petri net (N, M_0). As the Petri net is bounded, G is finite. Let G' be a bottom SCC of G and M be a marking in G'.

The reachability graph of (N, M) is the subgraph of G starting at M and therefore equal to G'. G' is a single SCC, therefore every marking in G' is a home marking of (N, M), including M.

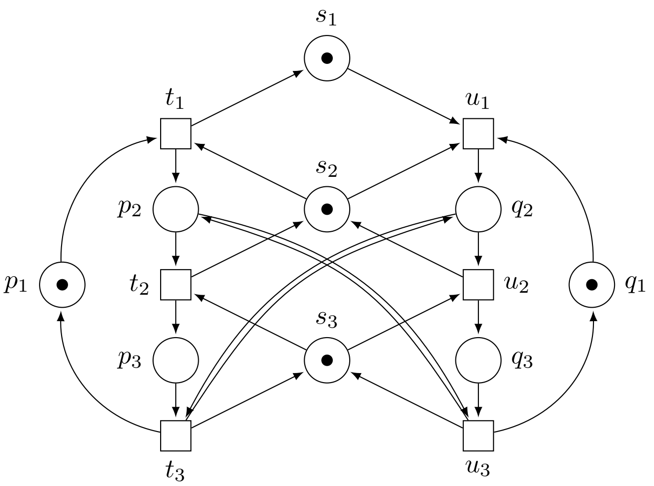
Exercise 2.5* **Live and bounded Petri net without home markings**

Note: This is a bonus exercise, as it is rather challenging. Only do it if you are interested and have enough time.

Exhibit a live and bounded Petri net without home markings.

Solution:

The following Petri net is live, bounded and has no home markings:



This can be verified with the following reachability graph, with the markings given as $M = (p_1, p_2, p_3 \mid q_1, q_2, q_3 \mid s_1, s_2, s_3)$: