Technische Universität München (I7)
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Thursday 16th April, 2015

Solution

Petri nets – Homework 1

Discussed on Wednesday 22nd and Thursday 23rd April, 2015.

For questions regarding the exercises, please send an email to meyerphi@in.tum.de or just drop by at room 03.11.042.

Exercise 1.1 Milner's scheduler

We want to specify a simple scheduler for a set of n agents $P_1, \dots, P_n$. Each agent P_i performs a task repeatedly, and the scheduler is required to ensure that they begin the task in cyclic order starting with P_1. The different task-performances need not exclude each other in time—for example P_2 can begin before P_1 finishes—but the scheduler is required to ensure that each agent finishes one performance before it begins another.

We assume that P_i requests task initiation by an action a_i and signals completion by an action b_i. The scheduler can then be specified by requiring that:

(1) It must perform $a_1, \dots, a_n$ cyclically, starting with a_1.

(2) It must perform a_i and b_i alternately, for each i.

However, a scheduler which imposes a fixed sequence, say $a_1 b_1 a_2 b_2 \dots$, is not good enough, the scheduler must allow *any* sequence of actions compatible with the conditions (1) and (2) above. For example, for $n = 2$, the sequences $a_1 a_2 b_1 b_2 a_1$ and $a_1 b_1 a_2 a_1 b_2 b_1$ are compatible with the specification, but the sequences $a_1 b_1 a_1$ and $a_1 b_1 a_2 a_1 a_2$ are not.

(a) For $n = 2$ agents, give a Petri net (N, M_0) which models a scheduler satisfying the given requirements.

The Petri net must have transitions $\{a_1, a_2, b_1, b_2\} \subseteq T$. Firing one of these transitions is equivalent to the execution of the corresponding action. Further, for any firing sequence σ enabled at M_0, the requirements above must hold when interpreting the firing sequence as a sequence of actions. The Petri net may also have additional transitions, which do not correspond to an action and may occur at any time.

The Petri net should also be live and bounded.

(b) How many reachable markings does the Petri net have?

(c) How can this solution be generalized to any number of n agents?

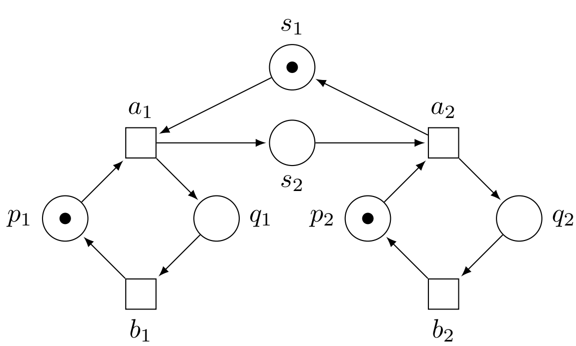
(d) How does the number of reachable markings grow as n increases?

Solution:

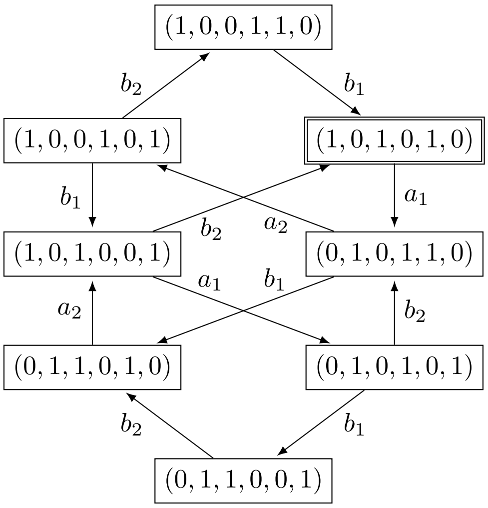
(a) The following Petri net models a scheduler satisfying the given requirements.

For requirement (1), the circuit $s_1 a_1 s_2 a_2 s_1$ with one token ensures that a_1 and a_2 are executed in cyclic order, while for requirement (2), the circuit $p_1 a_1 q_1 b_1 p_1$ and $p_2 a_2 q_2 b_2 p_2$ with one token each ensure that a_1 and b_1 as well as a_2 and b_2 are executed alternately.

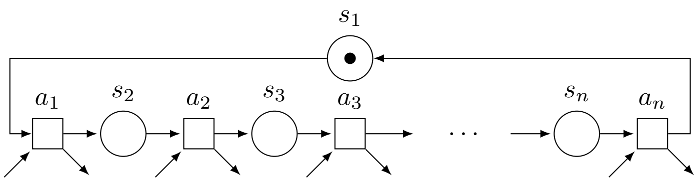
Liveness and boundedness can be verified by the reachability graph in the solution for (b).



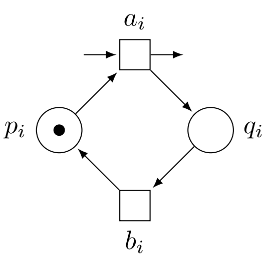
(b) The Petri net has 8 reachable markings. This can be verified by the following reachability graph, with markings given by $M = (s_1, s_2, p_1, q_1, p_2, q_2)$.



(c) The requirement (1) can be met by a circuit $s_1 a_1 s_2 a_2 s_3 a_3 \dots s_n a_n s_1$ with a token initially on s_1.



The requirement (2) can be met by a having a circuit $p_i a_i q_i b_i p_i$ with a token initially on p_i for each $1 \leq i \leq n$.



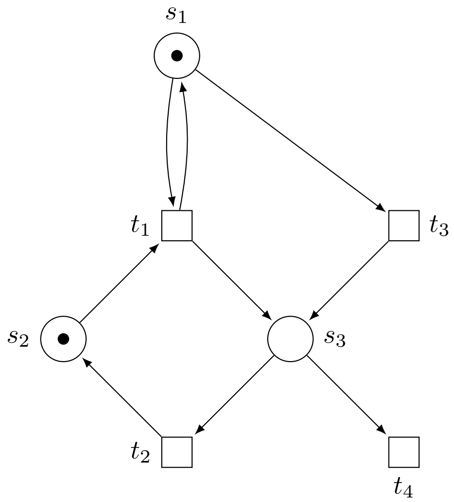
By combining these nets (taking the union of places, transitions and arcs), we obtain a Petri net modelling a scheduler for n agents.

(d) In all reachable markings, exactly one of $s_1, \dots, s_n$ is marked and for each $1 \leq i \leq n$, either p_i or q_i is marked. This gives us an upper bound of $n \cdot 2^n$ on the number of reachable markings.

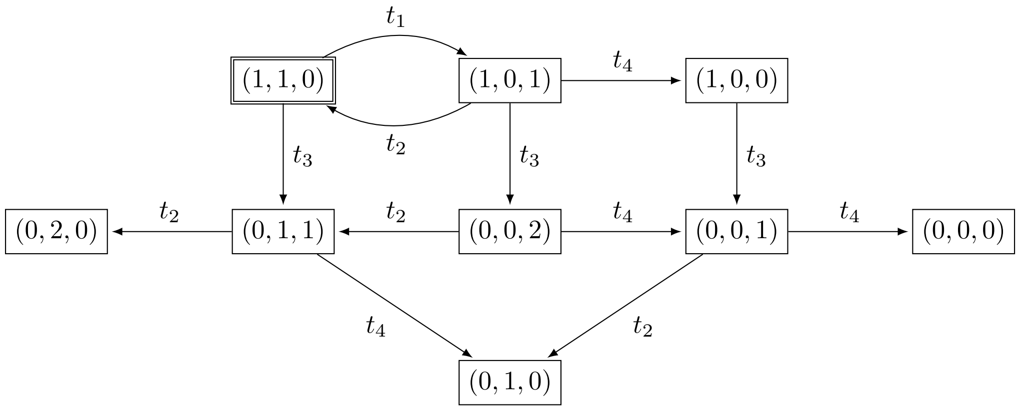
We show that all of these markings can be reached. For some $1 \leq i \leq n$, first we fire $a_1 \dots a_n b_1 \dots b_{i-1} a_1 \dots a_{i-1}$ to reach a marking where s_i and all q_j for $1 \leq j \leq n$ are marked. Then for each $1 \leq j \leq n$, we can either not fire b_j, keeping the token in q_j, or fire b_j to move the token to p_j. This gives us $n \cdot 2^n$ for the number of reachable markings and therefore an exponential growth.

Exercise 1.2 Reachability graph

Construct the reachability graph of the following Petri net. Is the empty marking $M = (0, 0, 0)$ reachable? If yes, give a firing sequence $M_0 \xrightarrow{\sigma} M$.



Solution: With the markings as a fixed vector $M = (s_1, s_2, s_3)$, the reachability graph is as follows. The inital marking is marked with a double border.

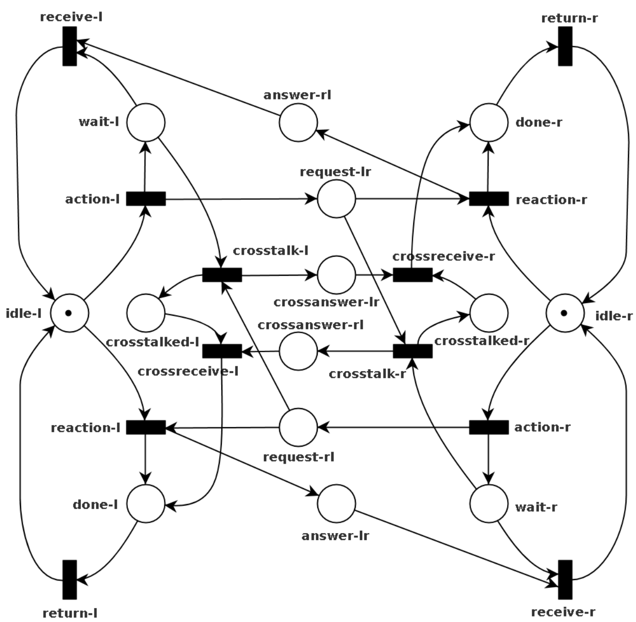


The empty marking $M = (0, 0, 0)$ is reachable with the firing sequence $\sigma = t_1 t_4 t_3 t_4$ or with any firing sequence matching the regular expression $(t_1 t_2)^* t_1 (t_4 t_3 + t_3 t_4) t_4$.

<u>Exercise 1.3</u> Petri net analysis with tools

Recall the third attempt for the solution of the action/reaction protocol (Figure 2.11 in the script). This system has a deadlock, and we would like to find the dead marking and a firing sequence leading to it. Due to the large size of the system and its reachability graph, we will use a tool to find them.

On the course website in the exercises section, you can find this net in suitable formats for the tools PIPE, APT and LoLA. Below the Petri net is shown as modelled in PIPE.



Install and use a tool of your choice to find a dead marking M and a firing sequence $M_0 \xrightarrow{\sigma} M$ leading to it.

- **PIPE** (`http://pipe2.sourceforge.net/`)

  To install PIPE, download the zip file from SourceForge and extract it. You can start it by executing `launch.sh` (Linux/OS X) or `launch.bat` (Windows).

  In PIPE, after opening the file, a reachability graph can be constructed by selecting the module "Reachability/Coverability Graph" on the left. Dead markings are colored red in the graph and a firing sequence can be read off the arcs from the initial marking (S0) to the dead marking.

  To verify this sequence, the animation mode in PIPE can be used. It is activated with the flag icon on the top. Enabled transitions are marked in red and can be fired by clicking on them. After firing the found sequence, no transition should be enabled.

- **APT** (`https://github.com/CvO-Theory/apt`)

  To install APT, clone the repository from GitHub or download the zip file and extract it. APT needs the JDK and Apache Ant to build it, for instructions refer to the readme on GitHub. It can then be run by executing `java -jar apt.jar`.

  APT can not directly find deadlocks. However, it can generate the reachability graph with the following command:
  `java -jar coverability_graph action-reaction-third-attempt.apt`
  The reachability graph can then be checked for a marking which has no outgoing arcs.

  APT can also check if the net is live and, if it is not, print a firing sequence after which some transition is not enabled any more. While this is not a sufficient condition for a deadlock, it is an indicator for one. The command for this is
  `java -jar apt.jar strongly_live action-reaction-third-attempt.apt`
  and then the sequence σ can be fired with
  `java -jar apt.jar fire_sequence "σ" action-reaction-third-attempt.apt`
  to give a marking which can be checked if it is dead.

- **LoLA** (`http://service-technology.org/lola/`)

  For installing LoLA, you need a working C++ compiler such as GCC or Clang. On Linux and OS X, it can be compiled with `./configure` and `make`. On Windows, you might need a Unix-like environment via Cygwin.

  LoLA can answer reachability queries and can print a witness marking and firing sequence when a marking is reachable. To find a deadlock, one can use the command:
  `lola action-reaction-third-attempt.lola -f "EF DEADLOCK" -s -p`

Solution: There are two dead markings. One is the marking where the places {action-r, reaction-l, return-l, action-l, crosstalk-r} are marked and a possible firing sequence leading to it is wait-l, answer-lr, crosstalk-r, crosstalked-r.

The other dead marking marks {action-l, reaction-r, return-r, action-r, crosstalk-l} and a possible firing sequence leading to it is wait-r, answer-rl, crosstalk-l, crosstalked-l.

Exercise 1.4 Boundedness, liveness and deadlock freedom

For each of the three Petri nets A, B and C, check if the following properties hold. This can be done by constructing the reachability graphs of the Petri nets.

(a) **Boundedness**: Is the Petri net bounded, i.e., for every place s, is there a number $b \geq 0$ such that $M(s) \leq b$ for every reachable marking M?

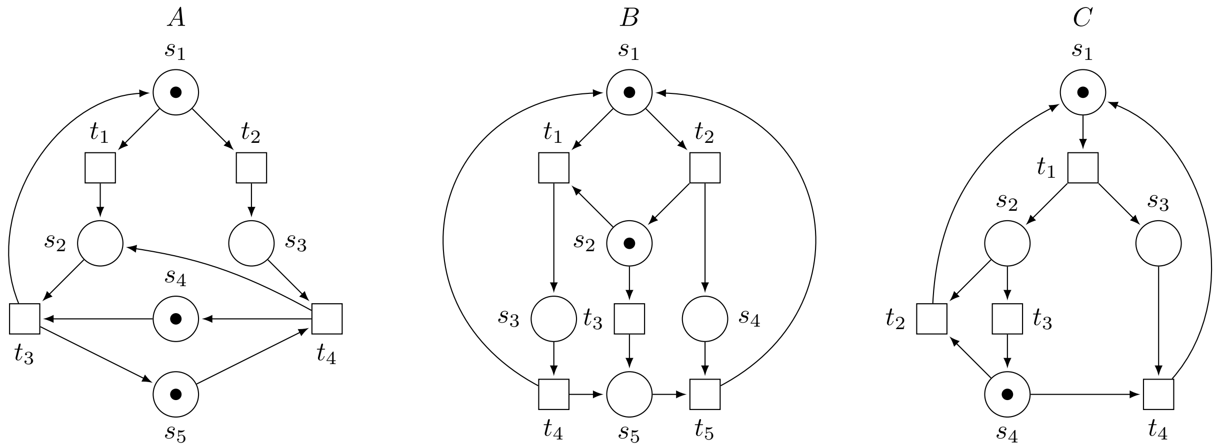
   If the Petri net is bounded, give the bound of each place s, that is, the maximal number of tokens in that place in all reachable markings.

(b) **Liveness**: Is the net live, i.e., for every reachable marking M and every transition t, is there a marking M' reachable from M that enables t?

   If the Petri net is not live, give a firing sequence σ and a transition t such that after firing σ from the initial marking M_0, we can never fire t again, that is, $M_0 \xrightarrow{\sigma} M$ and $M \stackrel{t}{\not\to}$ for all $M' \in [M\rangle$.

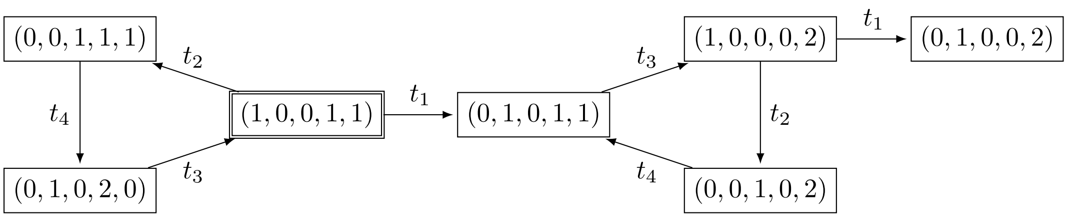
(c) **Deadlock freedom**: Is the Petri net deadlock-free, i.e., is there a reachable marking M that enables no transitions?

   If the Petri net has a deadlock, give a firing sequence σ that leads to a dead marking, that is, $M_0 \xrightarrow{\sigma} M$ and $M \stackrel{t}{\not\to}$ for all transitions t.



Solution:

The reachability graph for Petri net A, with the markings fixed as $M = (s_1, s_2, s_3, s_4, s_5)$, is:

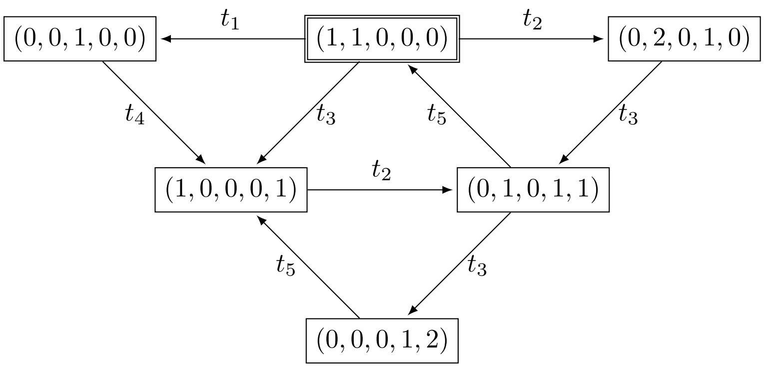


(a) The Petri net is bounded, the bound for s_1, s_2 and s_3 is 1 and for s_4 and s_5 is 2.

(b) The Petri net is not live. After firing $t_1t_3t_1$, no transition is enabled.

(c) The Petri net has a deadlock after firing $t_1t_3t_1$.

The reachability graph for Petri net B, with the markings fixed as $M = (s_1, s_2, s_3, s_4, s_5)$, is:

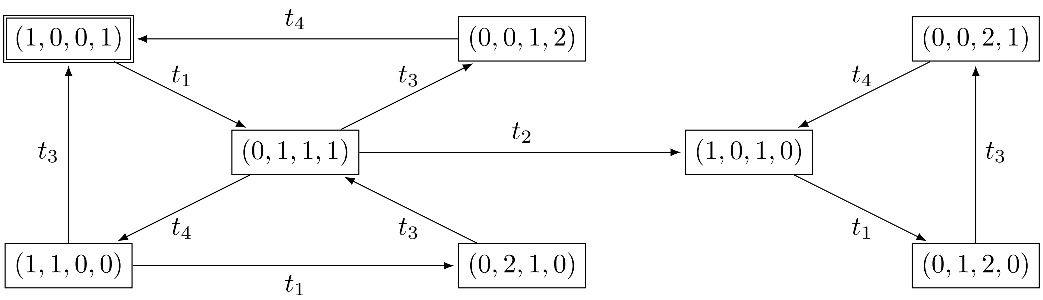


(a) The Petri net is bounded, the bound for s_1, s_3 and s_4 is 1 and for s_2 and s_5 is 2.

(b) The Petri net is live, as the reachability graph has only one strongly connected component which contains every transition.

(c) As the Petri net is live, it is deadlock-free.

The reachability graph for Petri net C, with the markings fixed as $M = (s_1, s_2, s_3, s_4)$, is:



(a) The Petri net is bounded, the bound for s_1 is 1 and for s_2, s_3 and s_4 is 2.

(b) The Petri net is not live. After firing t_1t_2, t_2 can never fire again, as only one of s_2 and s_4 can be marked again.

(c) The Petri net is deadlock-free, as every marking has a successor.

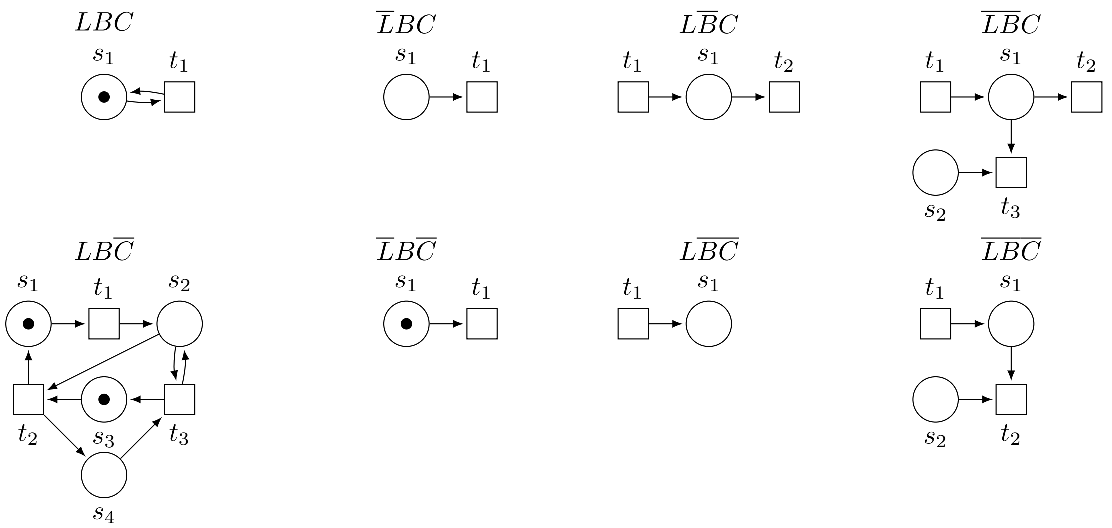
Exercise 1.5 Independence of boundedness, liveness and cyclicity

Definition 1.5.1 (Cyclic Petri nets)**.** A Petri net (N, M_0) is *cyclic* if, loosely speaking, it is always possible to return to the initial marking. Formally: $\forall M \in [M_0\rangle : M_0 \in [M\rangle$.

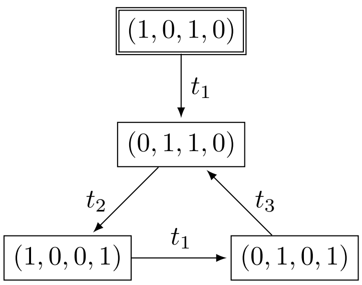
Show that the properties liveness, boundedness and cyclicity are independent of eath other by exhibiting eight Petri nets, one for each possible combination of the three properties and their negations.

Remark: Especially the live, bounded, but not cyclic Petri net is hard to find. However, it can be done with only 4 places and 3 transitions.

Solution: Let L,B and C be abbreviations for live, bounded and cyclic, and $\overline{L}$, $\overline{B}$ and $\overline{C}$ for their negations.



For most nets, it is clear that they have the desired properties. To see that the net $LB\overline{C}$ is indeed live, bounded, but not cyclic, we can construct the reachability graph, with the markings given as $M = (s_1, s_2, s_3, s_4)$:



Exercise 1.6 Strong Connectedness Theorem

Let (N, M_0) be a live and bounded Petri net. Show that N is strongly connected.

Hint: To show that the net is strongly connected, you need to show that for every arc $(x, y) \in F$, there is a path from y to x. Use liveness to construct a firing sequence containing the transition of the arc often enough and then use boundedness on the place of the arc to show that there needs to be a path back. You may also use the following lemma:

Lemma 1.6.1 (Exchange Lemma)**.** Let u and v be transitions of a net satisfying ${}^\bullet u \cap v^\bullet = \emptyset$. If $M \xrightarrow{vu} M'$ then $M \xrightarrow{uv} M'$.

Solution: Let $(x, y) \in F$. We distinguish between two cases:

Case 1: $x \in S$ and $y \in T$. Let V be the set of all transitions $v \in T$ for which there is a path from y to v and let $U = T \backslash V$. For $u \in U$ and $v \in V$ we have ${}^\bullet u \cap v^\bullet = \emptyset$.

Let b be the bound of x. Liveness implies that there exists a finite firing sequence $M_0 \xrightarrow{\sigma} M$ with $b + 1$ occurrences of y in σ. By Lemma 1.6.1, transitions of σ can repeatedly be swapped, resulting in firing sequences $M_0 \xrightarrow{\sigma_1} M' \xrightarrow{\sigma_2} M$ such that σ_1 contains only transitions in U and σ_2 contains only transitions in V.

Transition y is in the set V, so y occurs $b+1$ times in σ_2. Since $M'(x) \leq b$ and $y \in x^\bullet$, some transition $v \in {}^\bullet x$ occurs in σ_2. Since σ_2 contains only transitions of V, we have $v \in V$. By definition of V, there is a path from y to v and by extension also from y to x.

Case 2: $x \in T$ and $y \in S$. Let U be the set of all transitions $u \in T$ for which there is a path from u to x and let $V = T \backslash U$. For $u \in U$ and $v \in V$ we have ${}^\bullet u \cap v^\bullet = \emptyset$.

Let b be the bound of y. Liveness implies that there exists a finite firing sequence $M_0 \xrightarrow{\sigma} M$ with $b+1$ occurrences of x in σ. By Lemma 1.6.1, transitions of σ can repeatedly be swapped, resulting in firing sequences $M_0 \xrightarrow{\sigma_1} M' \xrightarrow{\sigma_2} M$ such that σ_1 contains only transitions in U and σ_2 contains only transitions in V.

Transition x is in the set U, so x occurs $b+1$ times in σ_1. Since $M'(y) \leq b$ and $x \in {}^\bullet y$, some transition $u \in y^\bullet$ occurs in σ_1. Since σ_1 contains only transitions of U, we have $u \in U$. By definition of U, there is a path from u to x and by extension also from y to x.