**Technische Universität München (I7)**
**Prof. J. Esparza / P. Meyer**

**Summer term 2015**

# SOLUTION

## Petri nets – Endterm

Last name: ______________________________

First name: ______________________________

Student ID no.: ______________________________

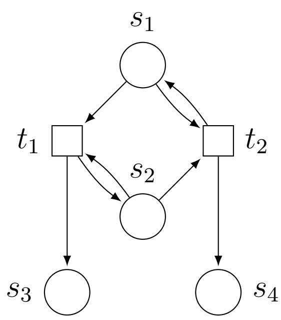
Signature: ______________________________

- If you feel ill, let us know immediately.
- Please, **do not write** until told so.
- You will be given **90 minutes** to fill in all the required information and write down your solutions.
- Don't forget to **sign**.
- Write with a non-erasable **pen**, do not use red or green color.
- You are not allowed to use **auxiliary means** other than your pen and a printout of the summary.
- You may answer in **English or German**.
- Please turn off your **cell phone**.
- Should you require additional **scrap paper**, please tell us.
- You can obtain **40 points** in the exam. You need **17 points** in total to pass (grade 4.0).
- Don't fill in the table below.
- Good luck!

| Ex 1 | Ex 2 | Ex 3 | Ex 4 | Ex 5 | $\sum$ |
|---|---|---|---|---|---|
| | | | | | |

## Exercise 1 **6P**

Apply the backwards-reachability algorithm to the net below and the marking $M = (0, 0, 1, 1)$ to decide if $M$ can be covered from the initial marking $M_0 = (1, 1, 0, 0)$. Record all intermediate steps.




**Solution:**

The table below records all sets $m$ with their minimal markings and the predecessors of the markings. New non-minimal markings are crossed out.

| $m$ | $M$ | $pre(M, t_1)$ | $pre(M, t_2)$ |
|---|---|---|---|
| $m_0$ | $(0, 0, 1, 1)$ | $(1, 1, 0, 1)$ | $(1, 1, 1, 0)$ |
| $m_1 \setminus m_0$ | $(1, 1, 0, 1)$ | ~~$(2, 1, 0, 1)$~~ | $(1, 2, 0, 0)$ |
| | $(1, 1, 1, 0)$ | $(2, 1, 0, 0)$ | ~~$(1, 2, 1, 0)$~~ |
| $m_2 \setminus m_1$ | $(1, 2, 0, 0)$ | ~~$(2, 2, 0, 0)$~~ | ~~$(1, 3, 0, 0)$~~ |
| | $(2, 1, 0, 0)$ | ~~$(3, 1, 0, 0)$~~ | ~~$(2, 2, 0, 0)$~~ |

We obtain $m_3 = m_2 = \{(0, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0), (1, 2, 0, 0), (2, 1, 0, 0)\}$. For no $M' \in m_3$, we have $M_0 \geq M'$, therefore $M$ cannot be covered from $M_0$.

## Exercise 2 **9P=3+3+3**

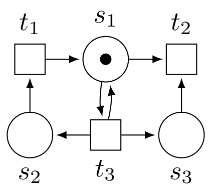
(a) Exhibit a <u>live</u> Petri net $(N, M_0)$ and a marking $M \geq M_0$ such that $(N, M)$ is <u>not live</u>. Argue succinctly why $(N, M)$ is not live (for example, by giving an occurrence sequence leading to a dead marking).

(b) Exhibit a connected net $N$ such that $I = (1, -1)$ is an S-invariant and $J = (1, 1)$ is a T-invariant of $N$. Explain your answer.

(c) Exhibit a Petri net $(N, M_0)$ and a marking $M$ such that the marking equation $M = M_0 + \mathbf{N} \cdot X$ has a solution $X : T \to \mathbb{Q}$ with $X \geq 0$, but no solution $X : T \to \mathbb{N}$. *Hint*: A net with 2 places and 2 transitions, with no tokens in $M_0$ and one token in $M$, suffices.

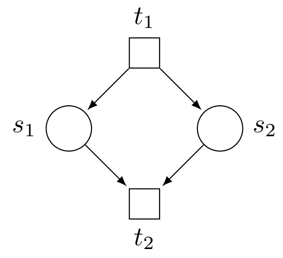
**Solution:**

(a) The following Petri net is live with $M_0 = (1, 0, 0)$, but not live with the initial marking $M = (1, 0, 1)$, as firing $t_2$ leads to a dead marking.

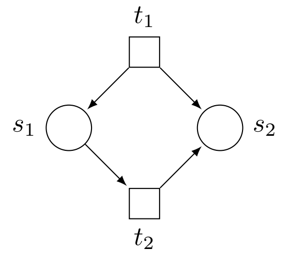



(b) The net below has $I = (1, -1)$ as an S-invariant and $J = (1, 1)$ as a T-invariant, as shown by the invariant equations:

$$t_1 : I(s_1) + I(s_2) = 0$$
$$t_2 : 0 = I(s_1) + I(s_2)$$
$$s_1 : J(t_1) = J(t_2)$$
$$s_2 : J(t_2) = J(t_1)$$

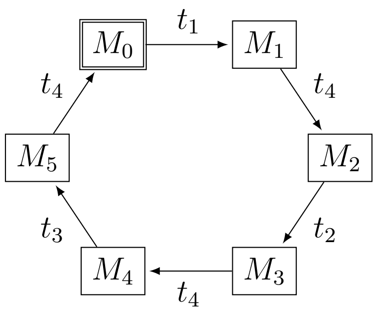



(c) The following Petri net with $M = (0, 1)$ only has $X = \left(\frac{1}{2}, \frac{1}{2}\right)$ as a solution to the marking equation.

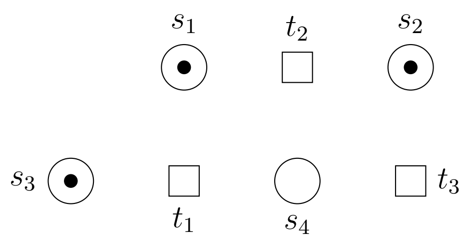


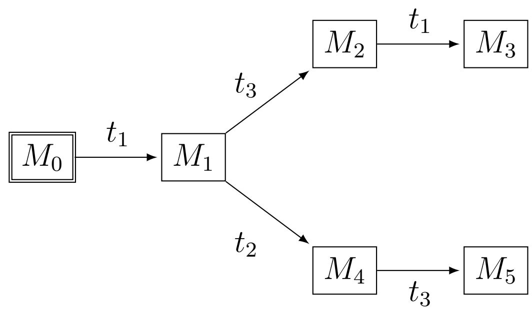

## Exercise 3 **8P=4+4**

(a) Exhibit a Petri net with the following reachability graph, where $M_0$ is the initial marking. Provide a clean drawing.



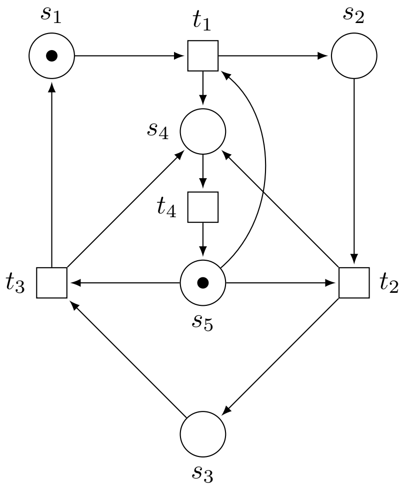

(b) Add arcs to the net below on the left such that it has the reachability graph given on the right, where $M_0$ is the initial marking. Provide a clean drawing.



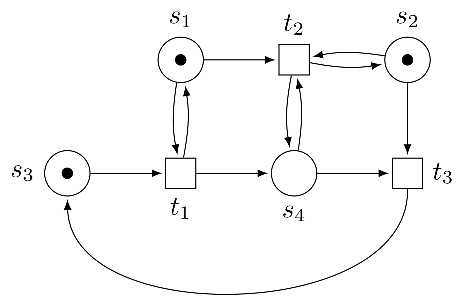




**Solution:**

(a)




(b)




## Exercise 4 

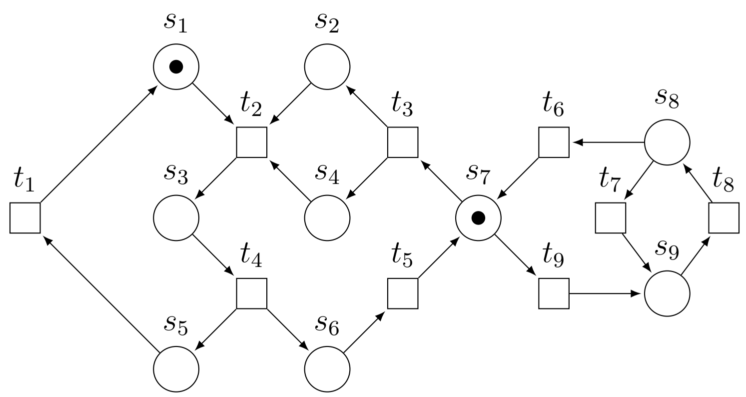
**8P=4+4**

For the following net, give:

(a) A positive S-invariant $I$ ($I$ is positive if $I(s) > 0$ for all places $s$).

(b) A positive T-invariant $J$ ($J$ is positive if $J(t) > 0$ for all transitions $t$).

Explain briefly the procedure you have followed to compute $I$ and $J$.




**Solution:**

(a) By following transitions, we can identify the S-components (which are subnet S-nets) $\{s_1, s_3, s_5\}$, $\{s_2, s_3, s_6, s_7, s_8, s_9\}$ and $\{s_3, s_4, s_6, s_7, s_8, s_9\}$, which give us the semi-positive S-invariants $I_1 = (1, 0, 1, 0, 1, 0, 0, 0, 0)$, $I_2 = (0, 1, 1, 0, 0, 1, 1, 1, 1)$ and $I_3 = (0, 0, 1, 1, 0, 1, 1, 1, 1)$. By adding them up, we get the positive S-invariant $I = (1, 1, 3, 1, 1, 2, 2, 2, 2)$.

(b) The left part of the net with transitions $\{t_1, t_2, t_3, t_4, t_5\}$ is a T-net, which gives us the semi-positive T-invariant $J_1 = \{1, 1, 1, 1, 1, 0, 0, 0, 0\}$. In the right part with transitions $\{t_6, t_7, t_8, t_9\}$, after firing each transition once, we need to fire $t_8$ once more for a net change of zero tokens, giving us $J_2 = \{0, 0, 0, 0, 0, 1, 1, 2, 1\}$. Adding these up gives us the positive T-invariant $J = (1, 1, 1, 1, 1, 1, 1, 2, 1)$.

**Exercise 5** **9P=3+3+3**

A vector $X$ is *semi-positive* if $X \geq 0$ and $X \neq 0$.

(a) Prove: Let $N$ be a net and $I$ a semi-positive S-invariant of $N$. The set $R = \{s \mid I(s) > 0\}$ of places is a trap of $N$.

(b) Prove: Every live T-system is cyclic.

(c) Let $N = (S, T, F)$ be a net. A *T-surinvariant* of $N$ is a vector $J : T \to \mathbb{Q}$ such that $\mathbf{N} \cdot J \geq 0$.

Prove: Let $(N, M_0)$ be a Petri net. If there is an infinite occurrence sequence $\sigma$ enabled at $M_0$, then $N$ has a semi-positive T-surinvariant. *Hint*: Use Dickson's lemma.

**Solution:**

(a) Let $R = \{s \mid I(s) > 0\}$. For $t \in R^\bullet$, there is an $s \in R$ with $s \in {}^\bullet t$. As $I(s) > 0$, $I(s') \geq 0$ for all $s' \in S$ and $\sum_{s \in {}^\bullet t} I(s) = \sum_{s' \in t^\bullet} I(s')$, there is an $s' \in t^\bullet$ with $I(s') > 0$. Therefore $s' \in R$ and $t \in {}^\bullet R$, so $R^\bullet \subseteq {}^\bullet R$ and $R$ is a trap.

(b) In a live T-system, a marking $M$ is reachable from $M_0$ iff $M_0 \sim M$. Let $M$ be a reachable marking. Then $M_0 \sim M$ and, as the relation is symmetric, $M \sim M_0$, so $M_0$ is reachable from $M$. Therefore the net is cyclic.

(c) Let $\sigma = t_1 t_2 t_3 \ldots$ with $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \ldots$. By Dickson's lemma, there are indices $i, j$ with $i < j$ and $M_i \leq M_j$. With $\sigma_1 = t_1 \ldots t_i$ and $\sigma_2 = t_{i+1} \ldots t_j$, we have $M_0 \xrightarrow{\sigma_1} M_i \xrightarrow{\sigma_2} M_j$. By the marking equation, we have $M_j = M_i + \mathbf{N} \cdot \vec{\sigma_2}$ and $\mathbf{N} \cdot \vec{\sigma_2} = M_j - M_i \geq 0$. As $\sigma_2$ is not empty, $\vec{\sigma_2}$ is a semi-positive T-surinvariant.