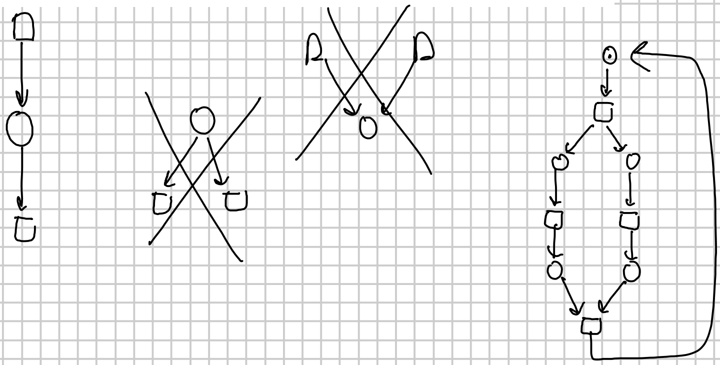
marked graphs synchronization graphs

Definition 3.12 *T-nets, T-systems*

A net is a T-net if $|{}^{\bullet}s| = 1 = |s^{\bullet}|$ for every place s.

A system (N, M_0) is a T-system if N is a T-net.



Definition 3.13 *Token counts of circuits*

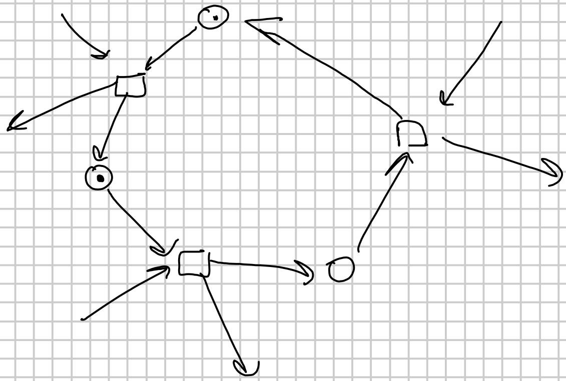
Let γ be a circuit of a net and let M be a marking. Let R be the set of places of γ. The token count $M(\gamma)$ of γ at M is defined as $M(R)$.

A circuit γ is marked at M if $M(\gamma) > 0$.

A circuit of a system is initially marked if it is marked at the initial marking.

Proposition 3.14 *Fundamental property of T-systems*

Let γ be a circuit of a T-system (N, M_0). For every reachable marking M, $M(\gamma) = M_0(\gamma)$.

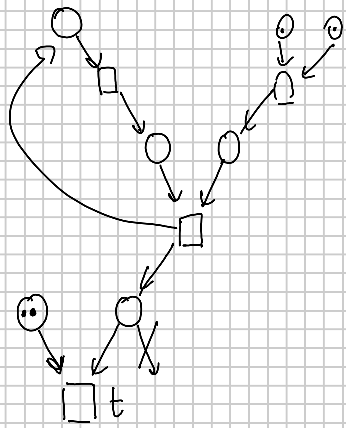


Theorem 3.15 *Liveness Theorem*

A T-system is live iff every circuit is initially marked.

Proof ($\Rightarrow$) If a circuit is initially unmarked, then it remains unmarked $\rightarrow$ the T-system is not live, because the transitions of the circuit can never occur.

($\Leftarrow$) Assume every circuit is initially marked.
Let M be an arbitrary reachable marking
Let t be an arbitrary transition
We show that t can be fired from M

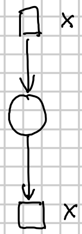


This backwards construction either terminates, and then firing all transitions from the top we finally fire t, or it doesn't terminate. But in this case the net necessarily contains a circuit without tokens, a contradiction.

Proposition 3.16 *T-invariants of T-nets*

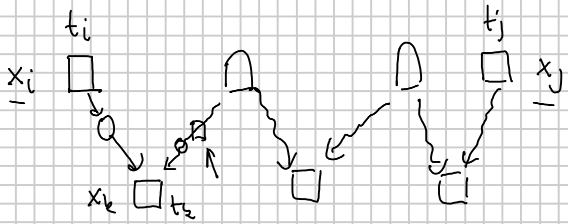
Let $N = (S, T, F)$ be a connected T-net. A vector $J : T \to \mathbb{Q}$ is a T-invariant iff $J = (\, x \ldots x \,)$ for some x. □

Proof We show that $(x, x, \ldots, x)$ is a T-invariant.



If $(x_1, x_2, \ldots, x_n)$ is a T-invariant, then

$x_1 = x_2 = \cdots = x_n$



Theorem 3.17 *Liveness in strongly connected T-systems*

Let (N, M_0) be a strongly connected T-system. The following statements are equivalent:

(a) (N, M_0) is live.

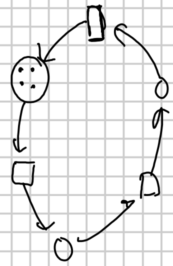
(b) (N, M_0) is deadlock-free.

(c) (N, M_0) has an infinite occurrence sequence.

Proof (c) $\Rightarrow$ (a)

We show that every infinite occurrence sequence contains all transitions infinitely often.

So every circuit contains at least one token, and so the T-system is live



$$M_0 \xrightarrow{\sigma_1} \underbrace{M_1 \xrightarrow{\sigma_2} M_1}_{(x,x,x,-,x)} \xrightarrow{\sigma_3} M_1 \xrightarrow{\sigma_4} M_1$$

Theorem 3.18 *Boundedness Theorem*

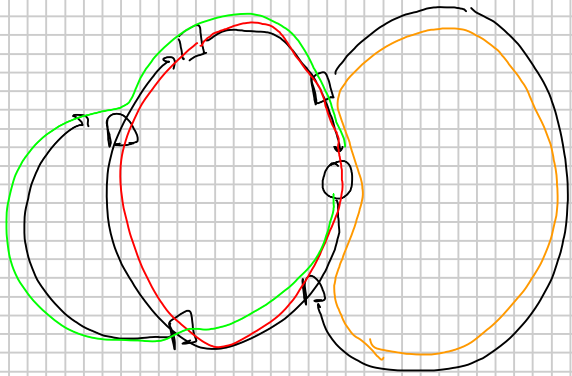
A live T-system (N, M_0) is b-bounded iff for every place s there exists a circuit γ which contains s and satisfies $M_0(\gamma) \leq b$.

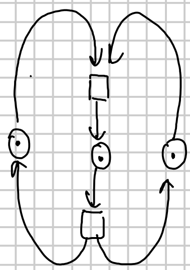
Proof Assume there exists a circuit γ containing s and satisfying $M_0(\gamma) \leq b$.

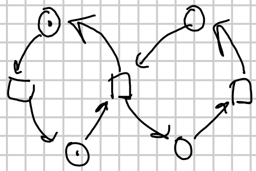
Then for every reachable marking we have $M(\gamma) = M_0(\gamma) \leq b$, and so the place s satisfies $M(s) \leq b$.

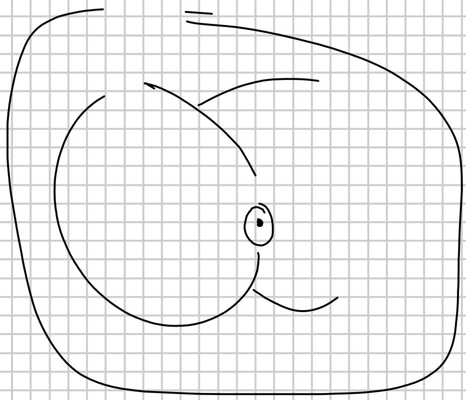
Assume that the system (N, M_0) is live and for every circuit γ containing s we have $M_0(\gamma) > b$. We show that some reachable marking M satisfies $M(s) > b$.

$b=1$









3

By liveness, we can put one token in the place s.

"Freeze" that token

The remaining tokens still make the net live (liveness theorem).

So we can put another token in s.

... and so on

Theorem 3.21 *Reachability Theorem*

Let (N, M_0) be a live T-system. A marking M is reachable iff it agrees with M_0 on all S-invariants.

Proof ($\Rightarrow$) Holds for all Petri nets

($\Leftarrow$) Assume M agrees with M_0 on all S-invariants

We know: there is a $X \in \mathbb{Q}^{|T|}$ such that

$$M = M_0 + C \cdot X$$

(a) There is $Y \in \mathbb{N}^{|T|}$ such that $M = M_0 + C \cdot Y$

Take $Y(t) = \lceil X(t) \rceil$ for every t. $(3.5, 2.3, 1.7) \downarrow (4, 3, 2)$

(If X contains negative component, consider the vector $X + \lambda(1, 1, \ldots, 1)$ for sufficiently large λ to make X nonnegative.

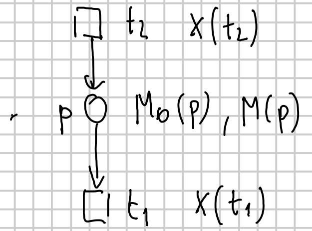
We have $M_0 + C \cdot (X + \lambda(1, \ldots, 1))$

$$= M_0 + C \cdot X + \lambda \cdot \underbrace{C \cdot (1, 1, \ldots, 1)}_{= 0}$$

$$= M_0 + C \cdot X$$

0 because $(1, 1, \ldots, 1)$ is a T-invariant)

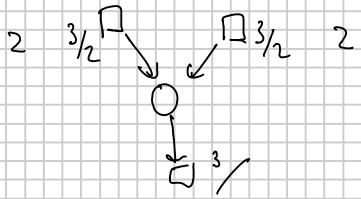
Let p be an arbitrary place



$M(p) = M_0(p) + X(t_2) - X(t_1)$

Since $M(p), M_0(p) \in \mathbb{N}$ we have $X(t_2) - X(t_1) \in \mathbb{N}$

It follows $\lceil X(t_2) \rceil - \lceil X(t_1) \rceil = X(t_2) - X(t_1)$



So $M_0 + C \cdot Y = M_0 + C \cdot X = M$.

(b) $\underbrace{M_0 \xrightarrow{*} M}$ $\qquad M \in [M_0\rangle$

By induction on $|Y|$

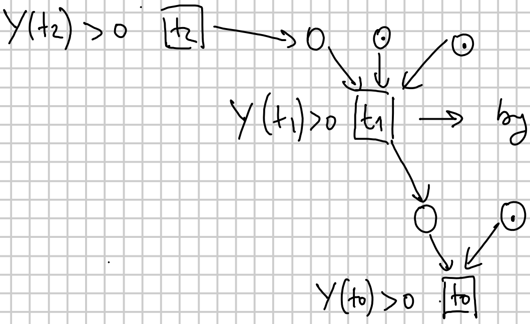
<u>Base</u> $|Y| = 0$ ✓

<u>Step</u> $|Y| > 0$.

(b1) M_0 enables some transition t s.t. $Y(t) > 0$

We use the "backwards construction" but starting at a transition t_0 such that $Y(t_0) > 0$

$M = M_0 + C.Y$



$\longrightarrow$ by the marking equation we get $Y(t_1) > 0$

The "backwards construction" cannot "run into a circuit" because the net is live, and so it must terminate with some transition t_i that is enabled at M_0.

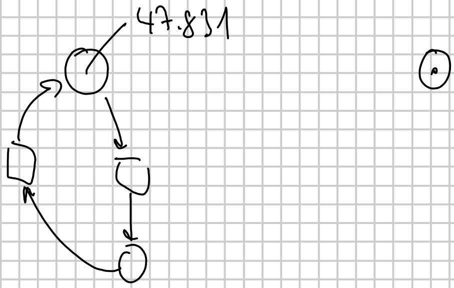
Take $t = t_i$

(b2) Let $M_0 \xrightarrow{t} M_1$ and let $Y_1 : T \rightarrow \mathbb{N}$ given by

$$Y_1(t') = \begin{cases} Y(t') & \text{if } t' \neq t \\ Y(t) - 1 & \text{if } t' = t \end{cases}$$

We have $M_0 + C.Y = M = M_1 + C.Y_1$

By induction hypothesis $(|Y_1| = |Y| - 1)$ we have $M_1 \xrightarrow{*} M$ and so $M_0 \xrightarrow{t} M_1 \xrightarrow{*} M$



"Lemma 3.23"

Let (N, M_0) be a T-system and let $M_0 \xrightarrow{\sigma_1 \sigma_2 t}$ such that

- $t \notin \mathcal{A}(\sigma_1)$ (where $\mathcal{A}(\sigma_1)$ denotes the set of transitions that occur in σ_1)
- $\mathcal{A}(\sigma_2) \subseteq \mathcal{A}(\sigma_1)$

Then $M_0 \xrightarrow{\sigma_1 t \sigma_2}$

Proof By induction on the length of σ_2

Base $|\sigma_2| = 0$ $\qquad \sigma_1 \sigma_2 t = \sigma_1 t \sigma_2$ ✓

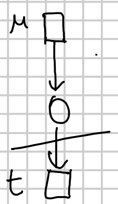
Step $\sigma_2 = \sigma_2' u$

We have to show $M_0 \xrightarrow{\sigma_1 t \sigma_2} \equiv M_0 \xrightarrow{\sigma_1 t \sigma_2' u}$

(a) We show $M_0 \xrightarrow{\sigma_1 \sigma_2' t u}$

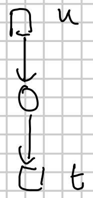
Two cases:

1) $u^\bullet \cap {}^\bullet t = \emptyset$ $\qquad u\, t$



Easy, because u "does not help" to enable t

2) $u^\bullet \cap {}^\bullet t \neq \emptyset$



$\sigma_1 \sigma_2' u\, t$

$\sigma_1 \sigma_2' t\, u$

Since $u \in \mathcal{A}(\sigma_2)$ and $\mathcal{A}(\sigma_2) \subseteq \mathcal{A}(\sigma_1)$ we have $u \in \mathcal{A}(\sigma_1)$.

So after $\sigma_1 \sigma_2' u$ there are at least two tokens in every place "between u and t" (because $t \notin \mathcal{A}(\sigma_1)$).

So t was already enabled before firing u, and so $\sigma_1 \sigma_2' t\, u$ is firable.

(b) $M_0 \xrightarrow{\sigma_1 t \sigma_2' u}$

By (a) we have $M_0 \xrightarrow{\sigma_1 \sigma_2' t u}$

By induction hypothesis, since $|\sigma_2'| < |\sigma_2|$ we get

$M_0 \xrightarrow{\sigma_1 t \sigma_2' u}$

" Lemma 3.24 + 3.25 "

Let (N, M_0) be a 1-safe T-system and $M_0 \xrightarrow{\sigma} M$, $|\sigma| \geq 1$.

Then there is σ_1, σ_2 such that

(a) $M_0 \xrightarrow{\sigma_1 \sigma_2} M$

(b) no transition occurs more than once in σ_1

(c) $\mathcal{A}(\sigma_2) \subset \mathcal{A}(\sigma_1)$

Proof

First we prove the result with $\subseteq$ in (c) instead of $\subset$ by induction on $|\sigma|$

Base $|\sigma| = 1$ Take $\sigma_1 = \sigma$ and $\sigma_2 = \epsilon$

Step $\sigma = \tau t$, $\tau \neq \epsilon$

By induction hypothesis $M_0 \xrightarrow{\tau_1 \tau_2 t} M$ where

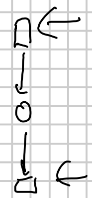
- no transition occurs more than once in τ_1,
- $\mathcal{A}(\tau_2) \subseteq \mathcal{A}(\tau_1)$

. If $t \in \mathcal{A}(\tau_1)$ then take $\sigma_1 = \tau_1$, $\sigma_2 = \tau_2 t$

. If $t \notin \mathcal{A}(\tau_1)$

$M_0 \xrightarrow{\tau_1 t \tau_2} M$ take $\sigma_1 = \tau_1 t$, $\sigma_2 = \tau_2$

. Now, assume $\mathcal{A}(\sigma_1) = \mathcal{A}(\sigma_2)$ $\sigma_1 \sigma_2$



- If $\mathcal{A}(\sigma_1)$ contains every transition, then replace σ_1 by ε! (because $M_0 \xrightarrow{\sigma_1} M_0$)

- If $\mathcal{A}(\sigma_1)$ does not contain every transition then the net is not 1-safe, contradiction.

"Theorem 3.27" (Shortest sequence theorem)

Let (N, M_0) be a 1-safe T-system and let M be reachable from M_0.

Then $M_0 \xrightarrow{\sigma} M$ with $|\sigma| \leq \frac{n(n-1)}{2}$ where n is the number of transitions.

1-safe n M $\qquad$ $M_0 \xrightarrow{\sigma} M$

$|\sigma| \in \Omega(2^n)$