1 Basic definitions

Definition 1.1 (Net, preset, postset)
A *net* $N = (S, T, F)$ consists of a finite set S of *places* (represented by circles), a finite set T of *transitions* disjoint from S (squares), and a *flow relation* (arrows) $F \subseteq (S \times T) \cup (T \times S)$.

The places and transitions of N are called *elements* or *nodes*. The elements of F are called *arcs*.

Given $x \in S \cup T$, the set ${}^\bullet x = \{y \mid (y, x) \in F\}$ is the *preset* of x and $x^\bullet = \{y \mid (x, y) \in F\}$ is the *postset* of x. For $X \subseteq S \cup T$ we denote ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$ and $X^\bullet = \bigcup_{x \in X} x^\bullet$.

Definition 1.2 (Subnet)
$N' = (S', T', F')$ is a *subnet* of $N = (S, T, F)$ if

- $S' \subseteq S$,
- $T' \subseteq T$, and
- $F' = F \cap ((S' \times T') \cup (T' \times S'))$ (not $F' \subseteq F \cap ((S' \times T') \cup (T' \times S'))$!).

Definition 1.3 (Path, circuit)
A path of a net $N = (S, T, F)$ is a finite, nonempty sequence $x_1 \dots x_n$ of nodes of N such that $(x_1, x_2), \dots, (x_{n-1}, x_n) \in F$. We say that a path $x_1 \dots x_n$ *leads from* x_1 *to* x_n.

A path is a *circuit* if $(x_n, x_1) \in F$ and $(x_i = x_j) \Rightarrow i = j$ for every $1 \leq i, j \leq n$.

N is *connected* if $(x, y) \in (F \cup F^{-1})^*$ for every $x, y \in S \cup T$, and *strongly connected* if $(x, y) \in F^*$ for every $x, y \in S \cup T$.

Proposition 1.4 *Let $N = (S, T, F)$ be a net.*

(1) N is connected iff there are no two subnets (S_1, T_1, F_1) and (S_2, T_2, F_2) of N such that

- *$S_1 \cup T_1 \neq \emptyset$, $S_2 \cup T_2 \neq \emptyset$;*
- *$S_1 \cup S_2 = S$, $T_1 \cup T_2 = T$, $F_1 \cup F_2 = F$;*
- *$S_1 \cap S_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$.*

(2) A connected net is strongly connected iff for every $(x, y) \in F$ there is a path leading from y to x.

Definition 1.5 (Markings)
Let $N = (S, T, F)$ be a net. A *marking* of N is a mapping $M : S \to \mathbb{N}$. Given $R \subseteq S$ we write $M(R) = \sum_{s \in R} M(s)$. A place s is *marked* at M if $M(s) > 0$. A set of places R is *marked* at M if $M(R) > 0$, that is, if at least one place of R is marked at M.

Definition 1.6 (Firing rule, dead markings)
A transition is *enabled* at a marking M if $M(s) \geq 1$ for every place $s \in {}^\bullet t$. If t is enabled, then it can *occur* or *fire*, leading from M to the marking M' (denoted $M \xrightarrow{t} M'$) given by:

$$M'(s) = \begin{cases} M(s) - 1 & \text{if } s \in {}^\bullet t \setminus t^\bullet \\ M(s) + 1 & \text{if } s \in t^\bullet \setminus {}^\bullet t \\ M(s) & \text{otherwise} \end{cases}$$

A marking is *dead* if it does not enable any transition.

Definition 1.7 (Firing sequence, reachable marking)
Let $N = (S, T, F)$ be a net and let M be a marking of N. A finite sequence $\sigma = t_1 \dots t_n$ is *enabled at a marking* M if there are markings $M_1, M_2, \dots, M_n$ such that $M \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \dots \xrightarrow{t_n} M_n$. We write $M \xrightarrow{\sigma} M_n$. The empty sequence ϵ is enabled at any marking and we have $M \xrightarrow{\epsilon} M$.

If $M \xrightarrow{\sigma} M'$ for some markings M, M' and some sequence σ, then we write $M \xrightarrow{*} M'$ and say that M' *is reachable from* M. $[M\rangle$ denotes the set of markings that are reachable from M.

An infinite sequence $\sigma = t_1 t_2 \dots$ is *enabled at a marking* if there are markings $M_1, M_2, \dots$ such that $M \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \longrightarrow \dots$

Proposition 1.8 *A (finite or infinite) sequence σ is enabled at M iff every finite prefix of σ is enabled at M.*

Lemma 1.9 *[Monotonicity lemma]*
Let M and L be two markings of a net.

(1) If $M \xrightarrow{\sigma} M'$ for a finite sequence σ, then $(M + L) \xrightarrow{\sigma} (M' + L)$ for every marking L.

(2) If $M \xrightarrow{\sigma}$ for an infinite sequence σ, then $(M + L) \xrightarrow{\sigma}$ for every marking L.

Definition 1.10 (Petri nets)
A *Petri net*, *net system*, or just a *system* is a pair (N, M_0) where N is a connected net $N = (S, T, F)$ with nonempty sets of places and transitions, and an *initial marking* $M_0 : S \to \mathbb{N}$. A marking M is *reachable in* (N, M_0) or *a reachable marking of* (N, M_0) if $M_0 \xrightarrow{*} M$.

Definition 1.11 (Reachability graph)
The *reachability graph* G of a Petri net (N, M_0) where $N = (S, T, F)$ is the directed, labeled graph satisfying:

- The nodes of G are the reachable markings of (N, M_0).
- The edges of G are labeled with transitions from T.
- There is an edge from M to M' labeled by t iff $M \xrightarrow{t} M$, that is, iff M enables t and the firing of t leads from M to M'.

2 Modelling with Petri nets

Definition 2.1 (Nets with place capacities)
A *net with capacities* $N = (S, T, F, K)$ consists of a net (S, T, F) and a mapping $K : S \to \mathbb{N}$.
A transition t is *enabled* at a marking M of N if
– $M(s) \geq 1$ for every place $s \in {}^\bullet t$ and
– $M(s) < K(s)$ for every place $s \in t^\bullet \setminus {}^\bullet t$
The notions of firing, Petri net with capacities, etc. are defined as in the capacity-free case.

Definition 2.2 (Nets with weighted arcs)
A *net with weighted arcs* $N = (S, T, W)$ consists of two disjoint sets of places and transitions and a *weight function* $W : (S \times T) \cup (T \times S) \to \mathbb{N}$. A transition t is *enabled* at a marking M of N if $M(s) \geq W(s, t)$ for every $s \in S$. If t is enabled then it can *occur* leading to the marking M' defined by

$$M'(s) = M(s) + W(t, s) - W(s, t)$$

for every place s. Other notions are defined as in the standard model.

Definition 2.3 (Nets with inhibitor arcs)
A *net with inhibitor arcs* $N = (S, T, F, I)$ consists of two disjoint sets of places and transitions, a set $F \subseteq (S \times T) \cup (T \times S)$ of arcs, and a set $I \subseteq S \times T$, disjoint with F, of *inhibitor arcs*. A transition t is *enabled* at a marking M of N if $M(s) > 0$ for every place s such that $(s,t) \in F$, and $M(s) = 0$ for every place s such that $(s,t) \in I$. If t is enabled then it can *occur* leading to the marking M', defined as for standard Petri nets.

Definition 2.4 (Nets with reset arcs)
A *net with reset arcs* $N = (S, T, F, R)$ consists of two disjoint sets of places and transitions, a set $F \subseteq (S \times T) \cup (T \times S)$ of arcs, and a set $R \subseteq S \times T$, disjoint with F, of *reset arcs*. A transition t is *enabled* at a marking M of N if $M(s) > 0$ for every place s such that $(s,t) \in F \cup R$. If t is enabled then it can *occur* leading to the marking obtained after the following operations:

- Remove one token from every place s such that $(s,t) \in F$.
- Remove *all* tokens from every place s such that $(s,t) \in R$.
- Add one token to every place s such that $(t,s) \in F$.

Definition 2.5 (System properties)
Let (N, M_0) be a Petri net.

(N, M_0) is *deadlock free* if every reachable marking enables at least one transition (that is, no reachable marking is dead).

(N, M_0) is *live* if for every reachable marking M and every transition t there is a marking $M' \in [M\rangle$ that enables t. (Intuitively: every transition can always fire again).

(N, M_0) is *bounded*, if for every place s there is a number $b \geq 0$ such that $M(s) \leq b$ for every reachable marking M. M_0 is a *bounded marking of* N if (N, M_0) is bounded. The *bound* of a place s of a bounded Petri net (N, M_0) is the number

$$max\{M(s) \mid M \in [M_0\rangle\}$$

(N, M_0) is *b-bounded* if every place has bound b.

In these notes we study the following problems:

- **Deadlock freedom**: is a given Petri net (N, M_0) deadlock-free?
- **Liveness**: is a given Petri net (N, M_0) live?
- **Boundedness**: is a given Petri net (N, M_0) bounded?
- **b-boundedness**: given $b \in \mathbb{N}$ and a Petri net (N, M_0), is (N, M_0) b-bounded?
- **Reachability**: given a Petri net (N, M_0) and a marking M of N, is M reachable?
- **Coverability**: given a Petri net (N, M_0) and a marking M of N, is there a reachable marking $M' \geq M$?

Proposition 2.6

(1) Liveness implies deadlock freedom.

(2) If (N, M_0) is bounded then there is a number b such that (N, M_0) is b-bounded.

(3) If (N, M_0) is bounded, then it has finitely many reachable markings.

Definition 2.7 (Well-formed nets)
A net N is *well formed* if there is a marking M_0 such that the Petri net (N, M_0) is live and bounded.

- **Well-formedness**: is a given net well formed?

```
COVERABILITY-GRAPH((S, T, F, M0))
  (V, E, v0) := ({M0}, ∅, M0);
  Work : set := {M0};
  while Work ≠ ∅
  do select M from Work;
     Work := Work \ {M};
     for t ∈ enabled(M)
     do M' := fire(M, t);
        M' := AddOmegas(M, t, M', V, E);
        if M' ∉ V
          then V := V ∪ {M'}
               Work := Work ∪ {M'};
        E := E ∪ {(M, t, M')};
  return (V, E, v0);
```

```
ADDOMEGAS(M, t, M', V, E)
  for M'' ∈ V
  do if M'' < M' and M'' −*→_E M
       then M' := M' + ((M' − M'') · ω);
  return M';
```

Figure 1: Algorithm for the construction of the coverability graph

3 Decision procedures

Lemma 3.1 (Königs lemma) *Let $G = (V, E)$ be the reachability graph of a Petri net (N, M_0). If V is infinite, then G contains an infinite simple path.*

Lemma 3.2 (Dickson's lemma) *For every infinite sequence $A_1 A_2 A_3 \ldots$ of vectors of $\mathbb{N}^k$ there is an infinite sequence $i_1 < i_2 < i_3 \ldots$ of indices such that $A_{i_1} \leq A_{i_2} \leq A_{i_3} \ldots$.*

Theorem 3.3 *(N, M_0) is unbounded iff there are markings M and L such that $L \neq 0$ and $M_0 \xrightarrow{*} M \xrightarrow{*} (M + L)$*

Theorem 3.4 **Boundedness** *is decidable.*

Theorem 3.5 COVERABILITY-GRAPH *terminates.*

Lemma 3.6 *For every ω-marking M' added by the algorithm to V and for every $k > 0$, there there is a reachable marking M'_k satisfying $M'_k(s) = M'(s)$ for every place s such that $M'(s) \in \mathbb{N}$, and $M'_k(s) > k$ for every place s such that $M'(s) = \omega$.*

Theorem 3.7 *Let (N, M_0) be a Petri net and let M be a marking of N. There is a reachable marking $M' \geq M$ iff the coverability graph of (N, M_0) contains an ω-marking $M'' \geq M$.*

Definition 3.8 (Integer nets) Let $N = (S, T, F)$ be a net. A *generalized marking* of N (*g-marking* for short) is a mapping $G\colon S \to \mathbb{Z}$. An *integer net* is a pair (N, G_0) where N is a net and G_0 is a g-marking. A g-marking G enables all transitions, and the occurrence of t at G leads to the marking G' given by

$$G'(s) = \begin{cases} G(s) - 1 & \text{if } s \in {}^\bullet t \setminus t^\bullet \\ G(s) + 1 & \text{if } s \in t^\bullet \setminus {}^\bullet t \\ G(s) & \text{otherwise} \end{cases}$$

We denote by $G \overset{t}{\hookrightarrow} G'$ that firing t at G yields to G'.

An *integer firing sequence* of an integer net is a sequence $G_0 \overset{t_1}{\hookrightarrow} G_1 \overset{t_2}{\hookrightarrow} \cdots \overset{t_n}{\hookrightarrow} G_m$.

Definition 3.9 Let $G \in \mathbb{Z}^k$ be a g-marking of N and let $0 \le i \le k$. We say that G is *i-natural* if its first i-components are natural numbers, i.e., if $G(j) \ge 0$ for every $1 \le j \le i$. If moreover $G(j) < r$ for every $1 \le j \le i$, then we say that G is *(i,r)-natural*.

An integer sequence $\sigma = G_0 \overset{t_1}{\hookrightarrow} \cdots \overset{t_m}{\hookrightarrow} G_m$ is *i-natural* (respectively *(i,r)-natural*) if every generalized marking of σ is i-natural (respectively (i,r)-natural). Given a g-marking $G \in \mathbb{Z}^k$, we say that σ is *(i,G)-covering* if $G_m(j) \ge G(j)$ for every $1 \le j \le i$.

Theorem 3.10 *Let $n = \max(1, |G(1)|, \ldots |G(k)|)$. For every $G_0 \in \mathbb{Z}^k$, if (N, G_0) has a (k,G)-covering, k-natural sequence, then it has one of length at most $(n+1)^{(2k)^k}$.*

Lemma 3.11 *For every $G_0 \in \mathbb{Z}^k$ and for every $1 \le i \le k$, if (N, G_0) has an (i,G)-covering, i-natural sequence, then it has one of length at most $f(i)$, where f is inductively defined as follows:*

- $f(0) = 1$, *and*
- $f(i) = (nf(i-1))^i + f(i-1)$ *for every* $1 \le i \le k$.

Definition 3.12 (Upward-closed sets of markings)
A set $\mathcal{M}$ of markings of a net N is *upward closed* if $M \in \mathcal{M}$ and $M' \ge M$ imply $M' \in \mathcal{M}$.

A marking M of an upward closed set $\mathcal{M}$ is *minimal* if there is no $M' \in \mathcal{M}$ such that $M' \le M$ and $M' \ne M$.

Lemma 3.13 *Every upward-closed set of markings has finitely many minimal elements.*

Definition 3.14 Let $\mathcal{M}$ be a set of markings of a net $N = (S, T, F)$, and let $t \in T$ be a transition. We define

$$\begin{aligned} pre(\mathcal{M}, t) &= \{M' \mid M' \xrightarrow{t} M \text{ for some } M \in \mathcal{M}\} \\ pre(\mathcal{M}) &= \bigcup_{t \in T} pre(\mathcal{M}, t) \end{aligned}$$

and further

$$\begin{aligned} pre^0(\mathcal{M}) &= \mathcal{M} \\ pre^{i+1}(\mathcal{M}) &= pre\big(pre^i(\mathcal{M})\big) \text{ for every } i \ge 0 \\ pre^*(\mathcal{M}) &= \bigcup_{i=0}^{\infty} pre^i(\mathcal{M}) \end{aligned}$$

Lemma 3.15 *If $\mathcal{M}$ is upward closed, then $pre(\mathcal{M})$ is also upward closed.*

Theorem 3.16 *Let $\mathcal{M}$ be an upward-closed set of markings of a net N. Then there is $i \ge 0$ such that*

$$pre^*(\mathcal{M}) = \bigcup_{j=0}^{i} pre^j(\mathcal{M})$$

Definition 3.17 Given a set A, and a partial order $\preceq \subseteq A \times A$, we say that $\preceq$ is a well-quasi-order (wqo) if every infinite sequence $a_1 a_2 a_3 \cdots \in A^\omega$ contains an infinite chain $a_{i_1} \preceq a_{i_2} \preceq \cdots$ (where $i_1 < i_2 < i_3 \ldots$).

```
BACK1((S, T, F, M0), M)
  𝓜 := {M' | M' ≥ M};
  Old_𝓜 := ∅;
  while true
  do Old_𝓜 := 𝓜;
     𝓜 := 𝓜 ∪ pre(𝓜);
     if M0 ∈ 𝓜
       then return covered end
     if 𝓜 = Old_𝓜
       then return not covered end
```

```
BACK2((S, T, F, M0), M)
  m := {M};
  old_m := ∅;
  while true
  do old_m := m;
     m := min(m ∪ ⋃_{t∈T} pre(m, t));
     if ∃M' ∈ m : M0 ≥ M'
       then return covered end
     if m = old_m
       then return not covered end
```

Figure 2: Backwards reachability algorithm.

Definition 3.18 Let A be a set and let $\preceq A \times A$ be a wqo. A set $X \subseteq A$ is *upward closed* if $x \in X$ and $x \preceq y$ implies $y \in X$ for every $x, y \in A$. In particular, given $x \in A$, the set $\{y \in A \mid y \succeq x\}$ is upward-closed.

A relation $\rightarrow \subseteq A \times A$ is *monotonic* if for every $x \rightarrow y$ and every $x' \succeq x$ there is $y' \succeq y$ such that $x' \rightarrow y'$.

Given $X \subseteq A$, we define

$$pre(X) = \{y \in A \mid y \rightarrow x \text{ and } x \in X\}$$

Further we define:

$$\begin{aligned} pre^0(X) &= X \\ pre^{i+1}(X) &= pre\big(pre^i(X)\big) \text{ for every } i \ge 0 \\ pre^*(X) &= \bigcup_{i=0}^{\infty} pre^i(X) \end{aligned}$$

Theorem 3.19 *Let A be a set and let $\preceq A \times A$ be a wqo. Let $X_0 \subseteq A$ be an upward closed set and let $\rightarrow \subseteq A \times A$ be monotonic. Then there is $j \in \mathbb{N}$ such that*

$$pre^*(X) = \bigcup_{i=0}^{j} pre^i(X)$$

Theorem 3.20 **Deadlock freedom**, **Liveness**, **Boundedness**, **b-boundedness**, **Reachability**, *and* **Coverability** *are all undecidable for Petri nets with inhibitor arcs.*

Definition 3.21 (Semilinear set) A set $X \subseteq \mathbb{N}^k$ is *linear* if there is $r \in \mathbb{N}^k$ (the root) and a finite set $P \subseteq \mathbb{N}^k$ (the periods) such that

$$X = \{r + \sum_{p \in P} \lambda_p p\}$$

A *semilinear set* is a finite union of linear sets.

Theorem 3.22 *[Leroux 12] Let (N, M_0) be a Petri net and let M be a marking of M. If M is not reachable from M_0, then there exists a semilinear set $\mathcal{M}$ of markings of N such that*

(a) $M_0 \in \mathcal{M}$,

(b) if $M \in \mathcal{M}$ and $M \xrightarrow{t} M'$ for some transition t of N, then $M' \in \mathcal{M}$, and

(c) $M \notin \mathcal{M}$.

P: Given a Petri net (N, M_0) and a subset R of places of N, is there a reachable marking M such that $M(s) = 0$ for every $s \in R$?

Theorem 3.23 **Deadlock-freedom** *can be reduced to* **P**.

Theorem 3.24 **P** *can be reduced to* **Reachability**.

Proposition 3.25 *Let (N, M_0) be a bounded Petri net. (N, M_0) is live iff for every bottom SCC of the reachability graph of (N, M_0) and for every transition t, some marking of the SCC enables t.*

4 Semi-decision procedures

Definition 4.1 (Incidence matrix)
Let $N = (S, T, F)$ be a net. The *incidence matrix* $\mathbf{N} : (S \times T) \to \{-1, 0, 1\}$ is given by

$$\mathbf{N}(s,t) = \begin{cases} 0 & \text{if} \quad (s,t) \notin F \text{ and } (t,s) \notin F \text{ or} \\ & \phantom{\text{if}} \quad (s,t) \in F \text{ and } (t,s) \in F \\ -1 & \text{if} \quad (s,t) \in F \text{ and } (t,s) \notin F \\ 1 & \text{if} \quad (s,t) \notin F \text{ and } (t,s) \in F \end{cases}$$

The column $\mathbf{N}(-, t)$ is denoted by $\mathbf{t}$, and the row $\mathbf{N}(s, -)$ by $\mathbf{s}$.

Definition 4.2 (Parikh-vector of a sequence of transitions)
Let $N = (S, T, F)$ be a net and let σ be a finite sequence of transitions. The *Parikh-vector* $\vec{\sigma} : T \to \mathbb{N}$ *von* σ is defined by

$$\vec{\sigma}(t) = \text{ number of occurrences of } t \text{ in } \sigma$$

Lemma 4.3 (Marking Equation Lemma)
Let N be a net and let $M \xrightarrow{\sigma} M'$ be a firing sequence of N. Then $M' = M + \mathbf{N} \cdot \vec{\sigma}$.

Definition 4.4 (The Marking Equation)
The Marking Equation of a Petri net (N, M_0) is $M = M_0 + \mathbf{N} \cdot X$ with variables M and X.

Proposition 4.5 (A sufficient condition for boundedness)
Let (N, M_0) be a Petri net. If the optimization problem

$$\begin{array}{ll} \textit{maximize} & \sum_{s \in S} M(s) \\ \textit{subject to} & M = M_0 + \mathbf{N} \cdot X \end{array}$$

has an optimal solution, then (N, M_0) is bounded.

Proposition 4.6 (A sufficient condition for non-reachability)
Let (N, M_0) be a Petri net and let L be a marking of N. If the equation

$$L = M_0 + \mathbf{N} \cdot X \qquad (\textit{with only } X \textit{ as variable})$$

has no solution, then L is not reachable from M_0.

Proposition 4.7 (A sufficient condition for deadlock-freedom)
Let (N, M_0) be a 1-bounded Petri net where $N = (S, T, F)$. If the following system of inequations has no solution then (N, M_0) is deadlock-free.

$$\begin{array}{l} M = M_0 + \mathbf{N} \cdot X \\ \sum_{s \in \cdot t} M(s) < |{}^\bullet t| \textit{ for every transition } t. \end{array}$$

Definition 4.8 (S-invariants)
Let $N = (S, T, F)$ be a net. An S-invariant of N is a vector $I : S \to \mathbb{Q}$ such that $I \cdot \mathbf{N} = 0$.

Proposition 4.9 (Fundamental property of S-invariants)
Let (N, M_0) be a Petri net and let I be a S-invariant of N. If $M_0 \xrightarrow{*} M$, then $I \cdot M = I \cdot M_0$.

Proposition 4.10 *The S-invariants of a net form a vector space over the real numbers.*

Proposition 4.11 *I is an S-invariant of $N = (S, T, F)$ iff. $\forall t \in T : \sum_{s \in {}^\bullet t} I(s) = \sum_{s \in t^\bullet} I(s)$.*

Definition 4.12 (Semi-positive and positive S-invariants)
Let I be an S-invariant of $N = (S, T, F)$. I is *semi-positive* if $I \geq 0$ and $I \neq 0$, and *positive* if $I > 0$ (that is, if $I(s) > 0$ for every $s \in S$). The *support* of an S-invariant is the set $\langle I \rangle = \{s \in S \mid I(s) > 0\}$.

Proposition 4.13 *[A sufficient condition for boundedness]*
Let (N, M_0) be a Petri net. If N has a positive S-invariant I, then (N, M_0) is bounded. More precisely: (N, M_0) is n-bounded for

$$n = max \left\{ \frac{I \cdot M_0}{I(s)} \mid s \textit{ is a place of } N \right\}$$

Proposition 4.14 *[A necessary condition for liveness]*
If (N, M_0) is live, then $I \cdot M_0 > 0$ for every semi-positive S-invariant of N.

Definition 4.15 (The $\sim$ relation)
Let M and L be markings and let I be a S-invariant of a net N. M und L *agree on* I if $I \cdot M = I \cdot L$. We write $M \sim L$ if M and L agree on all invariants of N.

Proposition 4.16 *[A necessary condition for reachability]*
Let (N, M_0) be a Petri net. $M \sim M_0$ holds for every $M \in [M_0\rangle$.

Theorem 4.17 *Let N be a net and let M, L be two markings of N.*
$M \sim L$ iff the equation $M = L + \mathbf{N} \cdot X$ has a rational solution.

Definition 4.18 (T-invariants)
Let $N = (S, T, F)$ be a net. A vector $J : T \to \mathbb{Q}$ is a T-invariant of N if $\mathbf{N} \cdot J = 0$.

Proposition 4.19 *J is a T-invariant of $N = (S, T, F)$ iff $\forall s \in S : \sum_{t \in {}^\bullet s} J(t) = \sum_{t \in s^\bullet} J(t)$.*

Proposition 4.20 *[Fundamental property of T-invariants]*
Let N be a net, let M be a marking of N, and let σ be a sequence of transitions of N enabled at M. The vector $\vec{\sigma}$ is a T-invariant of N iff $M \xrightarrow{\sigma} M$.

Theorem 4.21 *[Necessary condition for well-formedness]*
Every well-formed net has a positive T-invariant.

Definition 4.22 (Siphon)
Let $N = (S, T, F)$ be a net. A set $R \subseteq S$ of places is a *siphon* of N if ${}^\bullet R \subseteq R^\bullet$. A siphon R is *proper* if $R \neq \emptyset$.

Proposition 4.23 *[Fundamental property of siphons]*
Let R be a siphon of a net N, and let $M \xrightarrow{\sigma} M'$ be a firing sequence of N. If $M(R) = 0$, then $M'(R) = 0$.

Corollary 4.24 *[A necessary condition for reachability]*
If M is reachable in (N, M_0), then for every siphon R, if $M_0(R) = 0$ then $M(R) = 0$.

> **Input:** A net $N = (S, T, F)$ and $R \subseteq S$.
> **Output:** The largest siphon $Q \subseteq R$.
> **Initialization:** $Q := R$.
>
> **begin**
>   **while** there are $s \in Q$ and $t \in {}^\bullet s$ such that $t \notin Q^\bullet$ **do**
> $Q\colon = Q \setminus \{s\}$
>   **endwhile**
> **end**

Proposition 4.25 *[A necessary condition for liveness]*
If (N, M_0) is live, then M_0 marks every proper siphon of N.

Proposition 4.26 *If M is a dead marking of N, then the set of places unmarked at M is a siphon of N.*

Corollary 4.27 *[A sufficient condition for deadlock-freedom] Let (N, M_0) be a Petri net. If every reachable marking marks all siphons of N, then (N, M_0) is deadlock-free.*

Definition 4.28 (Trap)
Let $N = (S, T, F)$ be a trap. A set $R \subseteq S$ of places is a *trap* if $R^\bullet \subseteq {}^\bullet R$. A trap R is *proper* if $R \neq \emptyset$.

Proposition 4.29 *[Fundamental property of traps]*
Let R be a trap of a net N and let $M \xrightarrow{\sigma} M'$ be a firing sequence of N. If $M(R) > 0$, then $M'(R) > 0$.

Corollary 4.30 *[A necessary condition for reachability]*
If M is reachable in (N, M_0), then for every trap R, if $M_0(R) > 0$ then $M(R) > 0$.

Proposition 4.31 *[A sufficient condition for deadlock-freedom]*
Let (N, M_0) be a Petri net. If every proper siphon of N contains a trap marked at M_0, then (N, M_0) is deadlock-free.

5 Petri net classes with efficient decision procedures

Definition 5.1 (S-nets, S-systems) A net $N = (S, T, F)$ is a *S-net* if $|{}^\bullet t| = 1 = |t^\bullet|$ for every transition $t \in T$. A Petri net (N, M_0) is a *S-system* if N if N is a S-net.

Proposition 5.2 (Fundamental property of S-systems)
Let (N, M_0) be a S-system with $N = (S, T, F)$. Then $M_0(S) = M(S)$ for every reachable marking M.

Theorem 5.3 *[Liveness Theorem] A S-system (N, M_0) where $N = (S, T, F)$ is live iff N is strongly connected and $M_0(S) > 0$.*

Theorem 5.4 *[Boundedness Theorem] A live S-system (N, M_0) where $N = (S, T, F)$ is b-bounded iff $M_0(S) \leq b$.*

Theorem 5.5 *[Reachability Theorem] Let (N, M_0) be a live S-system and let M be a marking of N. M is reachable from M_0 iff $M_0(S) = M(S)$.*

Proposition 5.6 *[S-invariants of S-nets] Let $N = (S, T, F)$ be a connected S-net. A vector $I : S \to \mathbb{Q}$ is a S-invariant of N iff $I = (x, \ldots, x)$ for some $x \in \mathbb{Q}$.*

Definition 5.7 (T-nets, T-systems) A net $N = (S; T, F)$ is a *T-net* if $|{}^\bullet s| = 1 = |s^\bullet|$ for every place $s \in S$. A system (N, M_0) is a T-system if N is a T-net.

Proposition 5.8 (Fundamental property of T-systems) *Let γ be a circuit of a T-systems (N, M_0) and let M be a reachable marking. Then $M(\gamma) = M_0(\gamma)$.*

Theorem 5.9 *[Liveness Theorem] A T-system (N, M_0) is live iff $M_0(\gamma) > 0$ for every circuit γ of N.*

Theorem 5.10 *[Boundedness Theorem] A place s of a live T-system (N, M_0) is b-bounded iff it belongs to some circuit γ such that $M_0(\gamma) \leq b$.*

Corollary 5.11 *Let (N, M_0) be a live T-system*

1. *A place of N is bounded iff it belongs to some circuit.*
2. *Let s be a bounded place. Then*

$$max\{M(s) \mid M_0 \xrightarrow{*} M\} = min\{M_0(\gamma) \mid \gamma \text{ contains } s\}$$

3. *(N, M_0) is bounded iff N is strongly connected.*

Proposition 5.12 *[T-invariants of T-nets] Let $N = (S, T, F)$ be a connected T-net. A vector $J\colon T \to \mathbb{Q}$ is a T-invariant iff $J = (x \ldots x)$ for some $x \in \mathbb{Q}$.*

Theorem 5.13 *[Reachability Theorem] Let (N, M_0) be a live T-system. A marking M is reachable from M_0 iff $M_0 \sim M$.*

Theorem 5.14 *Let N be a strongly connected T-net. For every marking M_0 the following statements are equivalent:*

(1) *(N, M_0) is live.*

(2) *(N, M_0) is deadlock-free.*

(3) *(N, M_0) has an infinite firing sequence.*

Theorem 5.15 *[Genrich's Theorem] Let N be a strongly connected T-net with at least one place and one transition. There is a marking M_0 such that (N, M_0) is live and 1-bounded.*

Definition 5.16 (Free-Choice nets, Free-Choice systems) A net $N = (S, T, F)$ is *free-choice* if $s^\bullet \times {}^\bullet t \subseteq F$ for every $s \in S$ and $t \in T$ such that $(s, t) \in F$. A Petri net (N, M_0) is *free-choice* if N is a free-choice net..

Proposition 5.17 *[Alternative definitions of free-choice nets]*

(1) A net is free-choice if for every two transitions t_1, t_2:

$$(t_1 \neq t_2 \wedge {}^\bullet t_1 \cap {}^\bullet t_2 \neq \emptyset) \Rightarrow {}^\bullet t_1 = {}^\bullet t_2$$

(2) A net is free-choice if for every two places s_1, s_2:

$$(s_1 \neq s_2 \wedge s_1^\bullet \cap s_2^\bullet \neq \emptyset) \Rightarrow s_1^\bullet = s_2^\bullet$$

Theorem 5.18 *[Commoner's Liveness Theorem] A free-choice system (N, M_0) is live iff every siphon of N contains a trap marked at M_0.*

Theorem 5.19 *[Complexity] The problem*

> *Given: A free-choice system (N, M_0)*
> *Decide: Is (N, M_0) not live?*

is NP-complete.

Definition 5.20 (S-component) Let $N = (S, T, F)$ be a net. A subnet $N' = (S', T', F')$ of N is an *S-component* of N if

1. $T' = {}^\bullet S' \cup S'^\bullet$ (where ${}^\bullet s = \{t \in T \mid (t, s) \in F\}$, and analogously for $s^\bullet$).

2. N' is a strongly connected S-net.

Proposition 5.21 *Let (N, M_0) be a Petri net and let $N' = (S', T', F')$ be an S-component of N. Then $M_0(S') = M(S')$ for every marking M reachable from M_0.*

Theorem 5.22 *[Hack's Boundedness Theorem] Let (N, M_0) be a live free-choice system. (N, M_0) is bounded iff every place of N belongs to a S-component.*

Proposition 5.23 *[Place bounds] Let (N, M_0) be a live and bounded free-choice system and let s be a place of N. We have*

$$max\{M(s) \mid M_0 \xrightarrow{*} M\} = min\{M_0(S') \mid S' \text{ is the set of places of a S-component of } N\}$$

Definition 5.24 (Cluster) Let $N = (S, T, F)$ be a net. A *cluster* is an equivalence class of the equivalence relation $((F \cap (S \times T)) \cup (F \cap (S \times T))^{-1})^*$.

Theorem 5.25 *[The Rank Theorem] A free-choice system (N, M_0) is live and bounded iff*

1. *N has a positive S-invariant.*

2. *N has a positive T-invariant.*

3. *The rank of the incidence matrix $(\mathbf{N})$ is equal to $c - 1$, where c is the number of clusters of N.*

4. *Every siphon of N is marked under M_0.*

Theorem 5.26 **Reachability** *is NP-complete for live and bounded free-choice nets.*

Theorem 5.27 *[Reachability Theorem] Let (N, M_0) be a live, bounded, and cyclic free-choice system. A marking M of N is reachable from M_0 iff $M_0 \sim M$.*

Corollary 5.28 *The problem*

> *Given: a live, bounded, and cyclic free-choice system (N, M_0) and a marking M*
> *Decide: Is M reachable?*

can be solved in polynomial time.

Theorem 5.29 *A live and bounded free-choice system (N, M_0) is cyclic iff M_0 marks every proper trap of N.*

6 Definitions and theorems from the exercises

Definition 6.1 (Cyclic Petri nets) *A Petri net (N, M_0) is* cyclic *if, loosely speaking, it is always possible to return to the initial marking. Formally: $\forall M \in [M_0\rangle : M_0 \in [M\rangle$.*

Lemma 6.2 (Exchange Lemma) *Let u and v be transitions of a net satisfying ${}^\bullet u \cap v^\bullet = \emptyset$. If $M \xrightarrow{vu} M'$ then $M \xrightarrow{uv} M'$.*

Theorem 6.3 (Strong Connectedness Theorem) *Let (N, M_0) be a live and bounded Petri net. N is strongly connected.*

***Definition 6.4 (*Home marking)** *Let (N, M_0) be a Petri net. A marking M of the net N is a* home marking *of (N, M_0) if it is reachable from every marking of $[M_0\rangle$.*

We say that (N, M_0) has a home marking if some reachable marking is a home marking.

Definition 6.5 (Nets with transfer arcs) *A* net with transfer arcs *$N = (S, T, F, R)$ consists of two disjoint sets of places and transitions, a set $F \subseteq (S \times T) \cup (T \times S)$ of arcs, and a set $R \subseteq (S \times T) \cup (T \times S)$, disjoint from F, of* transfer arcs.

A transition t is enabled *at a marking M of N if $M(s) > 0$ for every place s such that $(s, t) \in F \cup R$. If t is enabled then it can* occur *leading to the marking M' obtained after the following operations:*

1. *Let k be the sum of the tokens in all places s such that $(s, t) \in R$, i.e., $k := \sum_{\{s \in S \mid (s,t) \in R\}} M(s)$.*

2. *Remove one token from every place s such that $(s, t) \in F$.*

3. *Remove all tokens from every place s such that $(s, t) \in R$.*

4. *Add one token to every place s such that $(t, s) \in F$.*

5. *Add k tokens to every place s such that $(t, s) \in R$.*

Definition 6.6 (Zero-reachability problem) *For a Petri net (N, M_0), is there a marking $M \in [M_0\rangle$ with $M(s) = 0$ for all $s \in S$?*

Lemma 6.7 (Reproduction lemma) *Let (N, M_0) be a bounded system and let $M_0 \xrightarrow{\sigma}$ be an infinite occurence sequence.*

1. *There exists sequences $\sigma_1, \sigma_2, \sigma_3$ such that $\sigma = \sigma_1\sigma_2\sigma_3$, σ_2 is not the empty sequence and*

$$M_0 \xrightarrow{\sigma_1} M \xrightarrow{\sigma_2} M \xrightarrow{\sigma_3}$$

   for some marking M.

2. *There exists a semi-positive T-invariant J such that $\langle J\rangle \subseteq \mathcal{A}(\sigma)$, where $\mathcal{A}(\sigma)$ is the set of transitions appearing in σ.*

Proposition 6.8 *Let N be a net and R a set of places of N. R is a trap of N* iff *for all markings M of N, if $M(R) > 0$, then $M'(R) > 0$ for all $M' \in [M\rangle$.*

Corollary 6.9 *Let (N, M_0) be a live T-system, M a marking of N and $X : T \to \mathbb{N}$ a vector such that $M = M_0 + \mathbf{N} \cdot X$. There is an occurence sequence $M_0 \xrightarrow{\sigma} M$ such that $\vec{\sigma} = X$.*