Petri nets

SS 2013

Solution to the exam
\[\langle 1, 0, 0 \rangle\]

\[\downarrow h\]

\[\langle 0, 1, 1 \rangle\]

\[\begin{align*}
\langle 1, 0, 1 \rangle & \xrightarrow{t_2} \langle 1, 0, \omega \rangle \\
\langle 0, 1, \omega \rangle & \xrightarrow{t_2} \langle 2, 0, \omega \rangle \\
\langle 2, 0, \omega \rangle & \xrightarrow{t_3} \langle \omega, 0, \omega \rangle \\
\langle \omega, 0, \omega \rangle & \xrightarrow{t_3} \langle \omega, \omega, \omega \rangle \\
\langle \omega, \omega, \omega \rangle & \xrightarrow{t_2} \langle 1, 1, \omega \rangle \\
\langle 1, 1, \omega \rangle & \xrightarrow{t_2} \langle \omega, \omega, \omega \rangle \\
\langle \omega, \omega, \omega \rangle & \xrightarrow{t_2} \langle 1, \omega, 1 \rangle \\
\langle 1, \omega, 1 \rangle & \xrightarrow{t_2} \langle h, h, h \rangle
\end{align*}\]

(will continue with the addition of \(\omega\)’s)
(a) Incident matrix:
\[
\begin{pmatrix}
1 & -1 \\
1 & -1 \\
-1 & 1
\end{pmatrix}
\]
\[C = \begin{pmatrix}
1 \\
1
\end{pmatrix} \Rightarrow (1,1) \text{ is } T\text{-invariant.}
\]
\[(1,1,2), C = 0 \Rightarrow (1,1,2) \text{ is } P\text{-invariant.}
\]

(b) Incident matrix:
\[
\begin{pmatrix}
1 \\
1
\end{pmatrix} \Rightarrow (1,1,2) \text{ is } P\text{-invariant.}
\]

There are no vectors \(X, Y \neq 0\) such that \(C.X = 0, \ Y.C = 0\) \(\Rightarrow\) the net has no \(T\)-invariant and no \(P\)-invariant.

(c) live and bounded
\[
\begin{pmatrix}
1 & -1 \\
1 & -1 \\
-1 & 1
\end{pmatrix}
\]
non-live and bounded
\[
\begin{pmatrix}
1 & -1 \\
1 & -1 \\
-1 & 1
\end{pmatrix}
\]
non-live and unbounded
\[
\begin{pmatrix}
1 & -1 \\
1 & -1 \\
-1 & 1
\end{pmatrix}
\]
non-live and unbounded.
By Commooner's theorem, a free-choice Petri net is live if and only if every proper sub-net contains an initially marked trap.

Assume a free-choice net \((N, M_0)\) is live and \(M \geq M_0\).

By Commooner's theorem, every proper sub-net of \(N\) contains a trap marked at \(M_0\).

Since \(N \geq M_0\), every proper sub-net of \(N\) contains a trap marked at \(M_0\).

By Commooner's theorem, \((N, M)\) is live.
b) If the formula is satisfiable, then there is a truth assignment that satisfies all clauses. From the transitions corresponding to this assignment put at least one token (possibly up to three) in each of the places $C_i$. This allows to fire transitions $t_5$, and the transitions $gc_1, \ldots, gc_m$ can take all remaining tokens from $C_1, \ldots, C_m$, reaching the empty marking.

c) If the formula is unsatisfiable, then independently of how the transitions at the top occur, after one of each pair has fired at least one of the places $C_1, \ldots, C_m$ is not marked. So the transition $t_5$ can never occur, and so the empty marking is not reachable.
The number of places is linear in the size of the formula, and so is the number of transitions (number of places is bounded by $n+m+1$ for a formula with $n$ variables and $m$ clauses, and number of transitions is bounded by $2n+m+1$).

Moreover, the places and transitions can be constructed in linear time during a single scan of the formula.