Decision Procedures
An Algorithmic Point of View

SAT

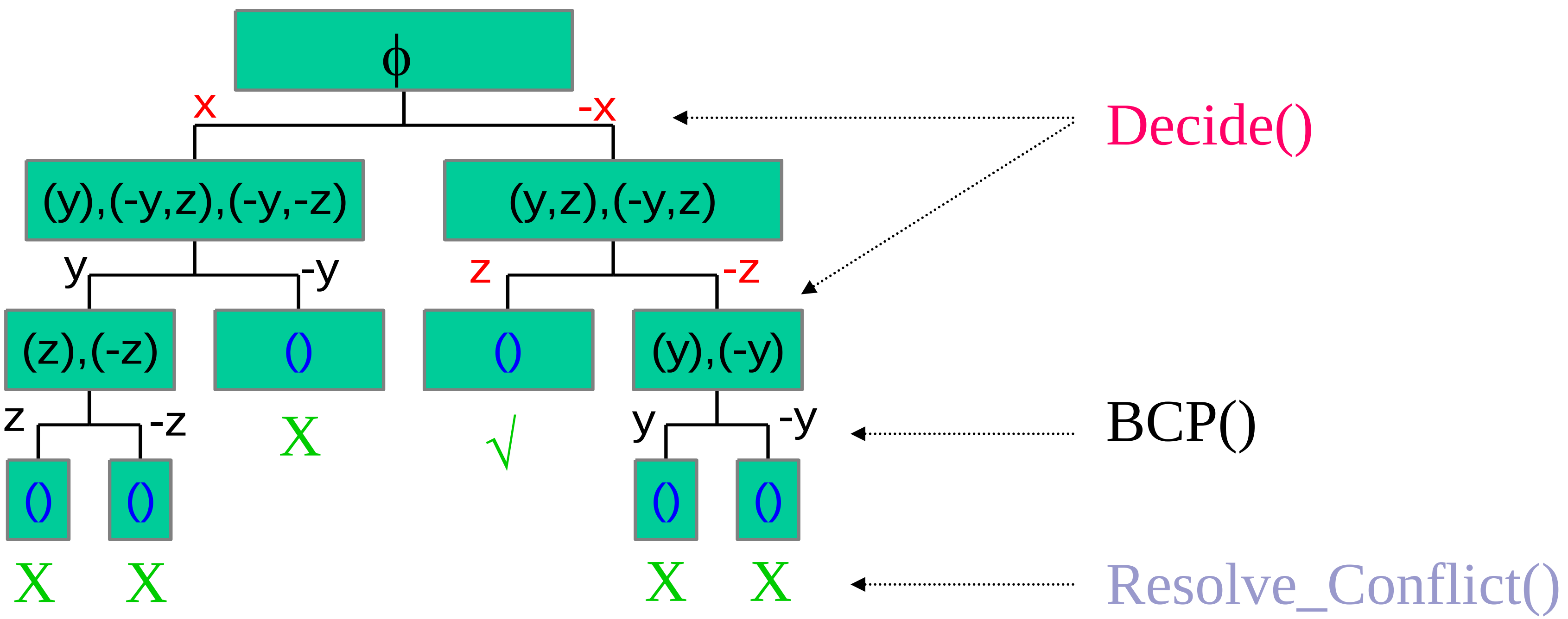
(slides from http://www.decision-procedures.org/)

Next: Deciding Propositional Formulas

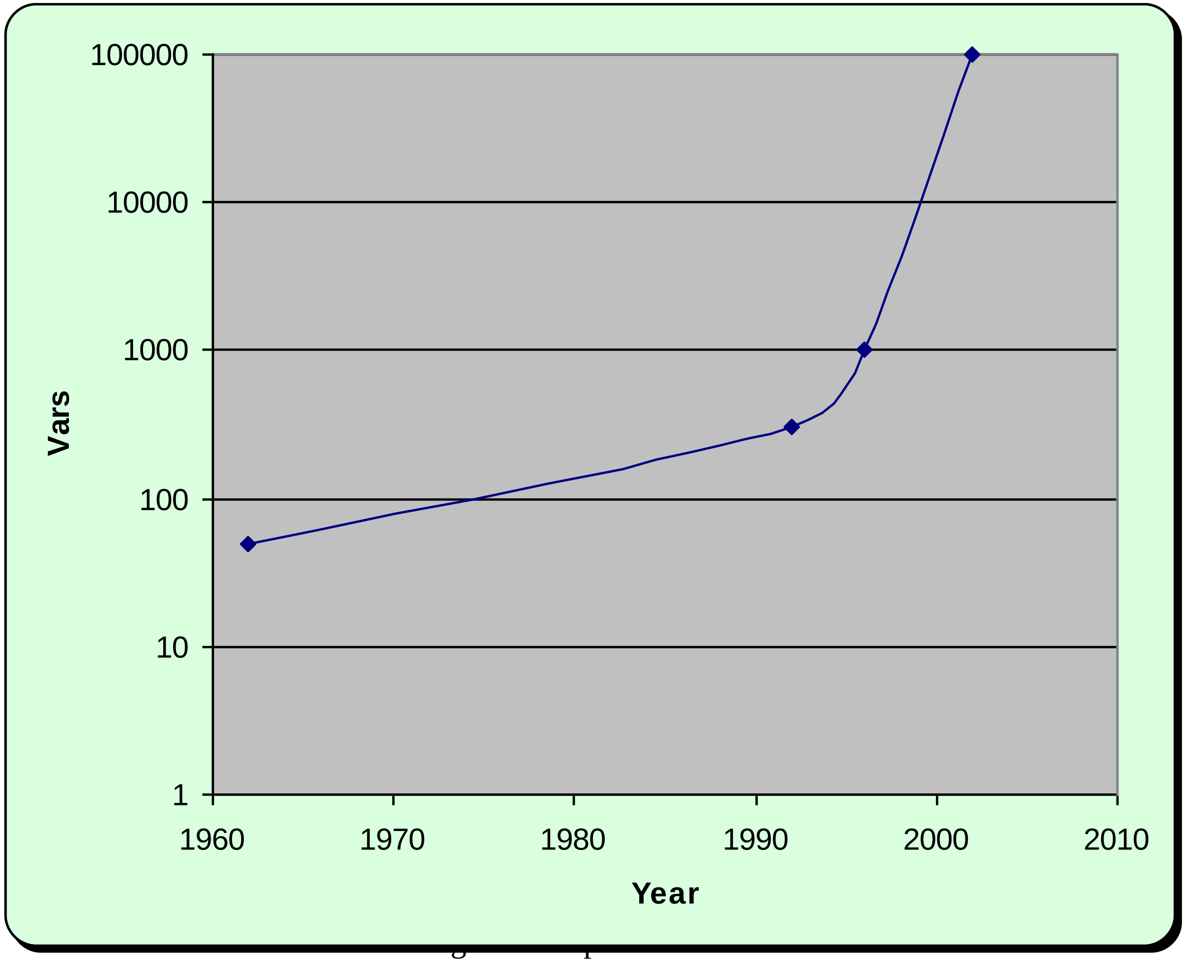
- SAT solvers
- Binary Decision Diagrams

A Basic SAT algorithm

- Given ϕ in CNF: (x,y,z),(-x,y),(-y,z),(-x,-y,-z)



SAT made some progress…



A Basic SAT algorithm

```
While (true)
{
        if (!Decide()) return (SAT);
        while (!BCP())
                if (!Resolve_Conflict()) return (UNSAT);
}
```

Choose the next variable and value. Return False if all variables are assigned

Apply repeatedly the *unit clause rule*. Return False if reached a conflict

Backtrack until no conflict. Return False if impossible

Basic Backtracking Search

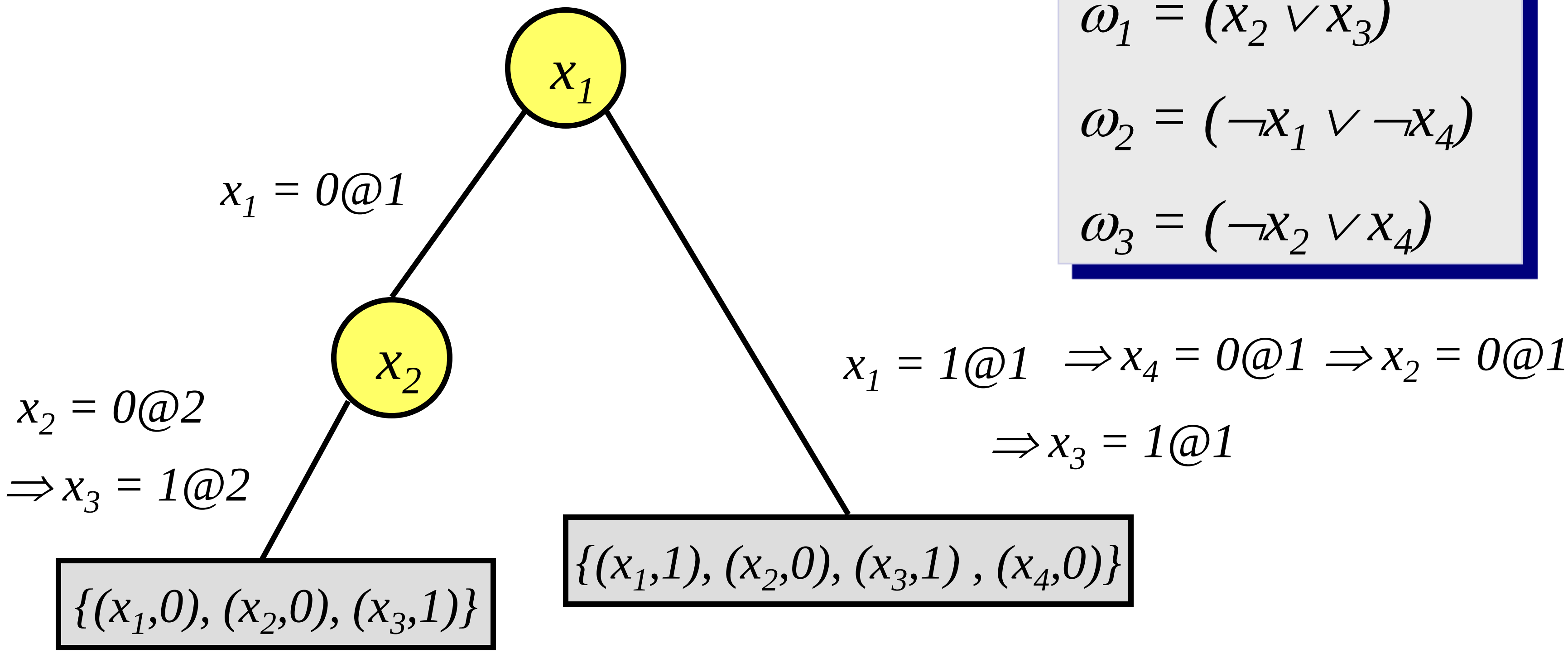
- Organize the search in the form of a decision tree
  - Each node corresponds to a decision
  - Definition: Decision Level (DL) is the depth of the node in the decision tree.
  - Notation: $x=v@d$
    $x \in \{0,1\}$ is assigned to v at decision level d

Backtracking Search in Action

$\omega_1 = (x_2 \vee x_3)$

$\omega_2 = (\neg x_1 \vee \neg x_4)$

$\omega_3 = (\neg x_2 \vee x_4)$



No backtrack in this example, regardless of the decision!

Backtracking Search in Action

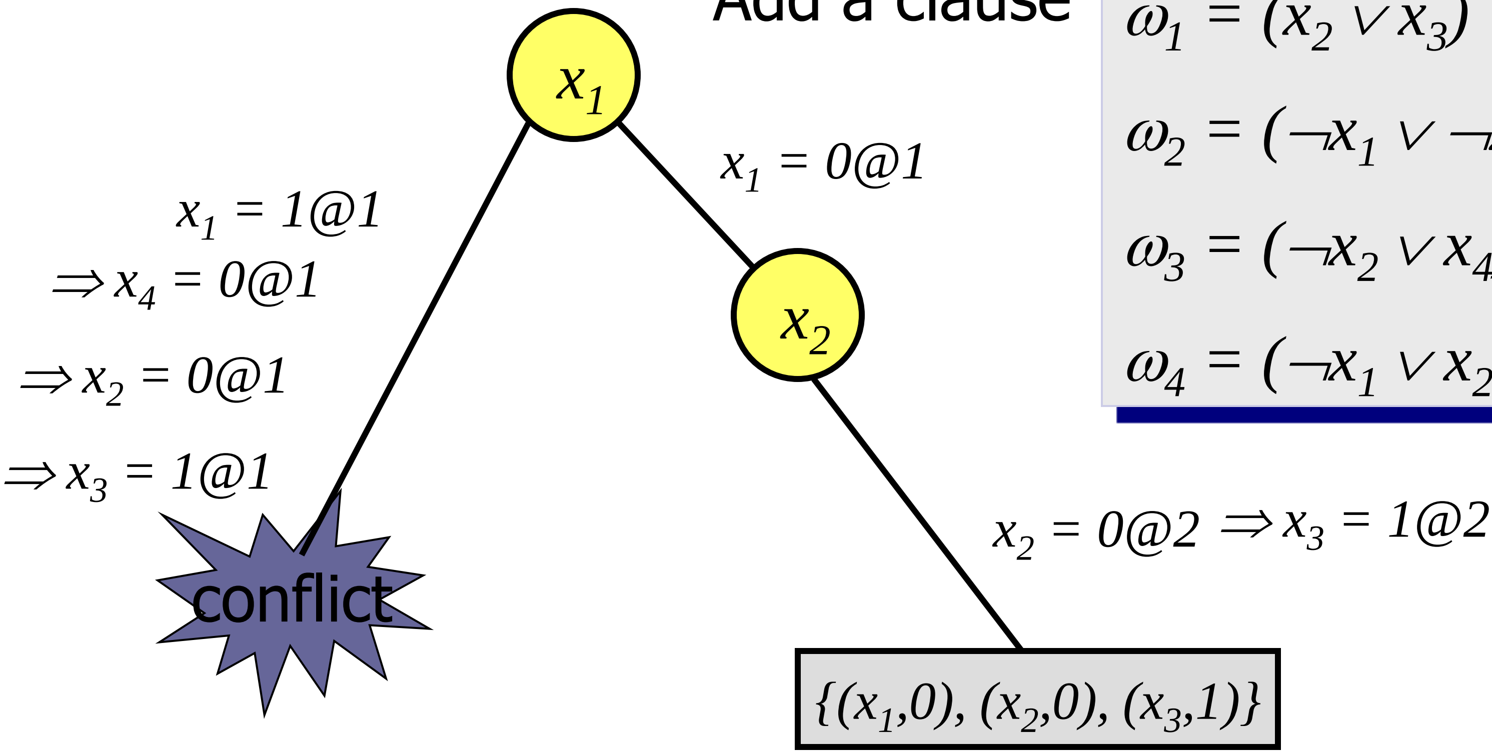
Add a clause

$\omega_1 = (x_2 \vee x_3)$

$\omega_2 = (\neg x_1 \vee \neg x_4)$

$\omega_3 = (\neg x_2 \vee x_4)$

$\omega_4 = (\neg x_1 \vee x_2 \vee \neg x_3)$



Status of a clause

- A clause can be
  - Satisfied: at least one literal is satisfied
  - Unsatisfied: all literals are assigned but non are satisfied
  - Unit: all but one literals are assigned but none are satisfied
  - Unresolved: all other cases
- Example: $C = (x_1 \vee x_2 \vee x_3)$

x_1	x_2	x_3	C
1	0		Satisfied
0	0	0	Unsatisfied
0	0		Unit
	0		Unresolved

Decision heuristics - DLIS

DLIS (Dynamic Largest Individual Sum) – choose the assignment that increases the most the number of satisfied clauses

- For a given variable x:
  - C_{xp} – # unresolved clauses in which x appears positively
  - C_{xn} - # unresolved clauses in which x appears negatively
  - Let x be the literal for which C_{xp} is maximal
  - Let y be the literal for which C_{yn} is maximal
  - If $C_{xp} > C_{yn}$ choose x and assign it TRUE
  - Otherwise choose y and assign it FALSE
- Requires l (#literals) queries for each decision.

Decision heuristics - JW

Jeroslow-Wang method

Compute for every clause ω and every variable l (in each phase):

- $J(l) := \sum_{l \in \omega, \omega \in \varphi} 2^{-|\omega|}$
- Choose a variable l that maximizes $J(l)$.
- This gives an exponentially higher weight to literals in shorter clauses.

Pause... ||

- We will see other (more advanced) decision Heuristics soon.
- These heuristics are integrated with a mechanism called Learning with Conflict-Clauses, which we will learn next.

Implication graphs and learning: option #1

Current truth assignment: $\{x_9=0@1, x_{10}=0@3, x_{11}=0@3, x_{12}=1@2, x_{13}=1@2\}$

Current decision assignment: $\{x_1=1@6\}$

$\omega_1 = (\neg x_1 \vee x_2)$

$\omega_2 = (\neg x_1 \vee x_3 \vee x_9)$

$\omega_3 = (\neg x_2 \vee \neg x_3 \vee x_4)$

$\omega_4 = (\neg x_4 \vee x_5 \vee x_{10})$

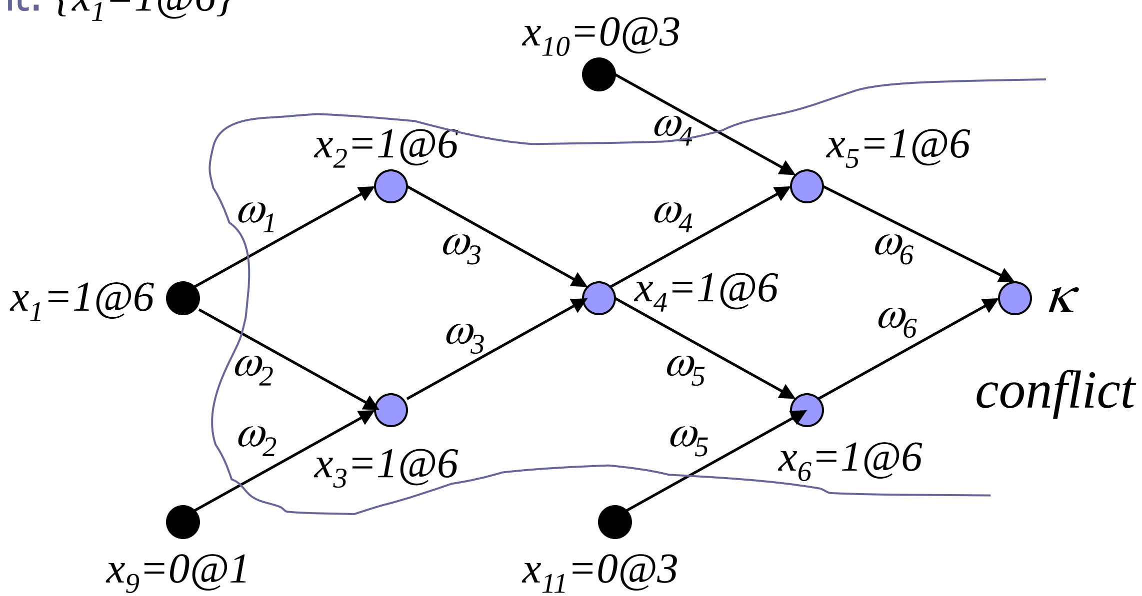
$\omega_5 = (\neg x_4 \vee x_6 \vee x_{11})$

$\omega_6 = (\neg x_5 \vee \neg x_6)$

$\omega_7 = (x_1 \vee x_7 \vee \neg x_{12})$

$\omega_8 = (x_1 \vee x_8)$

$\omega_9 = (\neg x_7 \vee \neg x_8 \vee \neg x_{13})$



We learn the *conflict clause* $\omega_{10} : (\neg x_1 \vee x_9 \vee x_{11} \vee x_{10})$

Implication graph, flipped assignment option #1

$\omega_1 = (\neg x_1 \vee x_2)$

$\omega_2 = (\neg x_1 \vee x_3 \vee x_9)$

$\omega_3 = (\neg x_2 \vee \neg x_3 \vee x_4)$

$\omega_4 = (\neg x_4 \vee x_5 \vee x_{10})$

$\omega_5 = (\neg x_4 \vee x_6 \vee x_{11})$

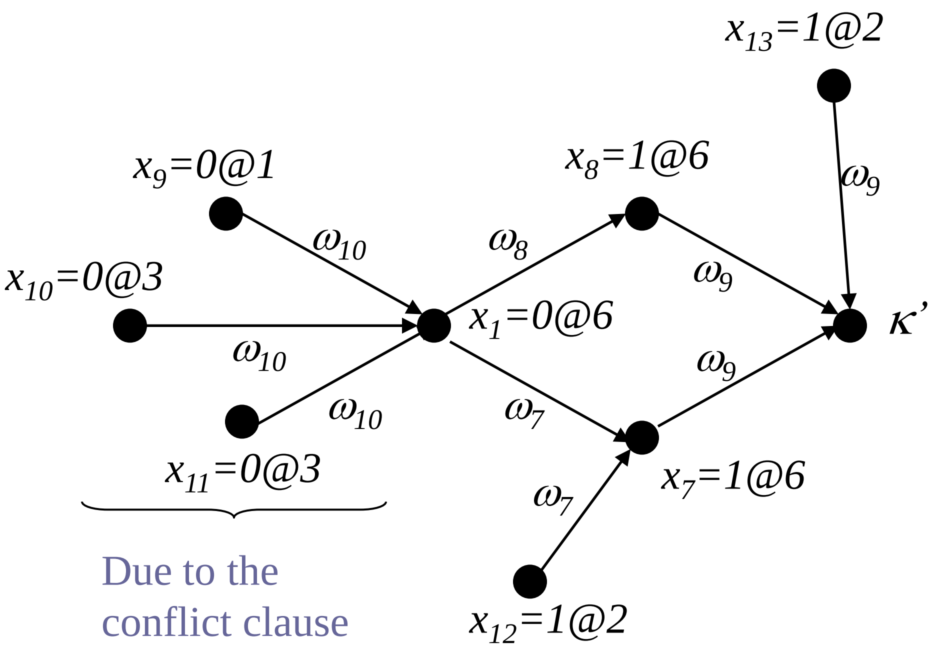
$\omega_6 = (\neg x_5 \vee x_6)$

$\omega_7 = (x_1 \vee x_7 \vee \neg x_{12})$

$\omega_8 = (x_1 \vee x_8)$

$\omega_9 = (\neg x_7 \vee \neg x_8 \vee \neg x_{13})$

$\omega_{10} : (\neg x_1 \vee x_9 \vee x_{11} \vee x_{10})$



No decision here

Another conflict clause: ω_{11}: $(\neg x_{13} \vee \neg x_{12} \vee x_{11} \vee x_{10} \vee x_9)$

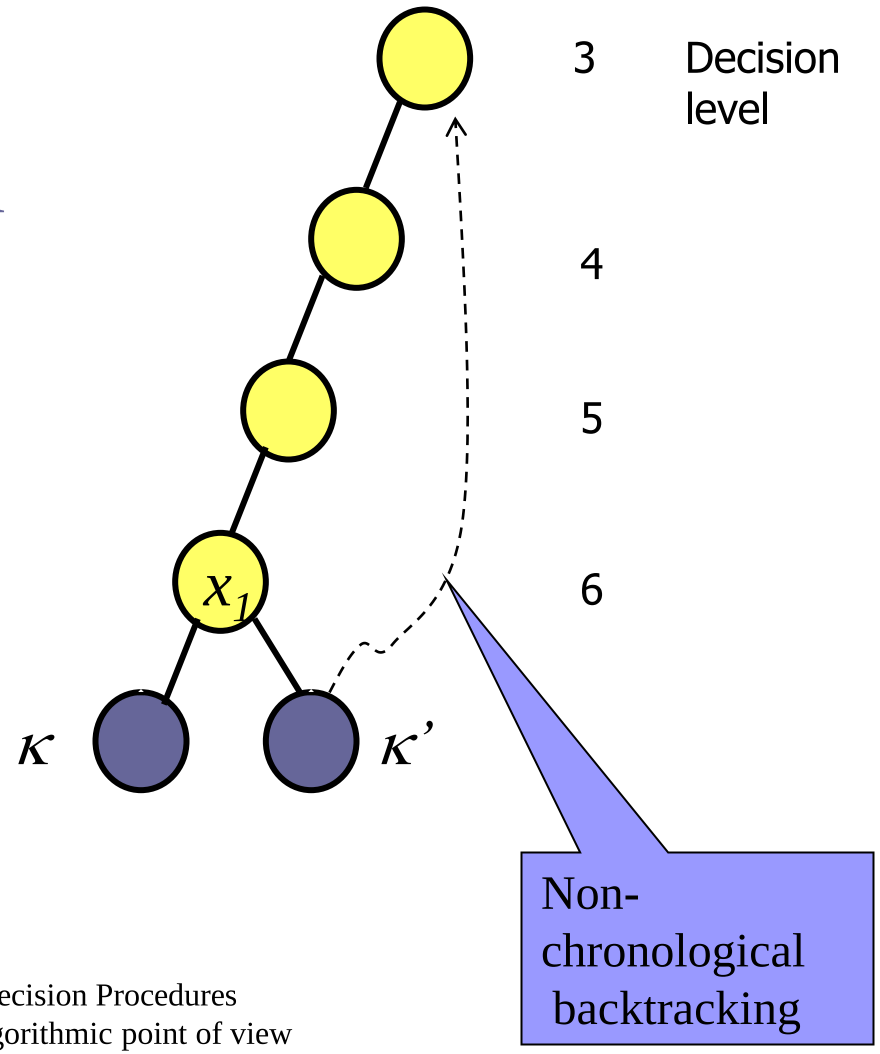
where should we backtrack to now ?

Non-chronological backtracking

Which assignments caused the conflicts ?

$x_9 = 0@1$
$x_{10} = 0@3$
$x_{11} = 0@3$
$x_{12} = 1@2$
$x_{13} = 1@2$

These assignments Are sufficient for Causing a conflict.

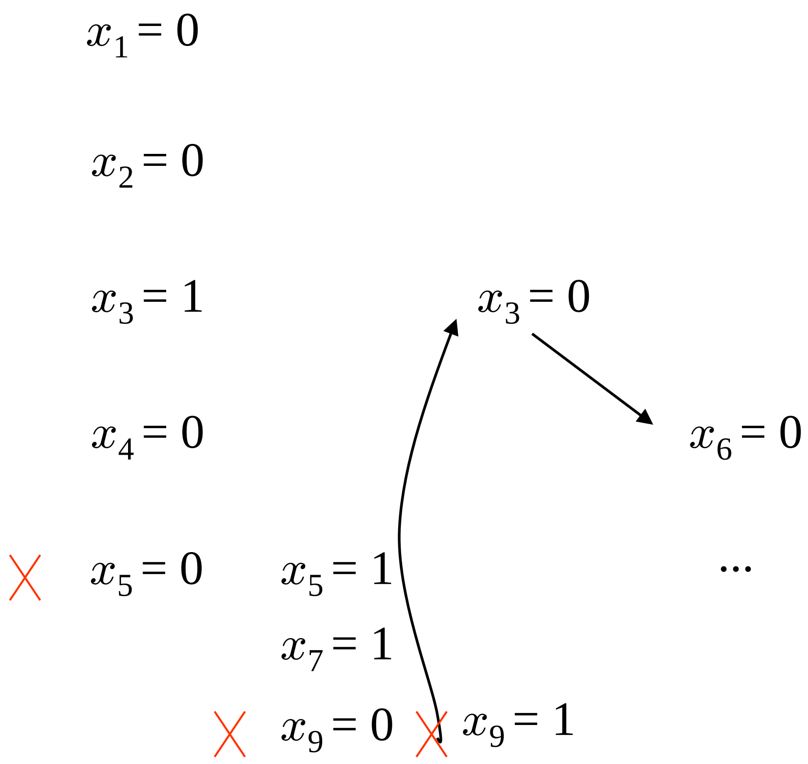


Backtrack to DL = 3

Non-chronological backtracking

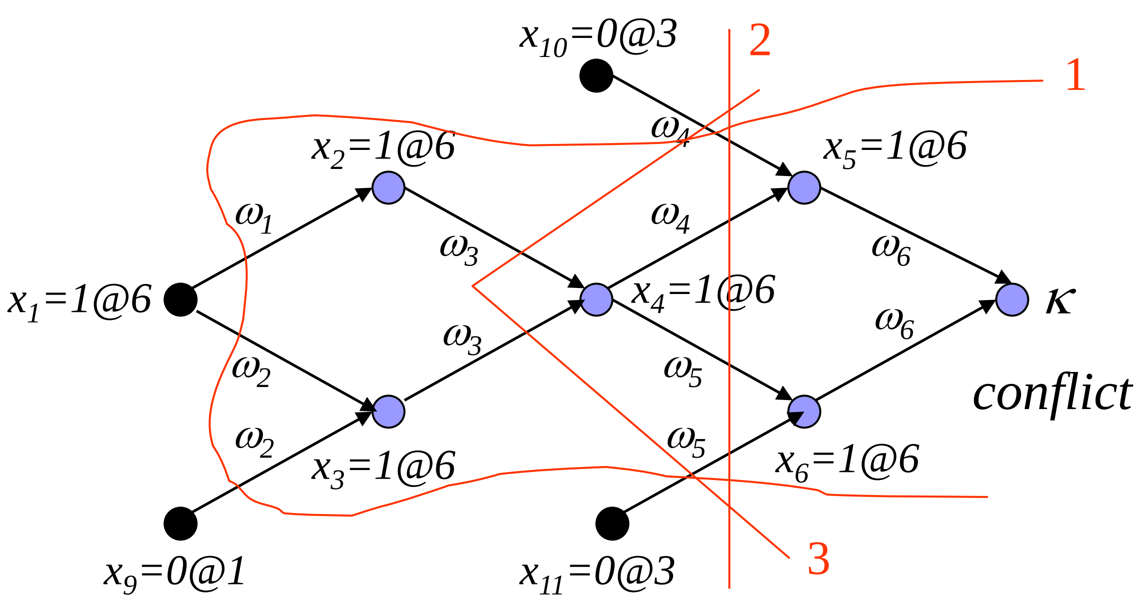
- So the rule is: backtrack to the largest decision level in the conflict clause.
- This works for both the initial conflict and the conflict after the flip.
- Q: What if the flipped assignment works?
  A: Change the decision retroactively.

Non-chronological Backtracking



More Conflict Clauses

- Def: A Conflict Clause is any clause implied by the formula
- Let L be a set of literals labeling nodes that form a cut in the implication graph, separating the conflict node from the roots.
- Claim: $\bigvee_{l \in L} \neg l$ is a Conflict Clause.



1. $(x_{10} \vee \neg x_1 \vee x_9 \vee x_{11})$

2. $(x_{10} \vee \neg x_4 \vee x_{11})$

3. $(x_{10} \vee \neg x_2 \vee \neg x_3 \vee x_{11})$

⋮

Conflict clauses

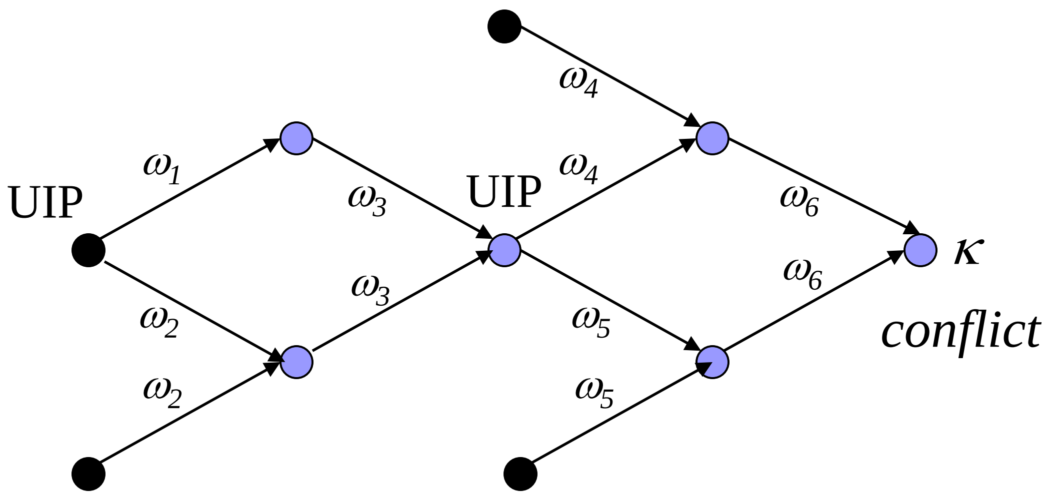
- How many clauses should we add ?
- If not all, then which ones ?
  - Shorter ones ?
  - Check their influence on the backtracking level ?
  - The most “influential” ?

Conflict clauses

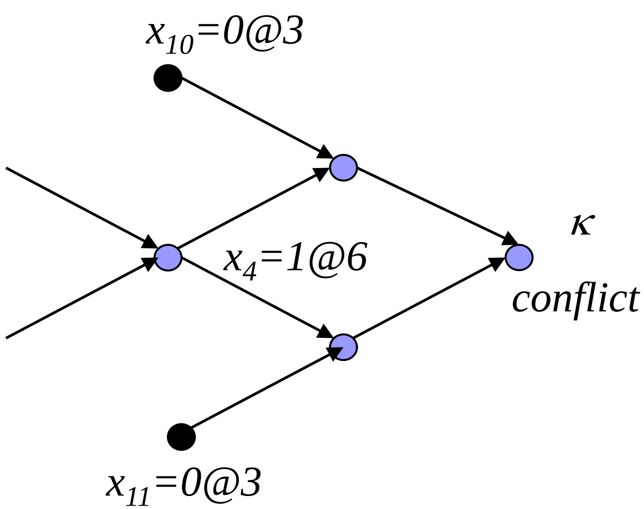
- Def: An Asserting Clause is a Conflict Clause with a single literal from the current decision level. Backtracking (to the right level) makes it a Unit clause.
- Asserting clauses are those that force an immediate change in the search path.
- Modern solvers only consider Asserting Clauses.

Unique Implication Points (UIP's)

- Definition: A Unique Implication Point (UIP) is an internal node in the Implication Graph that all paths from the decision to the conflict node go through it.
- The First-UIP is the closest UIP to the conflict.

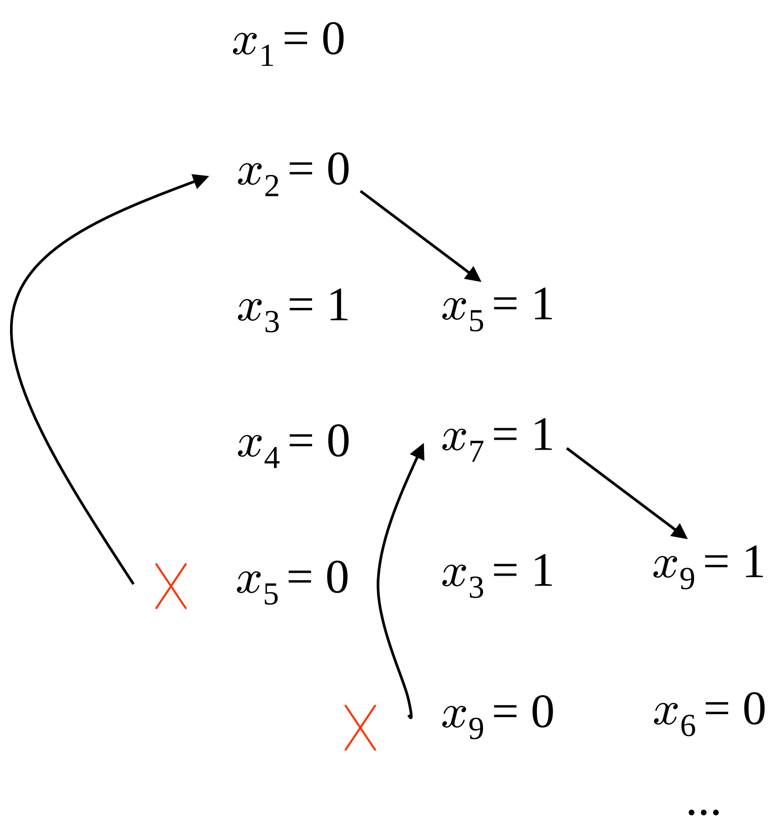


Conflict-driven backtracking (option #2)

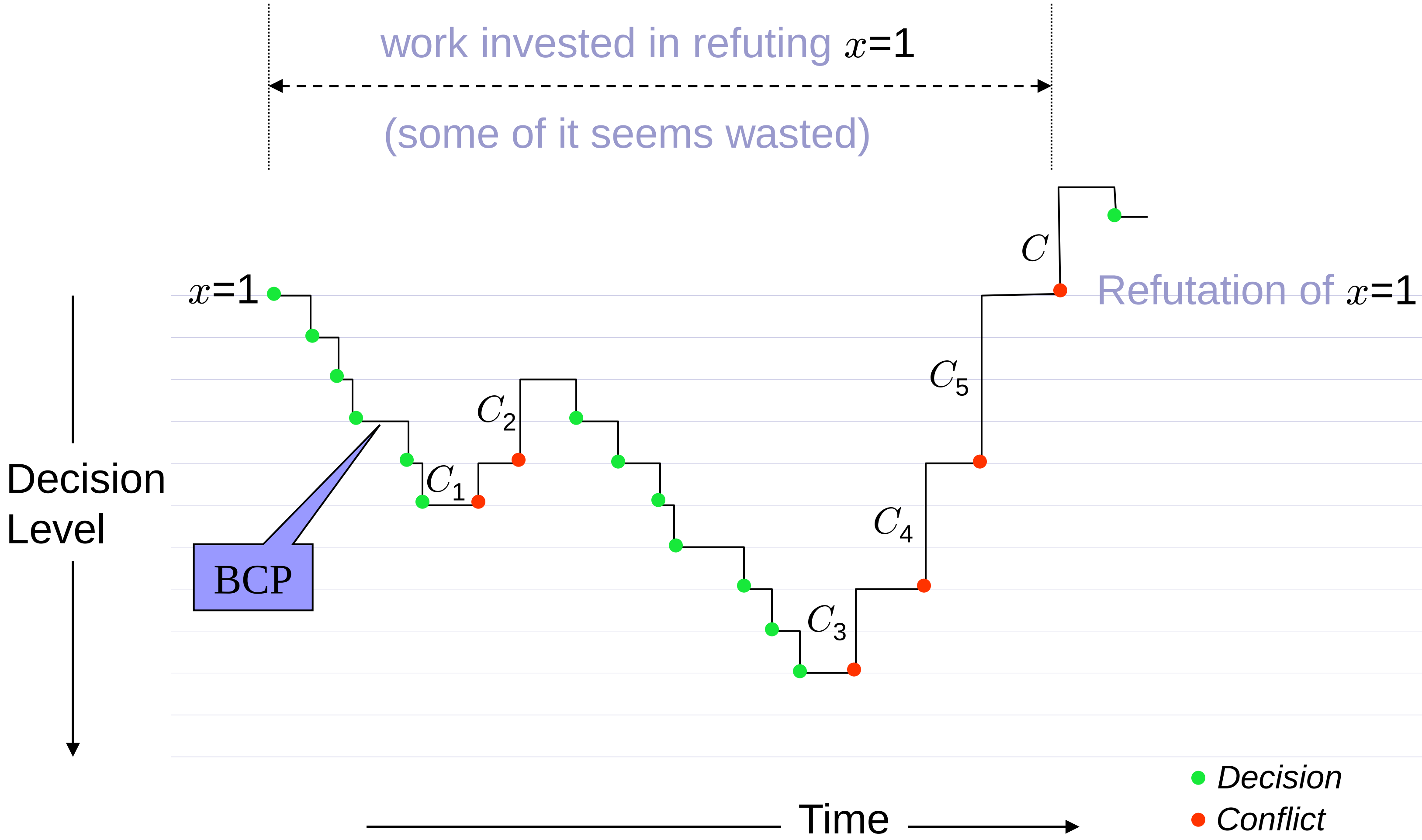


- Conflict clause: $(x_{10} \vee \neg x_4 \vee x_{11})$
- With standard Non-Chronological Backtracking we backtracked to DL = 6.
- Conflict-driven Backtrack: backtrack to the second highest decision level in the clause (without erasing it).
- In this case, to DL = 3.
- Q: why?

Conflict-driven Non-chronological Backtracking



Progress of a SAT solver



Conflict-Driven Backtracking

- So the rule is: backtrack to the second highest decision level dl, but do not erase it.
- This way the literal with the currently highest decision level will be implied in DL = dl.
- Q: what if the conflict clause has a single literal ?
  - For example, from $(x \vee \neg y) \wedge (x \vee y)$ and decision x=0, we learn the conflict clause (x).

Conflict clauses and Resolution

- The Binary-resolution is a sound inference rule:

$$\frac{(a_1 \vee \ldots \vee a_n \vee \beta) \qquad (b_1 \vee \ldots \vee b_m \vee (\neg\beta))}{(a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m)} \quad \text{(Binary Resolution)}$$

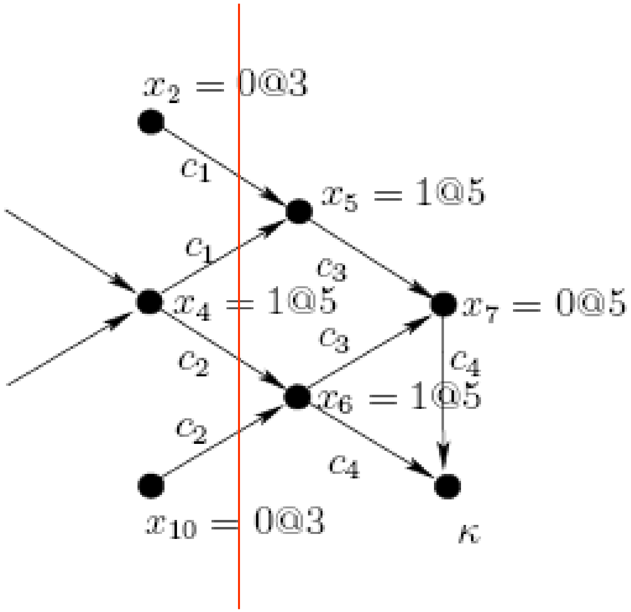
- Example:

$$\frac{(x_1 \vee x_2) \qquad (\neg x_1 \vee x_3 \vee x_4)}{(x_2 \vee x_3 \vee x_4)}$$

Conflict clauses and resolution

- Consider the following example:

$$
\begin{aligned}
c_1 &= (\neg x_4 \vee x_2 \vee x_5) \\
c_2 &= (\neg x_4 \vee x_{10} \vee x_6) \\
c_3 &= (\neg x_5 \vee \neg x_6 \vee \neg x_7) \\
c_4 &= (\neg x_6 \vee x_7) \\
&\vdots \qquad \vdots
\end{aligned}
$$

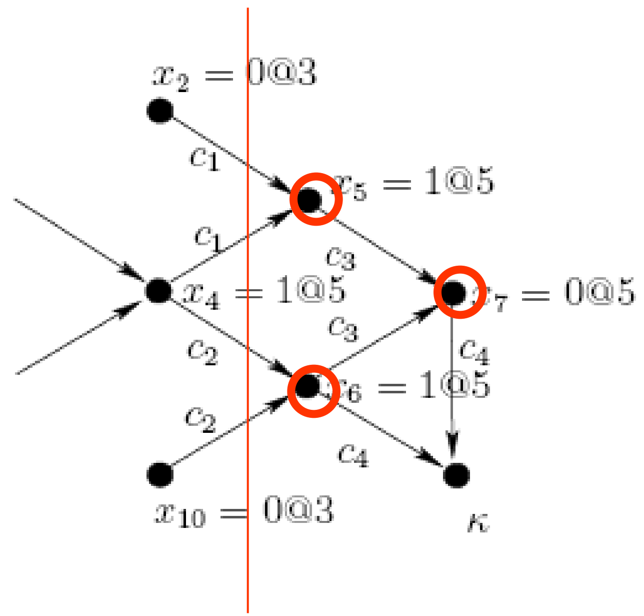


- Conflict clause: c_5: $(x_2 \vee \neg x_4 \vee x_{10})$

Conflict clauses and resolution

- Conflict clause: c_5: $(\boldsymbol{x_2} \vee \neg \boldsymbol{x_4} \vee \boldsymbol{x_{10}})$

$$\begin{array}{lcl} c_1 & = & (\neg x_4 \vee x_2 \vee x_5) \\ c_2 & = & (\neg x_4 \vee x_{10} \vee x_6) \\ c_3 & = & (\neg x_5 \vee \neg x_6 \vee \neg x_7) \\ c_4 & = & (\neg x_6 \vee x_7) \\ \vdots & & \vdots \end{array}$$



- Resolution order: x_4, x_5, x_6, x_7
  - T1 = Res(c_4, c_3, x_7) = $(\neg x_5 \vee \neg x_6)$
  - T2 = Res(T1, c_2, x_6) = $(\neg x_4 \vee \neg x_5 \vee X_{10})$
  - T3 = Res(T2, c_1, x_5) = $(\mathbf{x_2} \vee \neg \mathbf{x_4} \vee \mathbf{x_{10}})$

Finding the conflict clause:

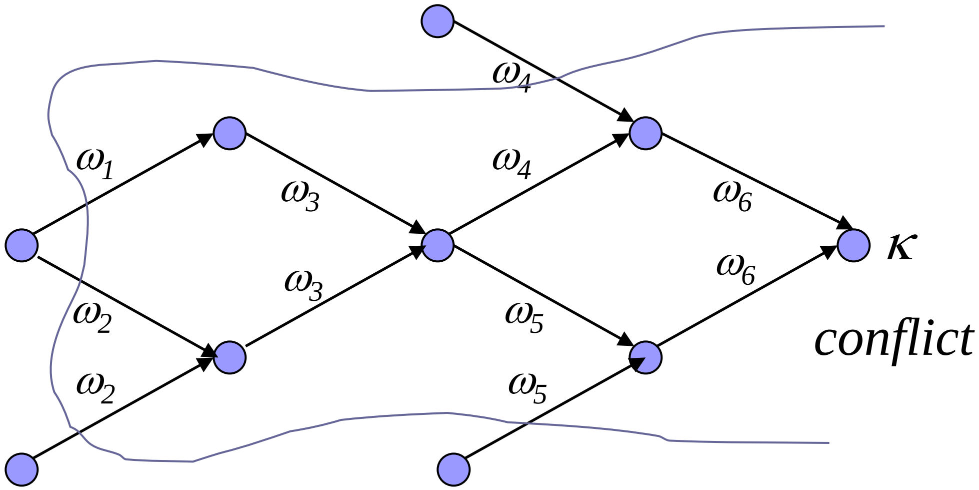
```
procedure Analyze-Conflict
    if current-decision-level = 0 then return -1;
    cl := current-conflicting-clause;
    while (¬Stop-criterion-met(cl)) do
        lit := Last-assigned-literal(cl);
        var := Variable-of-literal(lit);
        ante := Antecedent(var);
        cl := Resolve(cl, ante,var);
    add-clause-to-database(cl);
```

cl is asserting the first UIP

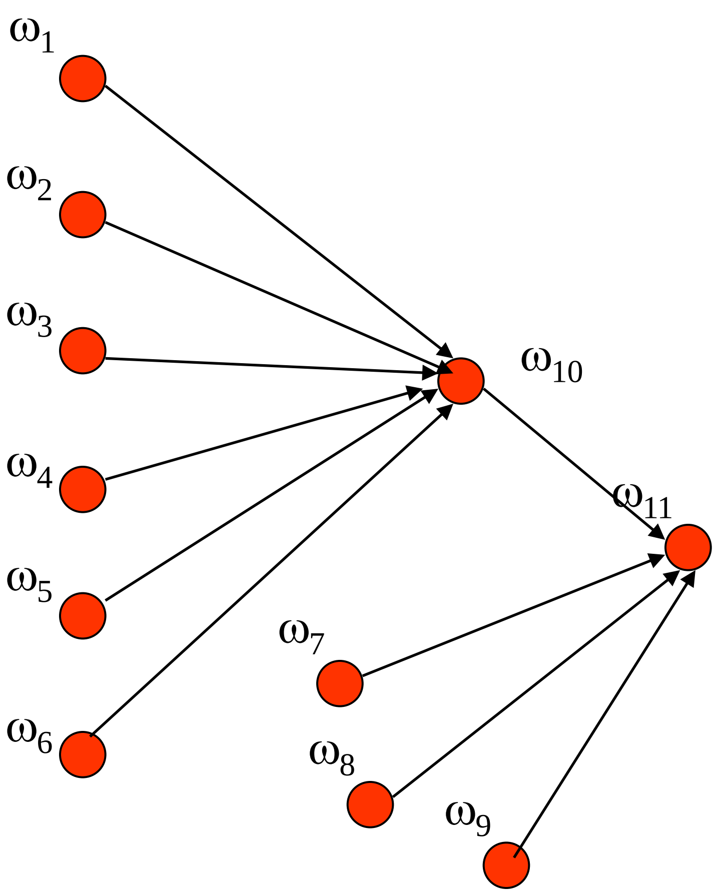
Applied to our example:

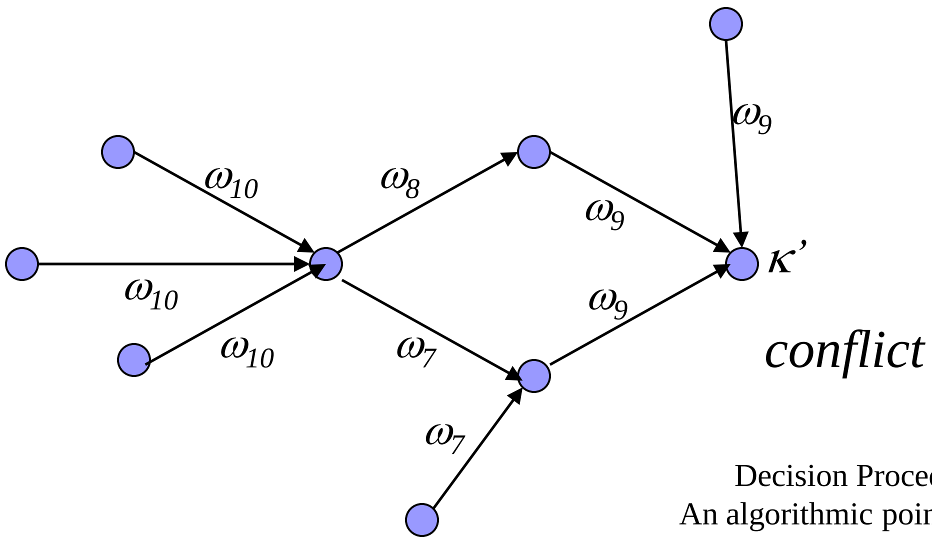
name	cl	lit	var	$ante$
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3
	$(\neg x_5 \vee \neg x_6)$	$\neg x_6$	x_6	c_2
	$(\neg x_4 \vee x_{10} \vee \neg x_5)$	$\neg x_5$	x_5	c_1
c_5	$(\neg x_4 \vee x_2 \vee x_{10})$			

The Resolution-Graph keeps track of the “inference relation”



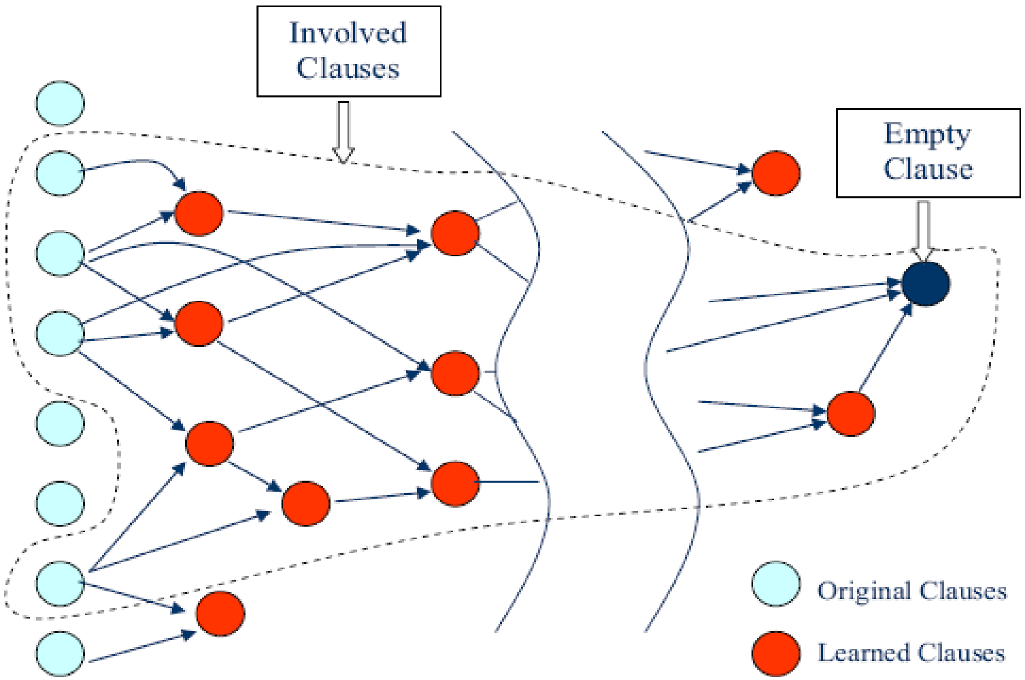
Resolution Graph





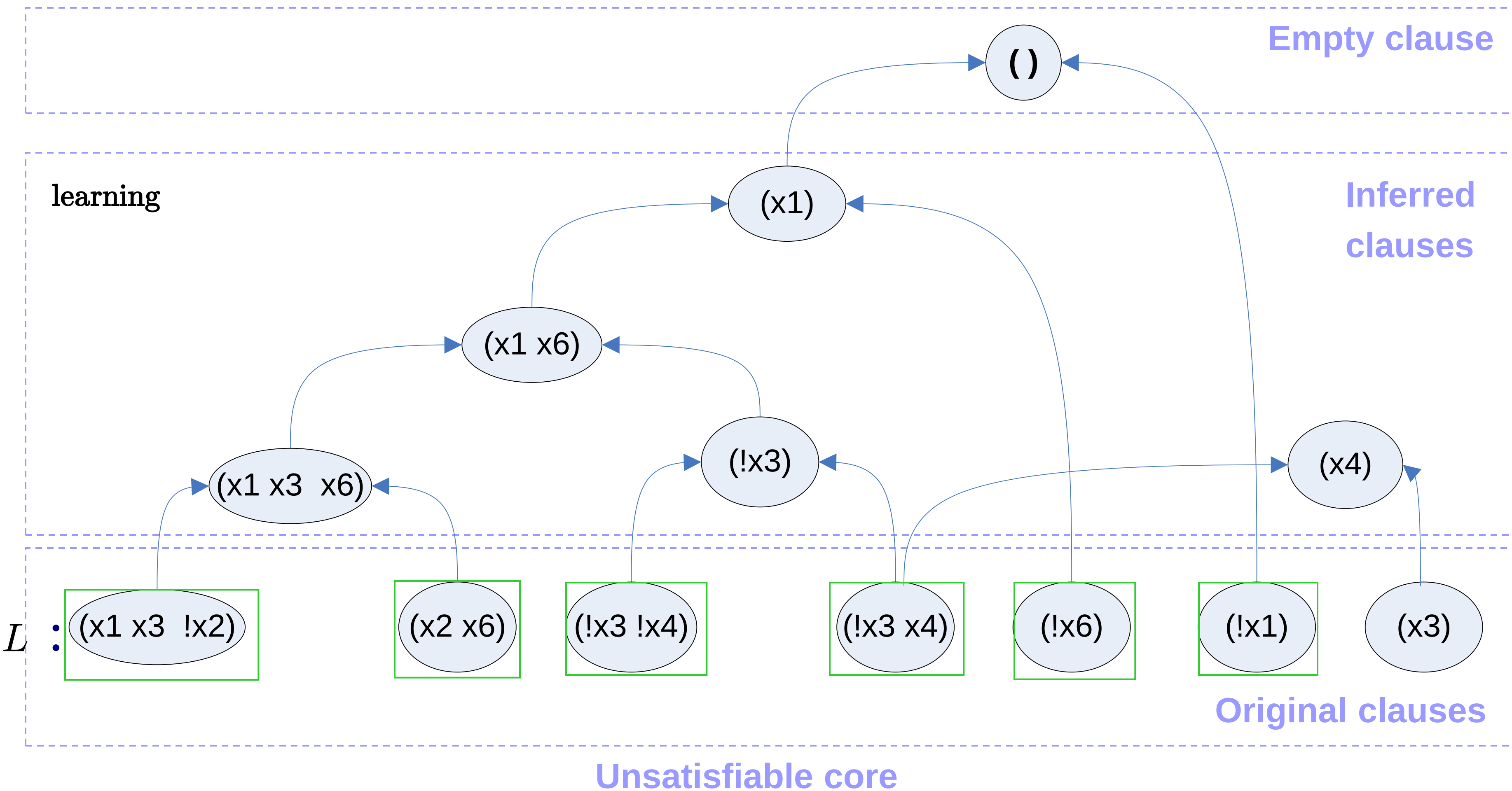
The resolution graph

What is it good for ?
Example: for computing an **Unsatisfiable core**



[Picture Borrowed from Zhang, Malik SAT'03]

Resolution graph: example



Decision heuristics - VSIDS

VSIDS (Variable State Independent Decaying Sum)

1. Each variable in each polarity has a counter initialized to 0.
2. When a clause is added, the counters are updated.
3. The unassigned variable with the highest counter is chosen.
4. Periodically, all the counters are divided by a constant.

(Implemented in **Chaff**)

Decision heuristics – VSIDS (cont’d)

- **Chaff** holds a list of unassigned variables sorted by the counter value.
- Updates are needed only when adding conflict clauses.
- Thus - decision is made in constant time.

Decision heuristics

VSIDS (cont'd)

VSIDS is a 'quasi-static' strategy:

- *static* because it doesn't depend on current assignment

- *dynamic* because it gradually changes. Variables that appear in recent conflicts have higher priority.

This strategy is a *conflict-driven* decision strategy.

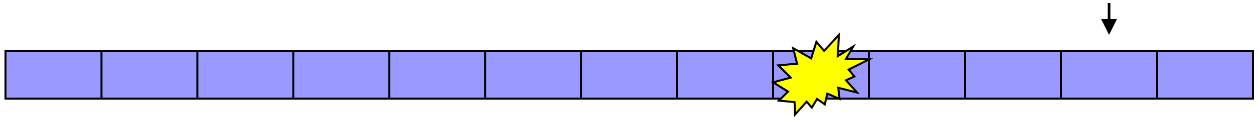
"..employing this strategy dramatically (i.e. an order of magnitude) improved performance ... "

Decision Heuristics - Berkmin

- Keep conflict clauses in a stack
- Choose the first unresolved clause in the stack
  - If there is no such clause, use VSIDS
- Choose from this clause a variable + value according to some scoring (e.g. VSIDS)

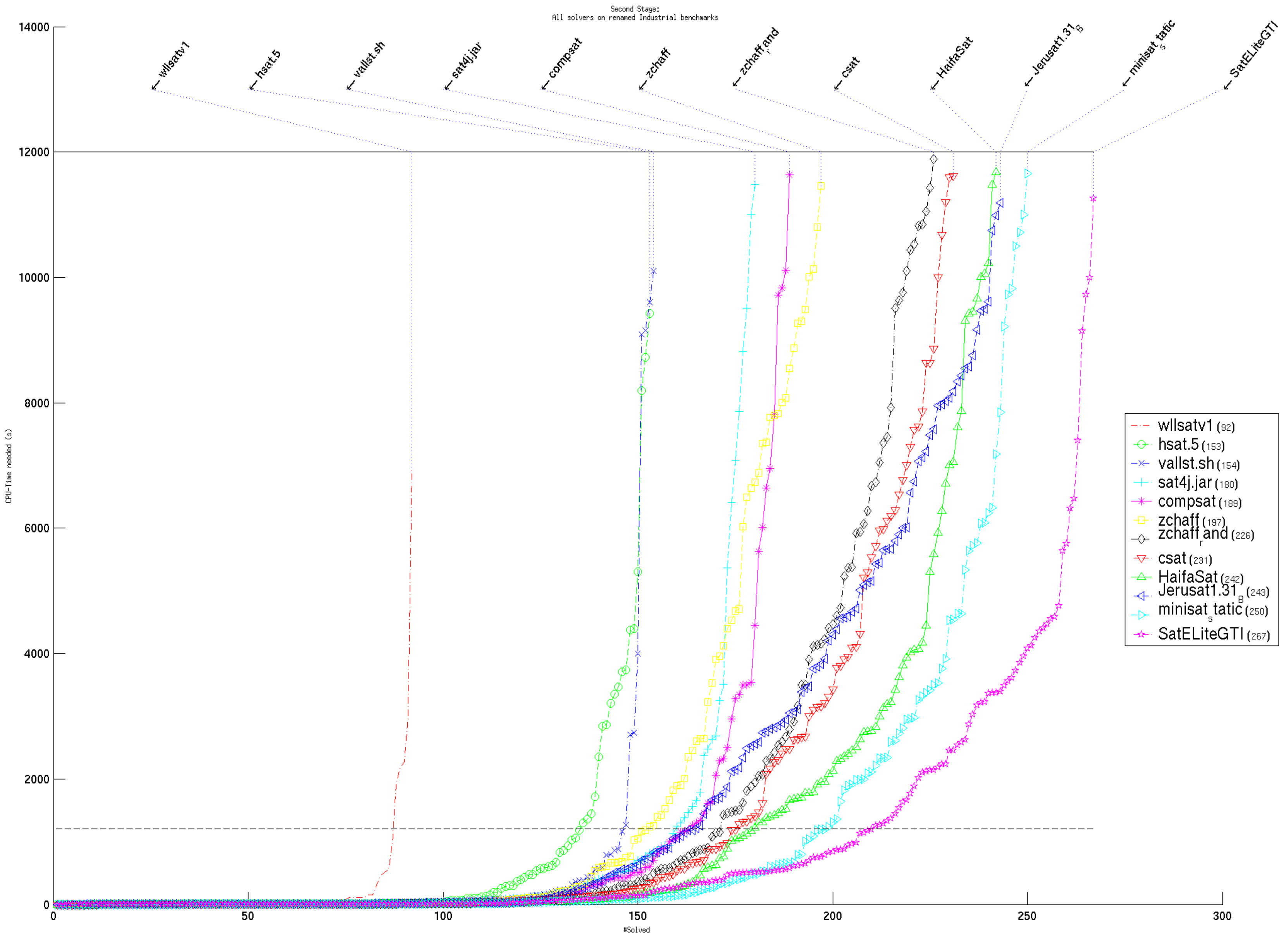
- This gives absolute priority to conflicts.

Berkmin heuristic



tail-
first conflict clause

The SAT competitions



- End of SAT (for now)
- Beginning of Binary Decision Diagrams