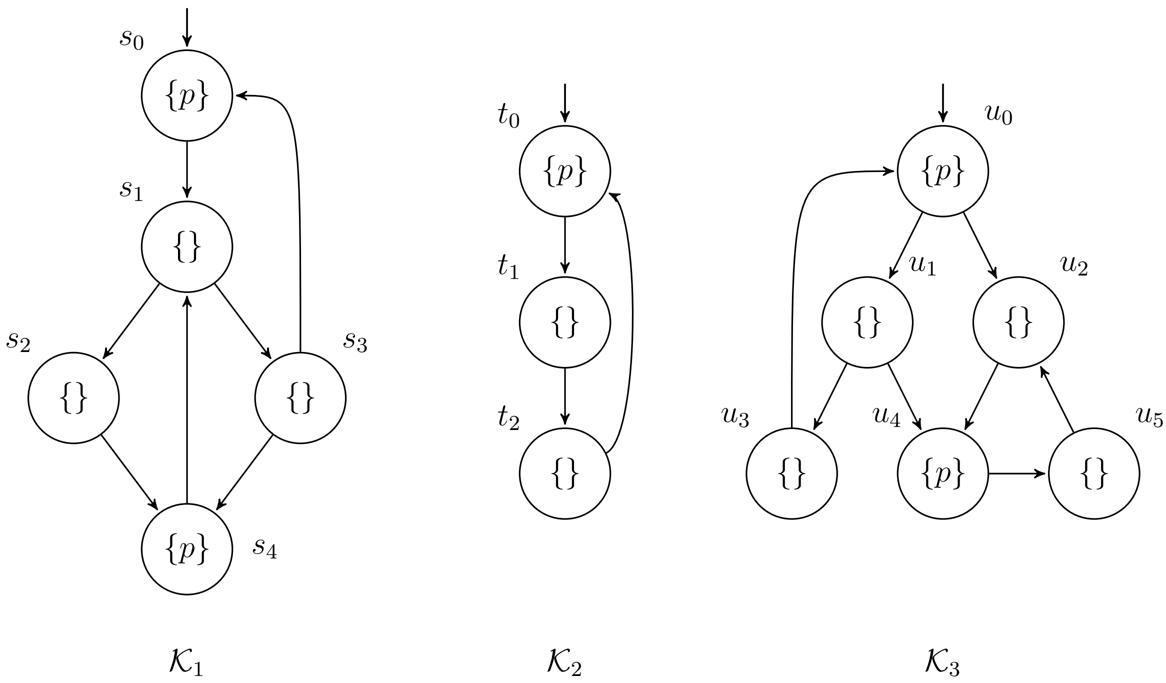
Model Checking – Exercise sheet 10

Exercise 10.1
Consider the following Kripke structures $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$, over $AP = \{p\}$:



(a) Does $\mathcal{K}_2$ simulate $\mathcal{K}_1$? If yes, give a simulation relation. Otherwise, explain why.

(b) Does $\mathcal{K}_2$ simulate $\mathcal{K}_3$? If yes, give a simulation relation. Otherwise, explain why.

(c) Does $\mathcal{K}_3$ simulate $\mathcal{K}_2$? If yes, give a simulation relation. Otherwise, explain why.

(d) Does $\mathcal{K}_3$ simulate $\mathcal{K}_1$? If yes, give a simulation relation. Otherwise, explain why.

Exercise 10.2
Let $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ be Kripke structures. Show that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are bisimilar, and $\mathcal{K}_2$ and $\mathcal{K}_3$ are bisimilar, then $\mathcal{K}_1$ and $\mathcal{K}_3$ are also bisimilar.

Exercise 10.3
(Taken from 'Principles of Model Checking')
Let $TS = (S, Act, \rightarrow, I, AP, L)$ be a transition system. A bisimulation for TS is a binary relation R on S such that for all $(s_1, s_2) \in R$:

- $L(s_1) = L(s_2)$.
- If $s_1' \in Post(s_1)$, then there exists an $s_2' \in Post(s_2)$ with $(s_1', s_2') \in R$.
- If $s_2' \in Post(s_2)$, then there exists an $s_1' \in Post(s_1)$ with $(s_1', s_2') \in R$.

States s_1 and s_2 are bisimulation-equivalent (or bisimilar), denoted $s_1 \sim_{TS} s_2$, if there exists a bisimulation R for TS with $(s_1, s_2) \in R$. The relations $\sim_n \subseteq S \times S$ are inductively defined by:

(a) $s_1 \sim_0 s_2$ iff $L(s_1) = L(s_2)$.

(b) $s_1 \sim_{n+1} s_2$ iff

- $L(s_1) = L(s_2)$,
- for all $s_1' \in Post(s_1)$ there exists $s_2' \in Post(s_2)$ with $s_1' \sim_n s_2'$,
- for all $s_2' \in Post(s2)$ there exists $s_1' \in Post(s_1)$ with $s_1' \sim_n s_2'$.

Show that for finite TS it holds that $\sim_{TS} = \bigcap_{n \geq 0} \sim_n$, i.e., $s_1 \sim_{TS} s_2$ if and only if $s_1 \sim_n s_2$ for all $n \geq 0$.

Solution 10.1

(a) Yes. $H = \{(s_0, t_0), (s_1, t_1), (s_2, t_2), (s_3, t_2), (s_4, t_0)\}$.

(b) No. If there exists a simulation H from $\mathcal{K}_3$ to $\mathcal{K}_2$, then we know that $(u_0, t_0) \in H$. Since $u_0 \to u_1$, we have $(u_1, t_1) \in H$. However, $u_1 \to u_4$ and u_4 satisfies p, but no successors of t_1 satisfy p, so H cannot exist.

(c) Yes. $H = \{(t_0, u_0), (t_1, u_1), (t_2, u_3\}$.

(d) Yes. $H = \{(s_0, u_0), (s_1, u_1), (s_2, u_3), (s_3, u_3), (s_4, u_0)\}$. Alternatively, we can also prove that $\mathcal{K}_1$ and $\mathcal{K}_2$ are bisimilar and use the result from (c).

Solution 10.2

Let H_{12} be a bisimulation between $\mathcal{K}_1$ and $\mathcal{K}_2$ and H_{23} be a bisimulation between $\mathcal{K}_2$ and $\mathcal{K}_3$. We define $H_{13} = \{(s, u) \mid \exists t : (s, t) \in H_{12} \wedge (t, u) \in H_{23}\}$ and show that H_{13} is a bisimulation between $\mathcal{K}_1$ and $\mathcal{K}_3$.

First, we prove that H_{13} is a simulation from $\mathcal{K}_1$ to $\mathcal{K}_3$. Basically, we need to prove that if $(s, u) \in H_{13}$ and $s \to_1 s'$, then there exists u' such that $u \to_3 u'$ and $(s', u') \in H_{13}$. From the definition of $(s, u) \in H_{13}$, we know that there exists t such that $(s, t) \in H_{12}$ and $(t, u) \in H_{23}$. Since $(s, t) \in H_{12}$ and $s \to_1 s'$, there must exist t' such that $t \to_2 t'$ and $(s', t') \in H_{12}$. Similarly, since $(t, u) \in H_{23}$ and $t \to_2 t'$, there must exist u' such that $u \to_3 u'$ and $(t', u') \in H_{23}$. Because $(s', t') \in H_{12}$ and $(t', u') \in H_{23}$, by the definition of H_{13} we have $(s', u') \in H_{13}$.

Analogously, we can prove that $\{(u, s) \mid (s, u) \in H_{13}\}$ is a simulation from $\mathcal{K}_3$ to $\mathcal{K}_1$.

Solution 10.3

First we'll show that $s_1 \sim_{TS} s_2 \implies s_1 \sim_n s_2$ for all $n \geq 0$ using induction on n. Base case is trivial since $s_1 \sim_{TS} s_2 \implies s_1 \sim_0 s_2$. For the general case we assume that $s_1 \sim_{TS} s_2 \implies s_1 \sim_{k-1} s_2$ and we will show that $s_1 \sim_{TS} s_2 \implies s_1 \sim_k s_2$. Now, for a pair of states such that $s_1 \sim_{TS} s_2$ there exist an R such that $(s_1, s_2) \in R$ and for $s_1' \in Post(s_1)$, there exist $s_2' \in Post(s_2)$ with $(s_1', s_2') \in R$ which implies that $s_1' \sim_{TS} s_2'$. By using the induction assumption, this implies that $s_1' \sim_{k-1} s_2'$. Hence, the second condition in the definition of $s_1 \sim_k s_2$ is satisfied. Similiarily, we can show that the third condition will also be satisfied.

For the other direction, we define a relation $R := \{(s_1, s_2) \mid s_1 \sim_n s_2, \forall n \geq 0\}$. We shall now show that this is a bisimulation relation. We first claim that $s_1 \sim_n s_2 \implies s_1 \sim_k s_2$ for all $k \leq n$ (Use induction on k). Now, since the TS is finite, there exist $N \in \mathbb{N}$ such that $\sim_k = \sim_N$ for all $k \geq N$ (Why?). Assume $(s_1, s_2) \in R$ then trivially $L(s_1) = L(s_2)$ and if $s_1' \in Post(s_1)$ then pick some $n_0 > N$ and since $s_1 \sim_{n_0} s_2$, there exists $s_2' \in Post(s_2)$ with $s_1' \sim_{n_0 - 1}$ which implies that $s_1' \sim_n s_2'$ for all $n \geq 0$. This means that $(s_1', s_2') \in R$ and the second condition for R to be a bisimulation is satisfied. Similarly, R satisfies the third condition as well.