Technische Universität München (I7) **Summer Semester 2019**
J. Esparza / P. Ashok / T. Meggendorfer **13.06.2019**

Model Checking – Exercise sheet 7

Exercise 7.1

Given two CTL formulas ϕ_1 and ϕ_2, we write $\phi_1 \Rightarrow \phi_2$ iff for every Kripke structure $\mathcal{K}$ we have $(\mathcal{K} \models \phi_1) \Rightarrow (\mathcal{K} \models \phi_2)$. Furthermore, we define an *implication graph* as a directed graph whose nodes are CTL formulas, and that contains an edge from ϕ_1 to ϕ_2 iff $\phi_1 \Rightarrow \phi_2$. Let $AP = \{p\}$.

(a) Draw an implication graph with the nodes: $\mathbf{EFEF}p$, $\mathbf{EGEG}p$, $\mathbf{AFAF}p$, $\mathbf{AGAG}p$.

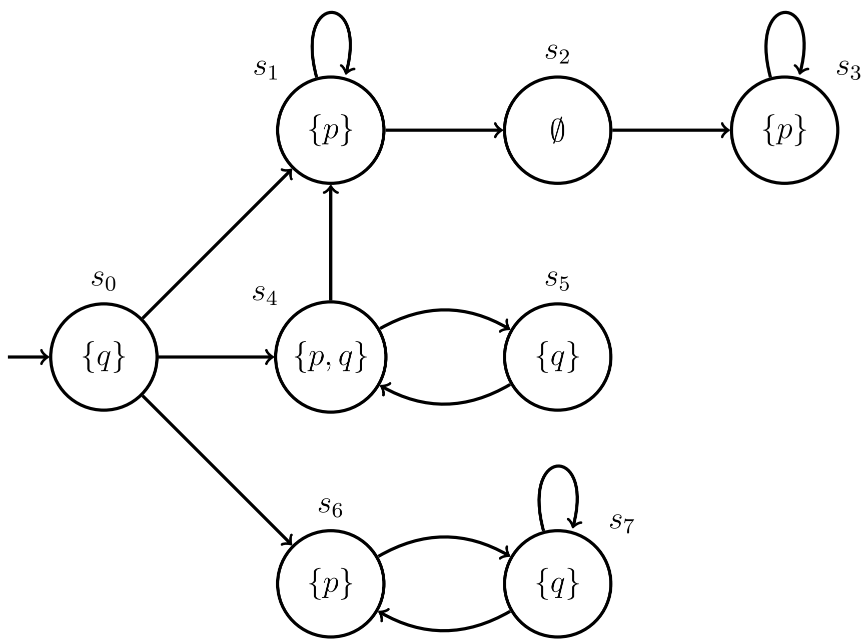
(b) For each implication $\phi_1 \Rightarrow \phi_2$ obtained in (a), give a Kripke structure $\mathcal{K}$ that satisfies ϕ_2 but not ϕ_1, i.e. give a $\mathcal{K}$ such that $\mathcal{K} \models \phi_2$ and $\mathcal{K} \not\models \phi_1$.

(c) Add the following CTL formulas to the implication graph obtained in (a): $\mathbf{AFEF}p$, $\mathbf{EFAF}p$, $\mathbf{AGEG}p$, $\mathbf{EGAG}p$.

(d) Complete the graph obtained in (c) with the nodes: $\mathbf{AGAF}p$, $\mathbf{AFAG}p$, $\mathbf{AGEF}p$, $\mathbf{EGAF}p$, $\mathbf{AFEG}p$, $\mathbf{EFAG}p$, $\mathbf{EFEG}p$, $\mathbf{EGEF}p$.

Exercise 7.2

Consider the following Kripke structure over $AP = \{p, q\}$:



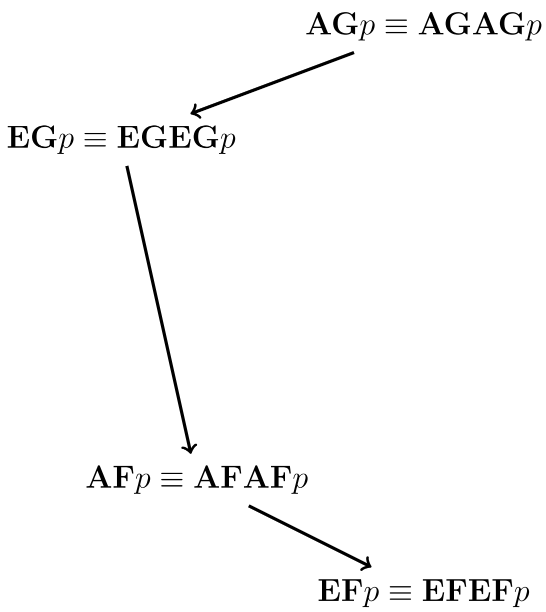
(a) Compute $[\![\mathbf{EG}q]\!]$ and $[\![\mathbf{EF}q]\!]$.

(b) Compute $[\![\mathbf{AGAF}p]\!]$ and $[\![\mathbf{EFAG}\neg q]\!]$.

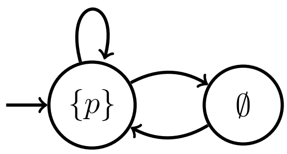
Solution 7.1

Note that the "⇒" relation is transitive, hence all transitive edges in (a), (b) and (d) are omitted.

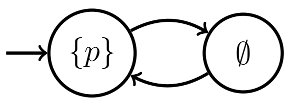
(a)



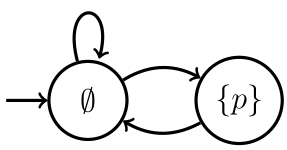
(b) The following Kripke structure satisfies $\mathbf{EG}p$, but not $\mathbf{AG}p$:



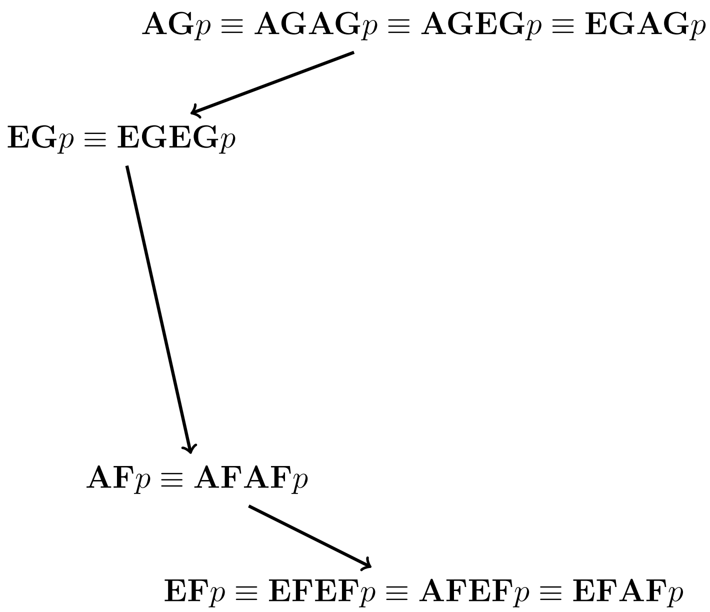
The following Kripke structure satisfies $\mathbf{AF}p$, but not $\mathbf{EG}p$:



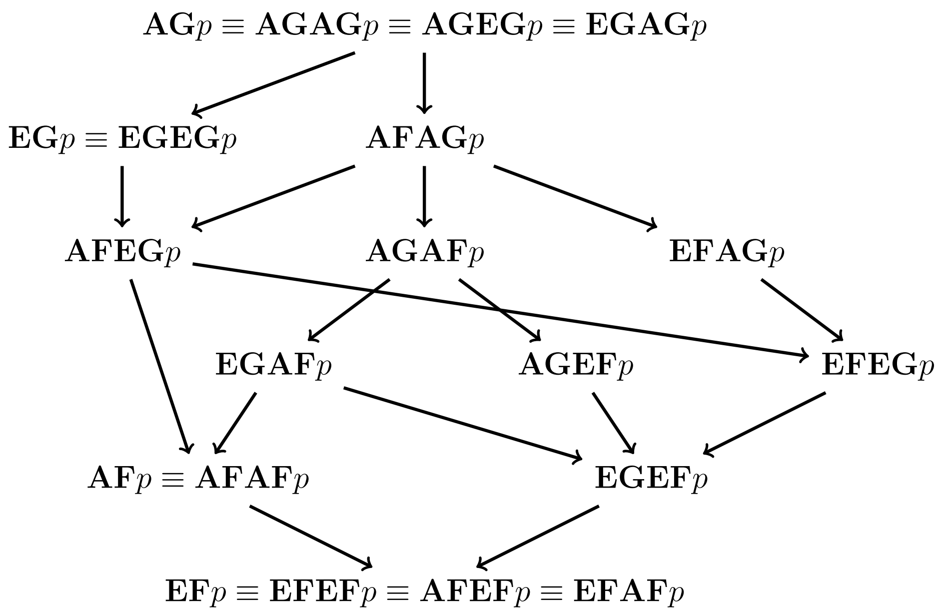
The following Kripke structure satisfies $\mathbf{EF}p$, but not $\mathbf{AF}p$:



(c)


AGp ≡ AGAGp ≡ AGEGp ≡ EGAGp
EGp ≡ EGEGp
AFp ≡ AFAFp
EFp ≡ EFEFp ≡ AFEFp ≡ EFAFp

(d)


AGp ≡ AGAGp ≡ AGEGp ≡ EGAGp
EGp ≡ EGEGp
AFAGp
AFEGp
AGAFp
EFAGp
EGAFp
AGEFp
EFEGp
AFp ≡ AFAFp
EGEFp
EFp ≡ EFEFp ≡ AFEFp ≡ EFAFp

Solution 7.2

Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.

(a)

- We compute the largest fixed point from the sequence

$$\pi^0(S), \pi^1(S), \pi^2(S), \ldots$$

where $\pi^0(S) = S$ and $\pi^{i+1}(S) = [\![q]\!] \cap pre(\pi^i(S))$. We obtain

$$\begin{aligned}\pi^0(S) &= S,\\ \pi^1(S) &= \{s_0, s_4, s_5, s_7\},\\ \pi^2(S) &= \{s_0, s_4, s_5, s_7\}.\end{aligned}$$

Therefore, $[\![\mathbf{EG}q]\!] = \{s_0, s_4, s_5, s_7\}$.

- We compute the smallest fixed point from the sequence

$$\xi^0(\emptyset), \xi^1(\emptyset), \xi^2(\emptyset), \ldots$$

where $\xi^0(\emptyset) = \emptyset$ and $\xi^{i+1}(\emptyset) = [\![q]\!] \cup pre(\xi^i(\emptyset))$. We obtain

$$\begin{aligned}\xi^0(\emptyset) &= \emptyset,\\ \xi^1(\emptyset) &= \{s_0, s_4, s_5, s_7\},\\ \xi^2(\emptyset) &= \{s_0, s_4, s_5, s_6, s_7\},\\ \xi^3(\emptyset) &= \{s_0, s_4, s_5, s_6, s_7\}.\end{aligned}$$

Therefore, $[\![\mathbf{EF}q]\!] = \{s_0, s_4, s_5, s_6, s_7\}$.

(b)

- Note that $\mathbf{AGAF}p \equiv \neg\mathbf{EFEG}\neg p$. Let us first compute $[\![\mathbf{EG}\neg p]\!]$ by computing the largest fixed point from the sequence $\pi^0(S), \pi^1(S), \pi^2(S), \ldots$. We obtain

$$\begin{aligned}\pi^0(S) &= S,\\ \pi^1(S) &= \{s_0, s_2, s_5, s_7\},\\ \pi^2(S) &= \{s_7\},\\ \pi^3(S) &= \{s_7\}.\end{aligned}$$

Therefore, $[\![\mathbf{EG}\neg p]\!] = \{s_7\}$. In general, $[\![\mathbf{EF}\varphi]\!]$ is the set of states that can reach some state of $[\![\varphi]\!]$. By setting $\varphi = \mathbf{EG}\neg p$, we obtain

$$\begin{aligned}[\![\mathbf{AGAF}p]\!] &= [\![\neg\mathbf{EFEG}\neg p]\!]\\ &= \overline{[\![\mathbf{EFEG}\neg p]\!]}\\ &= \overline{\{s_0, s_6, s_7\}}\\ &= \{s_1, s_2, s_3, s_4, s_5\}.\end{aligned}$$

- Note that $\mathbf{EFAG}\neg q \equiv \mathbf{EF}\neg\mathbf{EF}q$. By (a), $[\![\mathbf{EF}q]\!] = \{s_0, s_4, s_5, s_6, s_7\}$, and hence $[\![\neg\mathbf{EF}q]\!] = \overline{[\![\mathbf{EF}q]\!]} = \{s_1, s_2, s_3\}$. In general, $[\![\mathbf{EF}\varphi]\!]$ is the set of states that can reach some state of $[\![\varphi]\!]$. By setting $\varphi = \neg\mathbf{EF}q$, we obtain

$$[\![\mathbf{EFAG}\neg q]\!] = [\![\mathbf{EF}\neg\mathbf{EF}q]\!] = \{s_0, s_1, s_2, s_3, s_4, s_5\}.$$