Proposition 1. *For every LTL formula φ, the following holds:*

(a) $\sigma \models \mathbf{F}\varphi \iff \exists i\ \sigma^i \models \varphi$,

(b) $\sigma \models \mathbf{G}\varphi \iff \forall i\ \sigma^i \models \varphi$,

(c) $\mathbf{XF}\varphi = \mathbf{FX}\varphi$,

(d) $\mathbf{XG}\varphi = \mathbf{GX}\varphi$,

(e) $\mathbf{FF}\varphi = \mathbf{F}\varphi$,

(f) $\mathbf{GG}\varphi = \mathbf{G}\varphi$,

(g) $\mathbf{FGF}\varphi = \mathbf{GF}\varphi$,

(h) $\mathbf{GFG}\varphi = \mathbf{FG}\varphi$.

Proof.

(a)

$$\begin{aligned}\sigma \models \mathbf{F}\varphi &\iff \exists i \ (\sigma^i \models \varphi \wedge \forall k < i \ \sigma^k \models \mathbf{true}) \\ &\iff \exists i \ (\sigma^i \models \varphi \wedge \forall k < i \ \mathbf{true}) \\ &\iff \exists i \ \sigma^i \models \varphi. \end{aligned}$$

□

(b)

$$\begin{aligned}\sigma \models \mathbf{G}\varphi &\iff \sigma \models \neg\mathbf{F}\neg\varphi \\ &\iff \neg(\sigma \models \mathbf{F}\neg\varphi) \\ &\iff \neg(\exists i \ \sigma^i \models \neg\varphi) \\ &\iff \neg(\exists i \ \neg(\sigma^i \models \varphi)) \\ &\iff \forall i \ \sigma^i \models \varphi. \end{aligned}$$

□

(c)

$$\begin{aligned}\sigma \models \mathbf{XF}\varphi &\iff \sigma^1 \models \mathbf{F}\varphi \\ &\iff \exists i \ (\sigma^1)^i \models \varphi \\ &\iff \exists i \ (\sigma^i)^1 \models \varphi \\ &\iff \exists i \ \sigma^i \models \mathbf{X}\varphi \\ &\iff \sigma \models \mathbf{FX}\varphi. \end{aligned}$$

□

(d)

$$\begin{aligned}\mathbf{XG}\varphi &\equiv \mathbf{X}\neg\mathbf{F}\neg\varphi \\ &\equiv \neg\mathbf{XF}\neg\varphi \\ &\equiv \neg\mathbf{FX}\neg\varphi \\ &\equiv \neg\mathbf{F}\neg\mathbf{X}\varphi \\ &\equiv \mathbf{GX}\varphi. \end{aligned}$$

□

(e)

$$\begin{aligned}\sigma \models \mathbf{FF}\varphi &\iff \exists i \ \sigma^i \models \mathbf{F}\varphi \\ &\iff \exists i \ (\exists j \ (\sigma^i)^j \models \varphi) \\ &\iff \exists i, j \ \sigma^{i+j} \models \varphi \\ &\iff \exists k \ \sigma^k \models \varphi \\ &\iff \sigma \models \mathbf{F}\varphi. \end{aligned}$$

□

(f)

$$\begin{aligned}\mathbf{GG}\varphi &\equiv \neg\mathbf{F}\neg(\neg\mathbf{F}\neg\varphi)\\ &\equiv \neg\mathbf{FF}\neg\varphi\\ &\equiv \neg\mathbf{F}\neg\varphi\\ &\equiv \mathbf{G}\varphi.\end{aligned}$$

□

(g) Note that $\sigma \models \mathbf{GF}\varphi$ trivially implies $\sigma \models \mathbf{FGF}\varphi$. Assume $\sigma \models \mathbf{FGF}\varphi$. We have

$$\begin{aligned}\sigma \models \mathbf{FGF}\varphi &\iff \exists i\ \forall j\ \exists k\ \sigma^{i+j+k} \models \varphi\\ &\implies \forall j\ \exists k\ \sigma^{j+(i+k)} \models \varphi\\ &\implies \forall j\ \exists \ell\ \sigma^{j+\ell} \models \varphi\\ &\iff \sigma \models \mathbf{GF}\varphi.\end{aligned}$$

□

(h)

$$\begin{aligned}\mathbf{GFG}\varphi &\equiv \neg\mathbf{F}\neg\mathbf{F}\neg\mathbf{F}\neg\varphi\\ &\equiv \neg\mathbf{FGF}\neg\varphi\\ &\equiv \neg\mathbf{GF}\neg\varphi\\ &\equiv \mathbf{FG}\varphi.\end{aligned}$$

□