Technische Universität München (I7) **Summer Semester 2018**
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Model Checking – Exercise sheet 8

Exercise 8.1

Given two CTL formulas ϕ_1 and ϕ_2, we write $\phi_1 \Rightarrow \phi_2$ iff for every Kripke structure $\mathcal{K}$ we have $(\mathcal{K} \models \phi_1) \Rightarrow (\mathcal{K} \models \phi_2)$. Furthermore, we define an *implication graph* as a directed graph whose nodes are CTL formulas, and that contains an edge from ϕ_1 to ϕ_2 iff $\phi_1 \Rightarrow \phi_2$. Let $AP = \{p\}$.

(a) Draw an implication graph with the nodes: $\mathbf{EFEF}p$, $\mathbf{EGEG}p$, $\mathbf{AFAF}p$, $\mathbf{AGAG}p$.

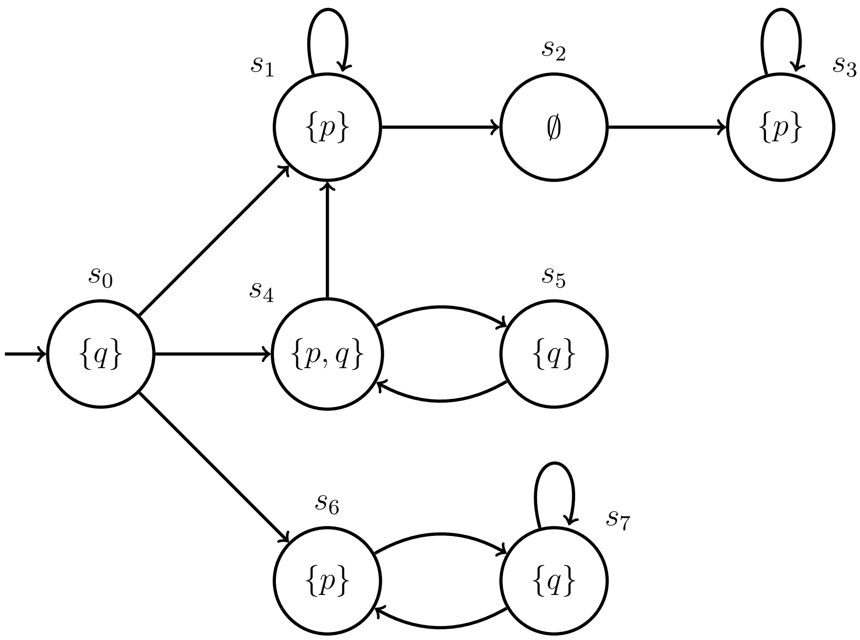
(b) For each implication $\phi_1 \Rightarrow \phi_2$ obtained in (a), give a Kripke structure $\mathcal{K}$ that satisfies ϕ_2 but not ϕ_1, i.e. give a $\mathcal{K}$ such that $\mathcal{K} \models \phi_2$ and $\mathcal{K} \not\models \phi_1$.

(c) Add the following CTL formulas to the implication graph obtained in (a): $\mathbf{AFEF}p$, $\mathbf{EFAF}p$, $\mathbf{AGEG}p$, $\mathbf{EGAG}p$.

(d) Complete the graph obtained in (c) with the nodes: $\mathbf{AGAF}p$, $\mathbf{AFAG}p$, $\mathbf{AGEF}p$, $\mathbf{EGAF}p$, $\mathbf{AFEG}p$, $\mathbf{EFAG}p$, $\mathbf{EFEG}p$, $\mathbf{EGEF}p$.

Exercise 8.2

Consider the following Kripke structure over $AP = \{p, q\}$:



(a) Compute $[\![\mathbf{EG}q]\!]$ and $[\![\mathbf{EF}q]\!]$.

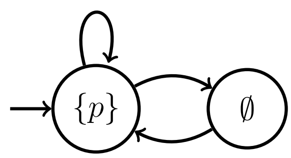
(b) Compute $[\![\mathbf{AGAF}p]\!]$ and $[\![\mathbf{EFAG}\neg q]\!]$.

Solution 8.1
Note that the "⇒" relation is transitive, hence all transitive edges in (a), (b) and (d) are omitted.

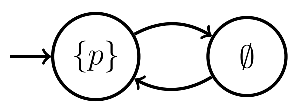
(a)

$$\begin{tikzcd}
 & & \mathbf{AG}p \equiv \mathbf{AGAG}p \arrow[dll] \\
\mathbf{EG}p \equiv \mathbf{EGEG}p \arrow[d] & & \\
\mathbf{AF}p \equiv \mathbf{AFAF}p \arrow[drr] & & \\
 & & \mathbf{EF}p \equiv \mathbf{EFEF}p
\end{tikzcd}$$

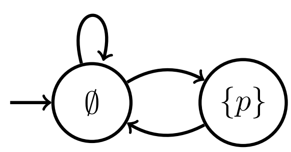
(b) The following Kripke structure satisfies $\mathbf{EG}p$, but not $\mathbf{AG}p$:



The following Kripke structure satisfies $\mathbf{AF}p$, but not $\mathbf{EG}p$:



The following Kripke structure satisfies $\mathbf{EF}p$, but not $\mathbf{AF}p$:



(c)

$$\begin{tikzcd}[column sep=tiny]
 & \mathbf{AG}p \equiv \mathbf{AGAG}p \equiv \mathbf{AGEG}p \equiv \mathbf{EGAG}p \arrow[dl] \\
\mathbf{EG}p \equiv \mathbf{EGEG}p \arrow[d] & \\
\mathbf{AF}p \equiv \mathbf{AFAF}p \arrow[dr] & \\
 & \mathbf{EF}p \equiv \mathbf{EFEF}p \equiv \mathbf{AFEF}p \equiv \mathbf{EFAF}p
\end{tikzcd}$$

(d)

$$\begin{tikzcd}[column sep=tiny]
 & & \mathbf{AG}p \equiv \mathbf{AGAG}p \equiv \mathbf{AGEG}p \equiv \mathbf{EGAG}p \arrow[dll] \arrow[d] & & & \\
\mathbf{EG}p \equiv \mathbf{EGEG}p \arrow[d] & & \mathbf{AFAG}p \arrow[dll] \arrow[d] \arrow[drr] & & & \\
\mathbf{AFEG}p \arrow[ddr] & & \mathbf{AGAF}p \arrow[dl] \arrow[dr] & & \mathbf{EFAG}p \arrow[dr] & \\
 & \mathbf{EGAF}p \arrow[d] & & \mathbf{AGEF}p \arrow[dr] & & \mathbf{EFEG}p \arrow[dl] \\
 & \mathbf{AF}p \equiv \mathbf{AFAF}p \arrow[drr] & & & \mathbf{EGEF}p \arrow[dl] & \\
 & & & \mathbf{EF}p \equiv \mathbf{EFEF}p \equiv \mathbf{AFEF}p \equiv \mathbf{EFAF}p & &
\end{tikzcd}$$

Solution 8.2

Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.

(a)
- We compute the largest fixed point from the sequence

$$\pi^0(S), \pi^1(S), \pi^2(S), \ldots$$

where $\pi^0(S) = S$ and $\pi^{i+1}(S) = [\![q]\!] \cap \mathit{pre}(\pi^i(S))$. We obtain

$$\begin{aligned} \pi^0(S) &= S, \\ \pi^1(S) &= \{s_0, s_4, s_5, s_7\}, \\ \pi^2(S) &= \{s_0, s_4, s_5, s_7\}. \end{aligned}$$

Therefore, $[\![\mathbf{EG}q]\!] = \{s_0, s_4, s_5, s_7\}$.

- We compute the smallest fixed point from the sequence

$$\xi^0(\emptyset), \xi^1(\emptyset), \xi^2(\emptyset), \ldots$$

where $\xi^0(\emptyset) = \emptyset$ and $\xi^{i+1}(\emptyset) = [\![q]\!] \cup \mathit{pre}(\xi^i(\emptyset))$. We obtain

$$\begin{aligned} \xi^0(\emptyset) &= \emptyset, \\ \xi^1(\emptyset) &= \{s_0, s_4, s_5, s_7\}, \\ \xi^2(\emptyset) &= \{s_0, s_4, s_5, s_6, s_7\}, \\ \xi^3(\emptyset) &= \{s_0, s_4, s_5, s_6, s_7\}. \end{aligned}$$

Therefore, $[\![\mathbf{EF}q]\!] = \{s_0, s_4, s_5, s_6, s_7\}$.

(b)
- Note that $\mathbf{AGAF}p \equiv \neg\mathbf{EFEG}\neg p$. Let us first compute $[\![\mathbf{EG}\neg p]\!]$ by computing the largest fixed point from the sequence $\pi^0(S), \pi^1(S), \pi^2(S), \ldots$. We obtain

$$\begin{aligned} \pi^0(S) &= S, \\ \pi^1(S) &= \{s_0, s_2, s_5, s_7\}, \\ \pi^2(S) &= \{s_7\}, \\ \pi^3(S) &= \{s_7\}. \end{aligned}$$

Therefore, $[\![\mathbf{EG}\neg p]\!] = \{s_7\}$. In general, $[\![\mathbf{EF}\varphi]\!]$ is the set of states that can reach some state of $[\![\varphi]\!]$. By setting $\varphi = \mathbf{EG}\neg p$, we obtain

$$\begin{aligned} [\![\mathbf{AGAF}p]\!] &= [\![\neg\mathbf{EFEG}\neg p]\!] \\ &= \overline{[\![\mathbf{EFEG}\neg p]\!]} \\ &= \overline{\{s_0, s_6, s_7\}} \\ &= \{s_1, s_2, s_3, s_4, s_5\}. \end{aligned}$$

- Note that $\mathbf{EFAG}\neg q \equiv \mathbf{EF}\neg\mathbf{EF}q$. By (a), $[\![\mathbf{EF}q]\!] = \{s_0, s_4, s_5, s_6, s_7\}$, and hence $[\![\neg\mathbf{EF}q]\!] = \overline{[\![\mathbf{EF}q]\!]} = \{s_1, s_2, s_3\}$. In general, $[\![\mathbf{EF}\varphi]\!]$ is the set of states that can reach some state of $[\![\varphi]\!]$. By setting $\varphi = \neg\mathbf{EF}q$, we obtain

$$[\![\mathbf{EFAG}\neg q]\!] = [\![\mathbf{EF}\neg\mathbf{EF}q]\!] = \{s_0, s_1, s_2, s_3, s_4, s_5\}.$$