Technische Universität München (I7) **Summer Semester 2018**
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Model Checking – Exercise sheet 4

Exercise 4.1
Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. Give Büchi automata recognizing the ω-languages over Σ defined by the following LTL formulas:

(a) $\mathbf{XG}\neg p$

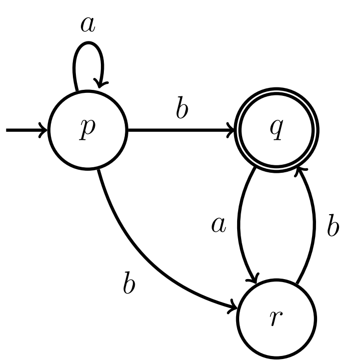
(b) $(\mathbf{GF}p) \to (\mathbf{F}q)$

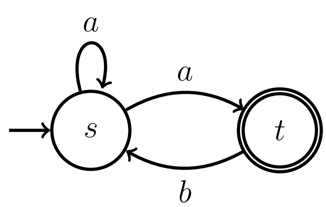
(c) $p \wedge \neg(\mathbf{XF}p)$

(d) $\mathbf{G}(p\ \mathbf{U}\ (p \to q))$

(e) $\mathbf{F}q \to (\neg q\ \mathbf{U}\ (\neg q \wedge p))$

Exercise 4.2
Let A and B be the following Büchi automata over $\Sigma = \{a, b\}$. Construct a Büchi automaton C such that $\mathcal{L}(C) = \mathcal{L}(A) \cap \mathcal{L}(B)$. Moreover, say whether there exists a deterministic Büchi automaton recognizing $\mathcal{L}(C)$. Justify your answer.





Exercise 4.3
Let $AP = \{p, q, r\}$ and $\Sigma = 2^{AP}$. For every $\sigma \in \Sigma^\omega$, let

$$P_\sigma = \{i \in \mathbb{N} : p \in \sigma(i)\},$$
$$Q_\sigma = \{i \in \mathbb{N} : q \in \sigma(i)\}.$$

We say that a sequence $\sigma \in \Sigma^\omega$ is *good* if there exists an injective function $f \colon P_\sigma \to Q_\sigma$ such that $i \leq f(i)$ for every $i \in P_\sigma$. Let $L = \{\sigma \in \Sigma^\omega : \sigma \text{ is good}\}$. Intuitively, L is the language of sequences where each occurrence of p is matched by a later occurrence of q.

(a) Show that $L \cap [\![\mathbf{GF}p]\!] = [\![(\mathbf{GF}p) \wedge (\mathbf{GF}q)]\!]$.

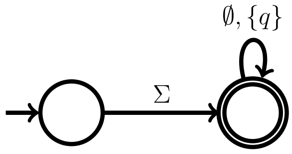
(b) Show that $L \cap \{p\}^*\{q\}^*\emptyset^\omega = L'$ where $L' = \{\{p\}^m\{q\}^n\emptyset^\omega : m \leq n\}$.

(c) Show that there is no Büchi automata recognizing L'. [Hint:]

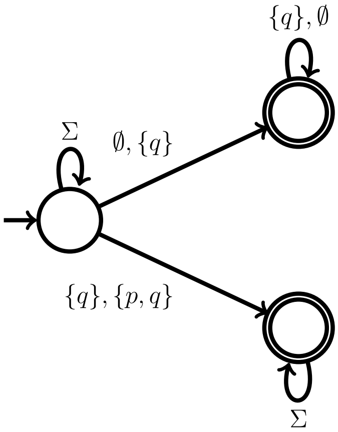
(d) Show that there is no Büchi automata recognizing L. [Hint:]

Solution 4.1

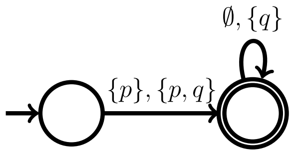
(a)



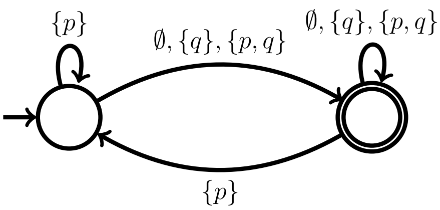
(b) Note that $(\mathbf{GF}p) \rightarrow (\mathbf{F}q) \equiv \neg(\mathbf{GF}p) \vee (\mathbf{F}q) \equiv (\mathbf{FG}\neg p) \vee (\mathbf{F}q)$. We construct Büchi automata for $\mathbf{FG}\neg p$ and $\mathbf{F}q$, and take their union:



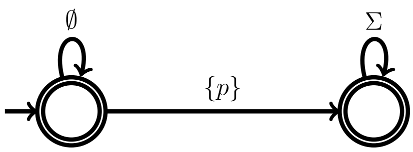
(c) Note that $p \wedge \neg(\mathbf{XF}p) \equiv p \wedge \mathbf{XG}\neg p$. We construct a Büchi automaton for $p \wedge \mathbf{XG}\neg p$:



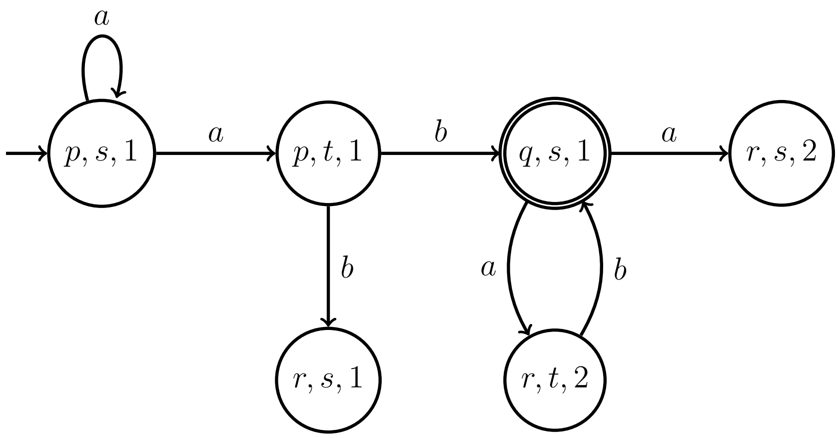
(d)



(e)



Solution 4.2



The above automaton is non deterministic. However, an equivalent deterministic automaton is obtained as follows:

- the self-loop from the first state is moved to the second state (from left to right),
- transitions $(p, t, 1) \xrightarrow{b} (r, s, 1)$ and $(q, s, 1) \xrightarrow{a} (r, s, 2)$ are removed.

Solution 4.3

(a) $\Rightarrow$) Let $\sigma \in L \cap [\![\mathbf{GF}p]\!]$. Let f be the injection that makes σ good. Since $\sigma \models \mathbf{GF}p$, P_σ is infinite. Since f is an injection from P_σ to Q_σ, the latter must also be infinite. Therefore, $\sigma \models \mathbf{GF}q$.

$\Leftarrow$) Let $\sigma \in [\![(\mathbf{GF}p) \wedge (\mathbf{GF}q)]\!]$. It follows immediately that $\sigma \in [\![\mathbf{GF}p]\!]$. It remains to show that $\sigma \in L$. Let $i_0 < i_1 < \cdots$ be the positions of σ where p holds. Note that $P_\sigma = \{i_j : j \in \mathbb{N}\}$. For every $j \in \mathbb{N}$, we define a function $f \colon P_\sigma \to Q_\sigma$ as follows:

$$f(i_j) = \begin{cases} \text{smallest } k \text{ s.t. } q \in \sigma(k) \text{ and } k \geq i_j & \text{if } j = 0, \\ \text{smallest } k \text{ s.t. } q \in \sigma(k) \text{ and } k \geq \max(i_j, f(i_{j-1}) + 1) & \text{otherwise.} \end{cases}$$

Note that f is well-defined because q holds infinitely often in σ. It is immediate that $i_j \leq f(i_j)$ for every $j \in \mathbb{N}$. Moreover, $f(i_0) < f(i_1) < f(i_2) < \cdots$, and hence f must be injective. Thus, $\sigma \in L$.

(b) $\Leftarrow$) Let $\sigma \in L'$. We have $\sigma = \{p\}^m\{q\}^n\emptyset^\omega$ for some $m \leq n$. We have $P_\sigma = \{0, 1, \ldots, m-1\}$ and $Q_\sigma = \{m, m+1, \ldots, m+n-1\}$. Let f be the function such that $f(i) = m + i$. Observe that f is an injection from P_σ to Q_σ and that $i \leq f(i)$ for every $i \in P_\sigma$. Therefore, $\sigma \in L$ and hence $\sigma \in L \cap \{p\}^*\{q\}^*\emptyset^\omega$.

$\Rightarrow$) Let $\sigma \in L \cap \{p\}^*\{q\}^*\emptyset^\omega$. There exist $m, n \in \mathbb{N}$ such that $\sigma = \{p\}^m\{q\}^n\emptyset^\omega$. If $m \leq n$, then we are done. Suppose for the sake of contradiction that $m > n$. Let f be the injection that makes σ good. Since $|P_\sigma| = m > n = |Q_\sigma|$, the pigeonhole principle implies the existence of $i, j \in P_\sigma$ such that $i \neq j$ and $f(i) = f(j)$. This is a contradiction since f is injective.

(c) For the sake of contradiction, suppose there exists a Büchi automaton $B = (Q, \Sigma, \delta, Q_0, F)$ such that $\mathcal{L}(B) = L'$. Let $m = |Q|$ and let $\sigma = \{p\}^m\{q\}^m\emptyset^\omega$. Since $\sigma \in \mathcal{L}(B)$, there exist $q_0, q_1, \ldots \in Q$ such that $q_0 \in Q_0$, there are infinitely many indices i such that $q_i \in F$ and

$$q_0 \xrightarrow{\sigma_0} q_1 \xrightarrow{\sigma_1} q_2 \cdots .$$

By the pigeonhole principle, there exist $0 \leq i < j \leq m$ such that $q_i = q_j$. Let $u = \sigma_0\sigma_1 \cdots \sigma_{i-1}$, $v = \sigma_i\sigma_{i+1} \cdots \sigma_{j-1}$ and $w = \sigma_j\sigma_{j+1} \cdots$. We have:

$$q_0 \xrightarrow{u} q_i \xrightarrow{v^{m+1}} q_j \xrightarrow{w} \cdots$$

Thus, $\sigma' \in \mathcal{L}(B)$ where $\sigma' = uv^{m+1}w$. Note that v solely consists of the letter $\{p\}$, hence $|P_{\sigma'}| \geq m + 1 > m = |Q_{\sigma'}|$, which contradicts $\sigma \in \mathcal{L}(B) = L'$.

(d) Suppose that L is recognized by some Büchi automaton. Since ω-regular languages are closed under intersection, it means that $L \cap \{p\}^*\{q\}^*\emptyset^\omega$ is also ω-regular, and hence that L' as well by (a). This contradicts (b).