Technische Universität München (I7) **Summer Semester 2016**
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Model Checking – Sample Solution 2

Exercise 2.1

For $\phi = \mathbf{G}\,\mathbf{F}\,p$ and $\psi = \mathbf{F}\,\mathbf{G}(p \wedge \neg q \rightarrow (r\ \mathbf{U}\ s))$:

σ	$\sigma \models \phi$	$\sigma \models \psi$
$\emptyset^\omega$	no	yes
$\{p\}^\omega$	yes	no
$\{p,q\}^\omega$	yes	yes
$\{p\}\{q\}^\omega$	no	yes
$(\{p\}\{q\})^\omega$	yes	no
$(\{p\}\{p,q\})^\omega$	yes	no
$(\{p\}\{r\})^\omega$	yes	no
$(\{q\}\{s\})^\omega$	no	yes
$(\{p,s\}\{q\})^\omega$	yes	yes
$(\{p\}\{q\}\{r\}\{s\})^\omega$	yes	no
$(\{p,q\}\{q\}\{r\}\{s\})^\omega$	yes	yes
$(\{p,r\}\{q\}\{r\}\{s\})^\omega$	yes	no
$(\{p,r\}\{q,r\}\{r\}\{r,s\})^\omega$	yes	yes

Exercise 2.2

In the following table, σ and σ' are two example sequences such that $\sigma \models \phi$ and $\sigma' \not\models \phi$.

	ϕ	σ	σ'
1.	$\mathbf{F}\,g$	$\{g\}\emptyset^\omega$	$\emptyset^\omega$
2.	$\mathbf{G}(g \rightarrow \mathbf{G}(\neg s \wedge \neg r))$	$\{g\}\emptyset^\omega$	$\{g,s\}\emptyset^\omega$
	or if "after" is strict		
	$\mathbf{G}(g \rightarrow \mathbf{X}\,\mathbf{G}(\neg s \wedge \neg r))$	$\{g\}\emptyset^\omega$	$\{g\}\{s\}\emptyset^\omega$
3.	$\mathbf{G}\,\mathbf{F}\,s$	$(\{s\}\{r\})^\omega$	$\{s\}\{s\}\{s\}\emptyset^\omega$
4.	$\mathbf{F}\,g \wedge \mathbf{G}(g \rightarrow \mathbf{X}\,\mathbf{G}\,\neg g)$	$\{g\}\emptyset^\omega$	$\{g\}\{g\}\emptyset^\omega$
5.	$\mathbf{G}(s \rightarrow \mathbf{X}\,\mathbf{F}\,r)$	$(\{s\}\{r\})^\omega$	$\{s\}\emptyset^\omega$
6.	$(\neg s \wedge \neg g)\ \mathbf{W}\ r$	$\{r\}\{g\}^\omega$	$\{g\}^\omega$

Exercise 2.3

$$\begin{aligned}
\sigma \models \mathbf{G}\,\phi \ &\Leftrightarrow\ \sigma \models \neg\,\mathbf{F}\,\neg\phi \\
&\Leftrightarrow\ \sigma \models \neg(\mathbf{true}\ \mathbf{U}\ \neg\phi) \\
&\Leftrightarrow\ \neg\exists i : (\sigma^i \models \neg\phi \wedge \forall k < i : \sigma^k \models \mathbf{true}) \\
&\Leftrightarrow\ \forall i : (\sigma^i \models \phi \vee \neg\forall k < i : \sigma^k \models \mathbf{true}) \\
&\Leftrightarrow\ \forall i : \sigma^i \models \phi
\end{aligned}$$

$$
\begin{aligned}
\sigma \models \phi \,\mathbf{R}\, \psi \quad &\Leftrightarrow \quad \sigma \models \neg(\neg\phi \,\mathbf{U}\, \neg\psi) \\
&\Leftrightarrow \quad \neg\exists i : (\sigma^i \models \neg\psi \wedge \forall k < i : \sigma^k \models \neg\phi) \\
&\Leftrightarrow \quad \forall i : (\sigma^i \models \psi \vee \neg(\forall k < i : \sigma^k \models \neg\phi)) \\
&\Leftrightarrow \quad \forall i : ((\forall k < i : \sigma^k \models \neg\phi) \rightarrow (\sigma^i \models \psi))
\end{aligned}
$$

$$
\begin{aligned}
\sigma \models \neg(\phi \,\mathbf{U}\, \psi) \quad &\Leftrightarrow \quad \sigma \models \neg\neg(\neg\phi \,\mathbf{R}\, \neg\psi) \\
&\Leftrightarrow \quad \sigma \models \neg\phi \,\mathbf{R}\, \neg\psi
\end{aligned}
$$

$$
\begin{aligned}
\sigma \models \phi \,\mathbf{U}\, \psi \quad &\Leftrightarrow \quad \exists i : (\sigma^i \models \psi \wedge \forall k < i : \sigma^k \models \phi) \\
&\Leftrightarrow \quad \exists i : (i = 0 \wedge \sigma^i \models \psi \vee i > 0 \wedge \sigma^i \models \psi \wedge \forall k < i : \sigma^k \models \phi) \\
&\Leftrightarrow \quad \exists i : (i = 0 \wedge \sigma^i \models \psi) \vee (\exists i > 0 \wedge \sigma^i \models \psi \wedge \forall k < i : \sigma^k \models \phi)) \\
&\Leftrightarrow \quad (\sigma^0 \models \psi) \vee (\exists i > 0 \wedge \sigma^i \models \psi \wedge \sigma^0 \models \phi \wedge \forall 0 < k < i : \sigma^k \models \phi)) \\
&\Leftrightarrow \quad (\sigma^0 \models \psi) \vee (\sigma^0 \models \phi \wedge \exists i' \wedge \sigma^{i'+1} \models \psi \wedge \forall k < i' : \sigma^{k+1} \models \phi)) \\
&\Leftrightarrow \quad \psi \vee (\phi \wedge \mathbf{X}(\phi \,\mathbf{U}\, \psi))
\end{aligned}
$$

$$
\begin{aligned}
\sigma \models \phi \,\mathbf{R}\, \psi \quad &\Leftrightarrow \quad \sigma \models \neg(\neg\phi \,\mathbf{U}\, \neg\psi) \\
&\Leftrightarrow \quad \neg(\exists i : \sigma^i \models \neg\psi \wedge \forall k < i : \sigma^k \models \neg\phi) \\
&\Leftrightarrow \quad \forall i : (\sigma^i \models \psi \vee \exists k < i : \sigma^k \models \phi) \\
&\Leftrightarrow \quad \forall i : (\sigma^i \models \neg\psi \rightarrow \exists k < i : \sigma^k \models \phi) \qquad (*)
\end{aligned}
$$

Case $\sigma \models \mathbf{G}\,\psi$ Obviously, both $\sigma \models \mathbf{G}\,\psi \vee \ldots$ and $\forall i : (\sigma^i \models \psi \vee \ldots)$ are true.

Case $\sigma \models \neg\,\mathbf{G}\,\psi$ Let j be the smallest position where ψ does not hold:

$$
\sigma \models \neg\,\mathbf{G}\,\psi \quad \Rightarrow \quad \exists j : \sigma^j \models \neg\psi \wedge \forall k < j : \sigma^k \models \psi
$$

Using the above fact, we prove the equivalence $(*)$ from both directions as follows:

$$
\begin{aligned}
\forall i : (\sigma^i \models \neg\psi \rightarrow \exists k < i : \sigma^k \models \phi) \quad &\Rightarrow \quad \exists k : ((\sigma^k \models \phi \wedge \psi) \wedge \forall k' < k : \sigma^{k'} \models \psi) \\
&\Rightarrow \quad \psi \,\mathbf{U}\, (\phi \wedge \psi)
\end{aligned}
$$

$$
\begin{aligned}
\psi \,\mathbf{U}\, (\phi \wedge \psi) \quad &\Rightarrow \quad \exists i : ((\sigma^i \models \phi \wedge \psi) \wedge \forall k < i : \sigma^k \wedge \psi) \\
&\Rightarrow \quad \exists j : \big(\sigma^j \models \neg\psi \wedge \exists i < j : ((\sigma^i \models \phi \wedge \psi) \wedge \forall k < j : \sigma^k \wedge \psi)\big) \\
&\qquad \text{(we know that such } j \text{ exists because } \sigma \models \neg\,\mathbf{G}\,\phi) \\
&\Rightarrow \quad \forall j : (\sigma^i \models \neg\psi \rightarrow \exists k < j : \sigma^k \models \phi)
\end{aligned}
$$

Exercise 2.4

From the definition: $\mathcal{K}_2 \models \phi$ iff $\forall \sigma \in [\![\mathcal{K}_2]\!] : \sigma \models \phi$. Since $\rightarrow_1 \subseteq \rightarrow_2$, we know that all paths in $\mathcal{K}_1$ are also paths in $\mathcal{K}_2$:

$$
\forall \sigma : \sigma \in [\![\mathcal{K}_1]\!] \rightarrow \sigma \in [\![\mathcal{K}_2]\!]
$$

Therefore, $\forall \sigma \in [\![\mathcal{K}_1]\!] : \sigma \models \phi$, which means $\mathcal{K}_1 \models \phi$.