

# Model Checking (Summer 2015)

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# Organization

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Language: English

Lecture: Tuesday 8:15–9:45 HS2, Wednesday 12:15–13:45, HS3  
(move by 15 minutes?)

Lecturer: Javier Esparza, [esparza@in.tum.de](mailto:esparza@in.tum.de)

Office hour: by appointment

Tutorials: Antoine Durand-Gasselín

Misc: 4V+2Ü

Exam at the end of semester

(please do take part in the exercises to prepare)

# Announcements

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## Web site:

`http://www7.in.tum.de/um/courses/mc/ss2015/`

Please check it regularly for slides, exercises, announcements etc

# Other

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## Preliminaries:

basic knowledge of logics, discrete structures, graph theory, ...

## Literature:

Baier, Katoen: Principles of Model Checking, MIT Press, 2008

Clarke, Grumberg, Peled: Model Checking, MIT Press, 1999

Emerson: Temporal and Modal Logic, chapter 16 in Handbook of Theoretical Computer Science, vol. B, Elsevier, 1991

Vardi: An Automata-Theoretic Approach to Linear Temporal Logic, LNCS 1043, 1996

Holzmann: The SPIN Model Checker, Addison-Wesley, 2003

# Part 1: Introduction

# What does “Model-Checking” mean?

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# What does “Model-Checking” mean?

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A technical term from logic

temporal logic: extension of predicate logic

in German: “Modellprüfung” (rarely used)

# Motivation

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Computer systems permeate more and more areas of our lives:

PCs, mobile phone, GPS, ...

control systems in cars, planes, ...

in banks (ATMs, credit risk assessment)

Correspondingly, we require more and more **dependable** hardware and software systems;

but the more complex a system grows, the more difficult it becomes to protect it against mistakes or attacks.

Bugs can have substantial economic impact or even endanger lives.

Estimated cost of bugs in the US: **60 bn dollars per year** (Source: Der Spiegel).

# Pentium bug (1994)

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The Pentium CPU computed wrong results for certain floating point operations, e.g.

$$4195835 - (4195835/3145727) \times 3145727 = 256$$

Cause: for efficiency reasons, the division operation used a table with 1066 pre-computed entries of which five were wrong.

Estimated cost for exchanging the CPUs: 500 million dollars

# Ariane 5 crash (1996)

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Ariane 5 began to disintegrate 39 seconds after launch because of aerodynamic loads resulting from an angle of attack of more than 20 degrees.

# Cause of the Ariane crash

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The angle of attack was caused by incorrect altitude data following a software exception.

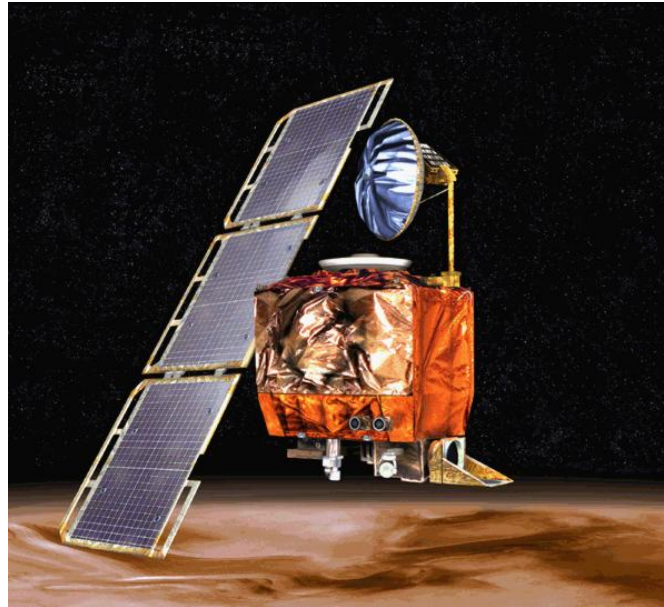
The software exception was raised by an overflow in the conversion of a 64-bit floating-point number to a 16-bit signed integer value. The result was an operand error.

The operand error occurred because of an unexpected high value of the horizontal velocity sensed by the platform. The value was much higher than expected because the early part of the trajectory of Ariane 5 differs from that of Ariane 4 and results in higher horizontal velocity values.

Direct cost 500.000.000 EUR, indirect cost 2.000.000.000 EUR

# Loss of Mars Climate Orbiter (1999)

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Cause: unchecked type mismatch of metric and imperial units.

# Power shutdown on USS Yorktown (1998)

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Cause: A sailor mistakenly typed 0 in a field of the kitchen inventory application. Subsequent division by this field cause an arithmetic exception, which propagated through the system, crashed all LAN consoles and remote terminal units, and led to power shutdown for about 3 hours.

# Unintended acceleration of Toyota cars (2009-2014)

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Electronic Throttle Control System (ETCS) suspected .

NASA team investigated it in 2010-11, results inconclusive.

Further investigation in 2012-13 found unprotected critical variables, stack overflows, memory corruption.

Jury found that ETCS defects caused a death, experts testified ETCS is unsafe.

Toyota fined 1.2 billion dollars for concealing safety defects in 2014, although not directly for software bugs.



# Apple's SSL bug (2014)

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Security check not performed due to a simple code error

```
...
if ((err = SSLFreeBuffer(&hashCtx)) != 0)
    goto fail;
if ((err = ReadyHash(&SSLHashSHA1, &hashCtx)) != 0)
    goto fail;
...
if ((err = SSLHashSHA1.update(&hashCtx, &signedParams)) != 0)
    goto fail;
    goto fail; /* MISTAKE! THIS LINE SHOULD NOT BE HERE */
if ((err = SSLHashSHA1.final(&hashCtx, &hashOut)) != 0)
    goto fail;
err = sslRawVerify(...);
...
```

(see also Heartbleed ...)

# Approaches to solving the problem

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## Avoid bugs:

Appropriate programming languages

Software engineering methods

## Detect bugs:

Simulation, testing

## Prove their absence:

Deductive methods (Hoare)

Program analysis

## Detect bugs and prove their absence:

Model checking

# Simulation and testing

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Can be used to find bugs in the design phase ([simulation](#)) or in the final product ([testing](#)).

Methods: Blackbox/whitebox testing, coverage metrics, etc

**Advantage:** can find (obvious) bugs quickly and cost-efficiently

**Disadvantage:** incompleteness

No coverage metric guarantees the absence of bugs, even at 100%, or even gives an estimate of the number of remaining bugs.

Achieving complete coverage becomes more difficult with growing complexity.

Concurrent systems very difficult to test.

# Program analysis

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Analyzes an overapproximation of the program

Symbolic execution on an **abstract domain** (like intervals)

## Advantages:

- Can prove absence of standard bugs (e.g. division by 0, array out of bounds).

- Often quite efficient, applicable to large systems.

## Disadvantages:

- Incompleteness, can produce large number of false alarms.

- Analyzes standard errors, not very specification oriented.

# Deductive verification

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Proofs using formal program **semantics** (Dijkstra, Hoare et al.)

Example: Hoare logic:

$$\{P\} S \{Q\}$$

Meaning: Whenever  $P$  holds before the execution of  $S$ , then  $Q$  holds afterwards.

Proof rules, e.g.

$$\{P\} \text{skip} \{P\} \quad \{P[x/e]\} x := e \{P\} \quad \frac{\{P\} S_1 \{Q\} \wedge \{Q\} S_2 \{R\}}{\{P\} S_1; S_2 \{R\}}$$

# Example: proof rule for loops

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$\{P\}$  while  $\beta$  do  $C$   $\{Q\}$

Show that there exists an **invariant**  $I$  with the following properties:

$$P \Rightarrow I \qquad \{I \wedge \beta\} C \{I\} \qquad I \wedge \neg\beta \Rightarrow Q$$

Termination: find a function  $f(x, y, \dots)$  of the program variables such that

$$\{\beta \wedge f(x, y, \dots) = k\} C \{f(x, y, \dots) < k\} \qquad f(x, y, \dots) \leq 0 \Rightarrow \neg\beta$$

The program  $C$  is deemed correct if  $\{true\} C \{P\}$  holds, where  $P$  is the function of interest.

# Advantages and disadvantages

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## Advantages:

Complete; its power is limited only by the (human) prover.

## Disadvantages:

see above

Onerous proofs “by hand” (help from [theorem provers](#)).

Very difficult for concurrent systems.

# Summary

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Simulation and testing can detect bugs but not prove their absence.

(They consider a **subset** of the possible executions.)

Deductive methods and program analysis can prove the absence of bugs, but can yield false positive.

(They consider a **superset** of the possible executions).

**Model checking** considers **all** and only the possible executions of a system

→ detection of bugs *and* proof of their absence is both possible (in principle).

→ computationally costly (but can be merged with program analysis)

→ particularly attractive for **concurrent systems**



# Reactive systems

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Examples: operating system, server, ATM, telephone switching system, . . .

Concurrent systems; no “function” is being computed, termination usually undesirable.

We are interested in certain properties of their executions, e.g.

- No deadlocks.

- No two processes can be in some “critical section” concurrently.

- Whenever a process wants to enter a critical section it will eventually be able to do so.

- All requests are eventually served.

⇒ formalization using [temporal logics](#)

# Propositional logic (Syntax)

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Formulae of propositional logic consist of **atomic propositions**, e.g.

$A \hat{=} \text{“Anna is an architect”}$

$B \hat{=} \text{“Bruno is a bear”}$

and **connectives**, e.g.  $\wedge$  (“and”),  $\vee$  (“or”),  $\neg$  (“not”),  $\rightarrow$  (“implies”).

# Examples

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Example formulae of propositional logic:

$A \wedge B$  (“Anna is an architect and Bruno is a bear”)

$\neg B$  (“Bruno is not a bear”)

Are such formulae true?

Answer: **It depends.**

Some formulae are always true ( $A \vee \neg A$ ) or always false ( $B \wedge \neg B$ ).

But in general, formulae are evaluated w.r.t. some **valuation** (or: “world”).

# Semantics of propositional logic

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A **valuation**  $\mathcal{B}$  is a function assigning a **truth value** (1 or 0) to each atomic proposition.

The **semantics** of a formula (defined inductively) is the set of valuations making the formula “true” and denoted  $\llbracket F \rrbracket$ . E.g.,

if  $F = A$  then  $\llbracket F \rrbracket = \{ \mathcal{B} \mid \mathcal{B}(A) = 1 \}$ ;

if  $F = F_1 \wedge F_2$  then  $\llbracket F \rrbracket = \llbracket F_1 \rrbracket \cap \llbracket F_2 \rrbracket$ ; ...

Other notations:  $\mathcal{B} \models F$  iff  $\mathcal{B} \in \llbracket F \rrbracket$ .

We say: “ $\mathcal{B}$  fulfils  $F$ ” or “ $\mathcal{B}$  is a model of  $F$ ”.

# The model-checking problem in propositional logic

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**Problem:** Given a valuation  $\mathcal{B}$  and a formula  $F$  of propositional logic; check whether  $\mathcal{B}$  is a model of  $F$ .

**Solution:**

Replace the atomic propositions by their truth values in  $\mathcal{B}$ , then use a truth table to evaluate to 1 or 0.

**Examples:** Let  $\mathcal{B}_1(A) = 1$  and  $\mathcal{B}_1(B) = 0$ . Then  $\mathcal{B}_1 \not\models A \wedge B$  and  $\mathcal{B}_1 \models \neg B$ .

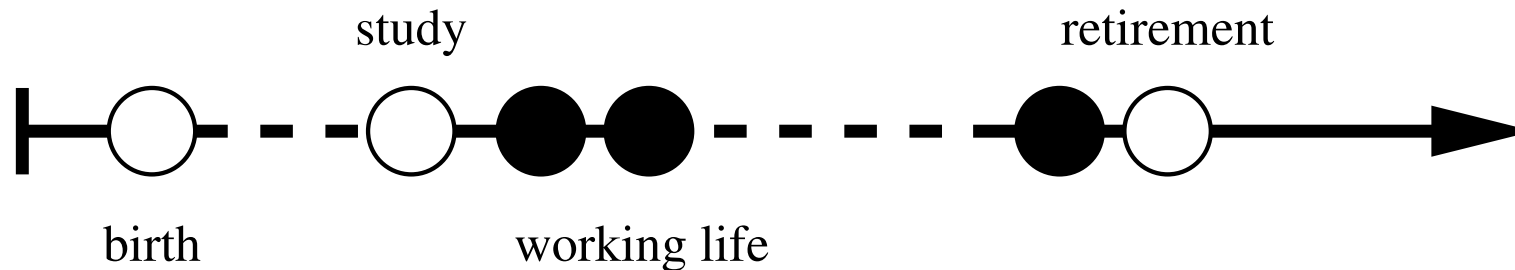
Let  $\mathcal{B}_2(A) = 1$  and  $\mathcal{B}_2(B) = 1$ . Then  $\mathcal{B}_2 \models A \wedge B$  and  $\mathcal{B}_2 \not\models \neg B$ .

# Temporal logic

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Takes into account that truth values of atomic proposition may change with time (the “world” transforms).

Example: Truth values of  $A$  in the course of Anna’s life:



Possible statements:

Anna will **eventually** be an architect (at some point in the future).

Anna is an architect **until** she retires.

⇒ Extension of propositional logic with temporal connectives (eventually, until)

# Preview

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## Linear-time temporal logics (example: LTL)

formulae with temporal operators

evaluated w.r.t. (infinite) sequences of valuations

Model-checking problem for LTL: Given an LTL formula and a sequence of valuations, check whether the sequence is a model of the formula.

## Computation-tree logic (CTL, CTL\*)

Considers (infinite) *trees* of valuations.

Interpretation: non-determinism; multiple possible developments.

# Connection with program verification

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**State space** of a program:

value of program counter

contents of variables

contents of stack, heap, ...

Possible atomic propositions:

“Variable  $x$  has a positive value.”

“The program counter is at label  $\ell$ .”

Given a set of atomic propositions, each program state gives rise to a valuation!



# Programs and temporal logics

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## Linear-time temporal logic:

Each program execution yields a **sequence** of valuations.

Interpretation of the program: the set of possible sequences

Question of interest: Do all sequences satisfy a given LTL formula?

## Computation-tree logic:

The program may branch at certain points, its possible executions yield a **tree** of valuations.

Interpretation of the program: tree with the (valuation of the) initial state as its root

Question of interest: Does this tree satisfy a given CTL formula?

Thus: **verification problem**  $\cong$  **model-checking problem**

# Model checking

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Apart from its definition in terms of logic, the term **model checking** is generally understood to mean methods that

**verify** whether a given system satisfies a given specification;

work **automatically**;

either prove **correctness** of the system w.r.t. to the specification;

or exhibit a **counterexample**, i.e. an execution violating the specification (at least in the linear-time framework).

# Pros and cons of model checking

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## Advantages:

works automatically(!)

suitable for reactive, concurrent, distributed systems

can check temporal-logic properties, not just reachability

## Disadvantages:

Programs are generally Turing-powerful → **undecidability**

Approach: concentrate on decidable subclasses, here: **finite automata**;  
interesting connections to automata theory!

State space often very large → **computationally expensive**

approach: **efficient algorithms and data structures**

# Problems with model checking

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For the aforementioned problems we cannot hope to verify arbitrary properties of arbitrary programs!

Possibly we must consider a simplified mathematical model of the system of interest that ignores its “unimportant” aspects.

Construction of such models and the specification as well as the actual verification require **effort** and (possibly high) **cost**.

⇒ useful in early design phases

⇒ economic gain for critical systems where failure is costly (CPUs, communication protocols, aircraft, ...)

# Success stories of model checking

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Since end of the 1970s: research on theoretical foundations

Since the 1990s: industrial applications

First hardware verification, later software verification:

verification of the **cache protocol in the IEEE Futurebus+** (1992)

The tool SMV was able to find several bugs after four years of trying to validate the protocol with other means.

verification of the **floating-point unit of the Pentium4** (2001)

Static Driver Verifier (Microsoft, 2000–2004) (**Windows device drivers**)

Research groups in big companies: IBM, Intel, Microsoft, OneSpin Solutions, ...

**Turing award 2007** for its “inventors”: Clarke, Emerson, Sifakis

# Goal of this course

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The course teaches the fundamentals of model checking, its theory and applications, especially **modelling** systems, formulating **specifications**, and **verifying** them.

**Modelling:** transition systems, Kripke structures; tools: Spin, SMV

**Specification:** temporal logics (LTL, CTL)

**Verification:** fundamental techniques and extensions (partial-order reduction, BDDs, abstraction, bounded model checking)

# Part 2: Kripke structures

# System model

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We shall use a very generic (and unspecific) model, i.e. **transition systems**, essentially directed graphs:

$$\mathcal{T} = (S, \rightarrow, r)$$

$S$   $\hat{=}$  **state space**; states that the system may attain  
(finite or infinite set)

$\rightarrow \subseteq S \times S$   $\hat{=}$  **transition relation**; describes which actions  
or “steps” are possible

$r \in S$   $\hat{=}$  **initial state** (“root”)



# Example 1: Producer/Consumer

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(Pseudocode) program with variables and concurrency:

```
var turn {0,1} init 0;  
cobegin {P || K} coend
```

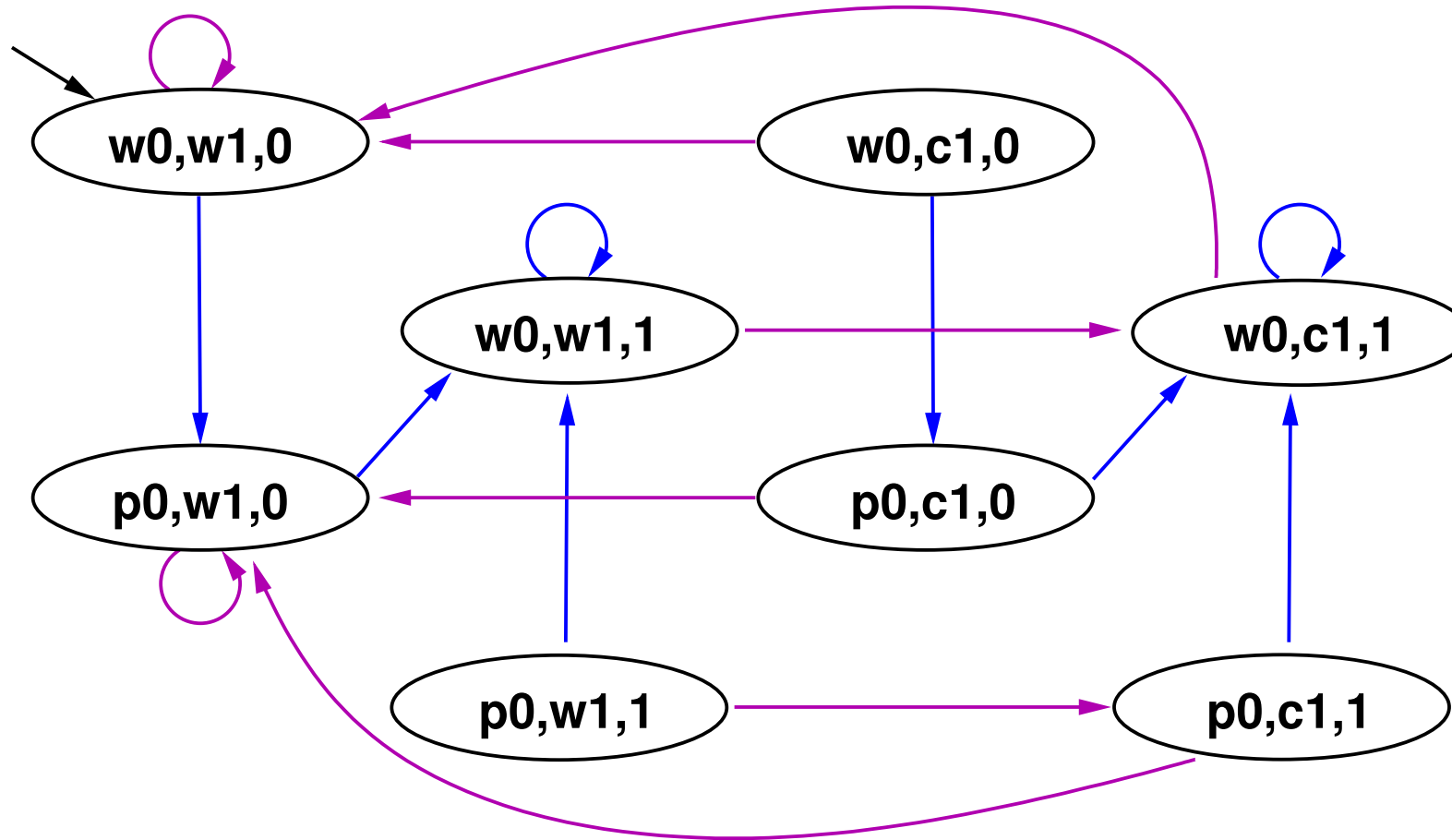
```
P = start;  
    while true do  
        w0: wait (turn = 0);  
        p0: /* produce */  
            turn := 1;  
    od;  
end
```

```
K = start;  
    while true do  
        w1: wait (turn = 1);  
        c1: /* consume */  
            turn := 0;  
    od;  
end
```

# Example 1: Corresponding transition system

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$S = \{w_0, p_0\} \times \{w_1, c_1\} \times \{0, 1\}$ ; initial state  $(w_0, w_1, 0)$



## Example 2: Recursive Program

```

procedure p;
p0: if ? then
p1:     call s;
p2:     if ? then call p; end if;
           else
p3:     call p;
           end if
p4: return
  
```

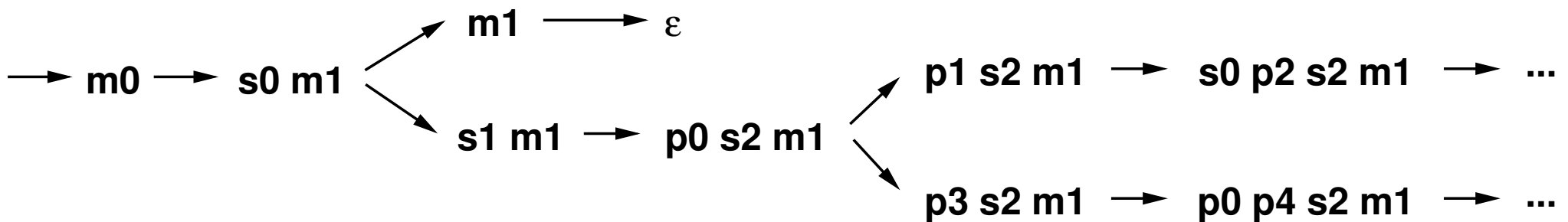
```

procedure s;
s0: if ? then return; end if;
s1: call p;
s2: return;

procedure main;
m0: call s;
m1: return;
  
```

$S = \{p_0, \dots, p_4, s_0, \dots, s_2, m_0, m_1\}^*$ ,

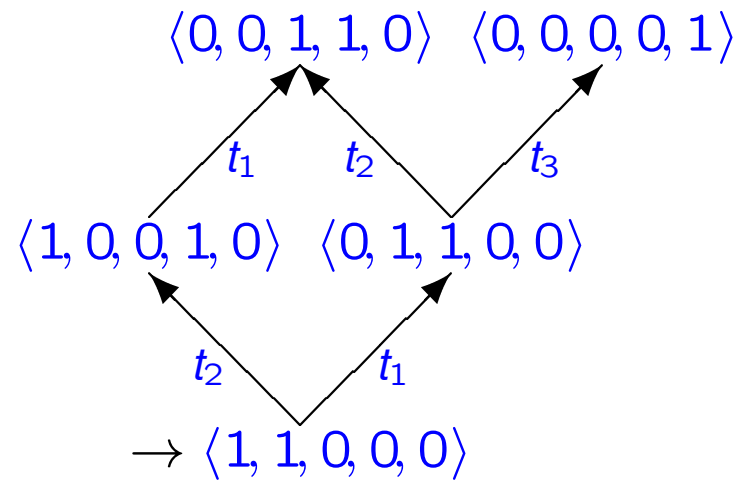
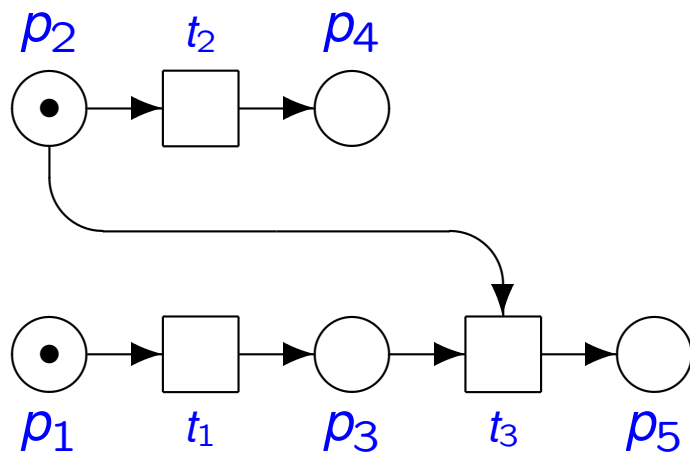
initial state  $m_0$



# Example 3: Petri net

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State space = set of markings



# Implicitly given transition systems

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Quite often, a transition system is given to us “implicitly”, i.e. in the form of a program, from which we extract the transition system.

In such a setting, we are given an initial state and a function for computing the direct successor states of a given state, such that transitions are computed only on demand.

Some of our analysis methods will be suitable for such a setting.

# Notation for transition systems

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We write  $s \rightarrow t$  if  $(s, t) \in \rightarrow$ .

If  $s \rightarrow t$  then  $s$  is called a **direct predecessor** of  $t$  and  $t$  a **direct successor** of  $s$ .

$S^*$  denotes the *finite*,  $S^\omega$  the *infinite* sequences (words) over  $S$ .

$w = s_0 \dots s_n$  is a **path** of length  $n$  if  $s_i \rightarrow s_{i+1}$  for all  $0 \leq i < n$ .

$\rho = s_0 s_1 \dots$  is an **infinite path** if  $s_i \rightarrow s_{i+1}$  for all  $i \geq 0$ .

## Notation for transition systems II

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$\rho(i)$  denotes the  $i$ -th element of  $\rho$  and  $\rho^i$  the suffix starting at  $\rho(i)$ .

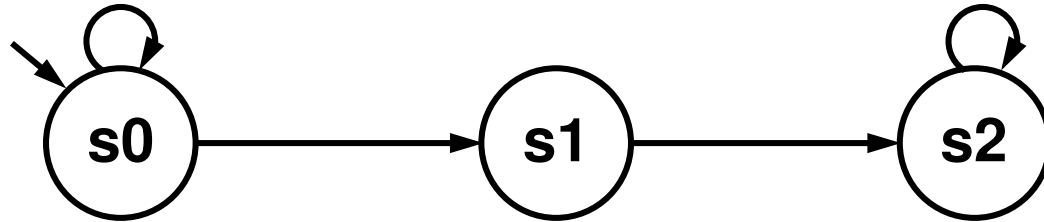
$s \rightarrow^* t$  if there is a path from  $s$  to  $t$ .

$s \rightarrow^+ t$  if there is a path from  $s$  to  $t$  with at least one transition.

If  $s \rightarrow^* t$  then  $s$  is a predecessor of  $t$  and  $t$  a successor of  $s$ .

# Example

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$S = \{s_0, s_1, s_2\}$ ; initial state  $s_0$

$s_0 \rightarrow s_0$      $s_0 \rightarrow s_1$      $s_1 \rightarrow s_2$      $s_2 \rightarrow s_2$

$s_0s_1s_2$  is a path of length 2, i.e.  $s_0 \rightarrow^* s_2$  and  $s_0 \rightarrow^+ s_2$

$s_1 \rightarrow^* s_1$  but  $s_1 \not\rightarrow^+ s_1$

$\rho = s_0s_0s_1s_2s_2s_2 \dots$  is an infinite path.

$\rho(2) = s_1$      $\rho^1 = s_0s_1s_2s_2s_2 \dots$



# Finite and infinite transition systems

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In principle, a transition system may have infinitely many states. Some of the possible reasons are:

**Data:** integers, reals, lists, trees, pointer structures, ...

**Control:** (recursive) procedures, dynamic thread creation, ...

**Communication:** unbounded FIFO channels, ...

**Unknown parameters:** number of participants in a protocol, ...

**Real time:** discrete or continuous time

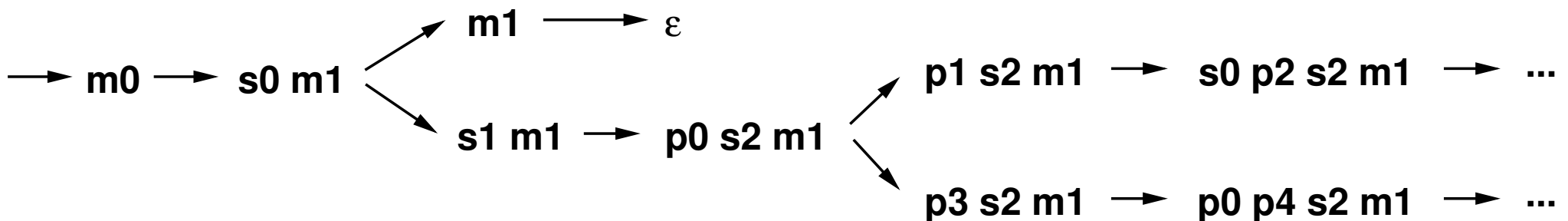
Some (not all!) of these features lead to **Turing-powerful** computation models (and thus **undecidable** verification problems).

# Example: Recursive program

```
procedure p;  
p0: if ? then  
p1:     call s;  
p2:     if ? then call p; end if;  
      else  
p3:     call p;  
      end if  
p4: return
```

```
procedure s;  
s0: if ? then return; end if;  
s1: call p;  
s2: return;  
  
procedure main;  
m0: call s;  
m1: return;
```

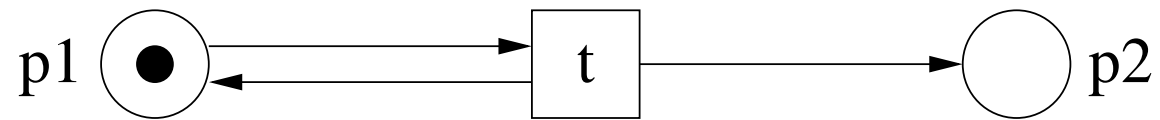
The state space of this example is infinite (stack!), however, LTL and CTL model checking remain decidable.



# Example: Petri net

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Petri nets may have an infinite state space, too:



Reachable states are:  $\langle 1, 0 \rangle$ ,  $\langle 1, 1 \rangle$ ,  $\langle 1, 2 \rangle$ , ...

Reachability and LTL decidable, CTL undecidable.

# Finite state spaces

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For now, we restrict ourselves to finite state spaces.

Finite systems: e.g. hardware systems, programs with finite data types (Boolean programs), certain communication protocols, ...

Finite systems may also be obtained by **abstracting** an infinite system.

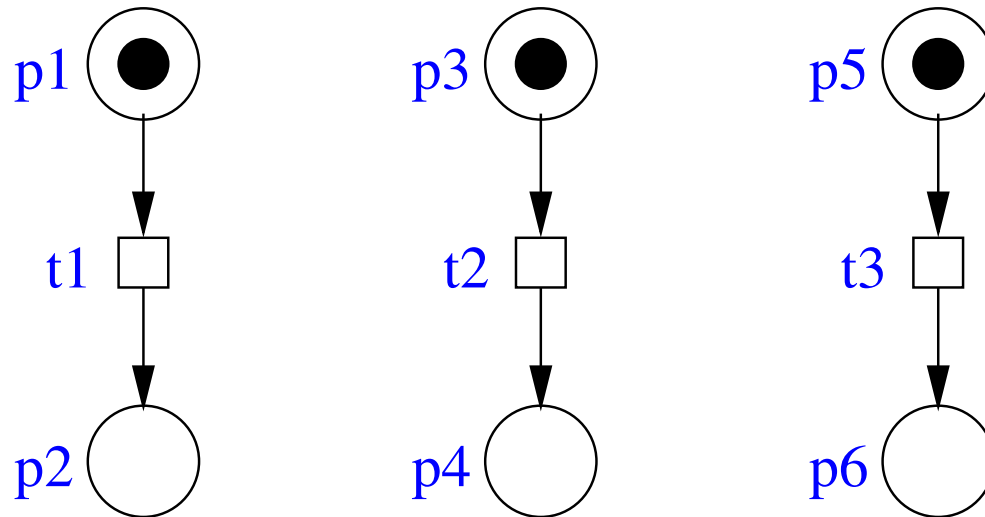
Remaining problem: **state-space explosion**, systems may be finite but VERY large.

# Reasons for state-space explosion (1)

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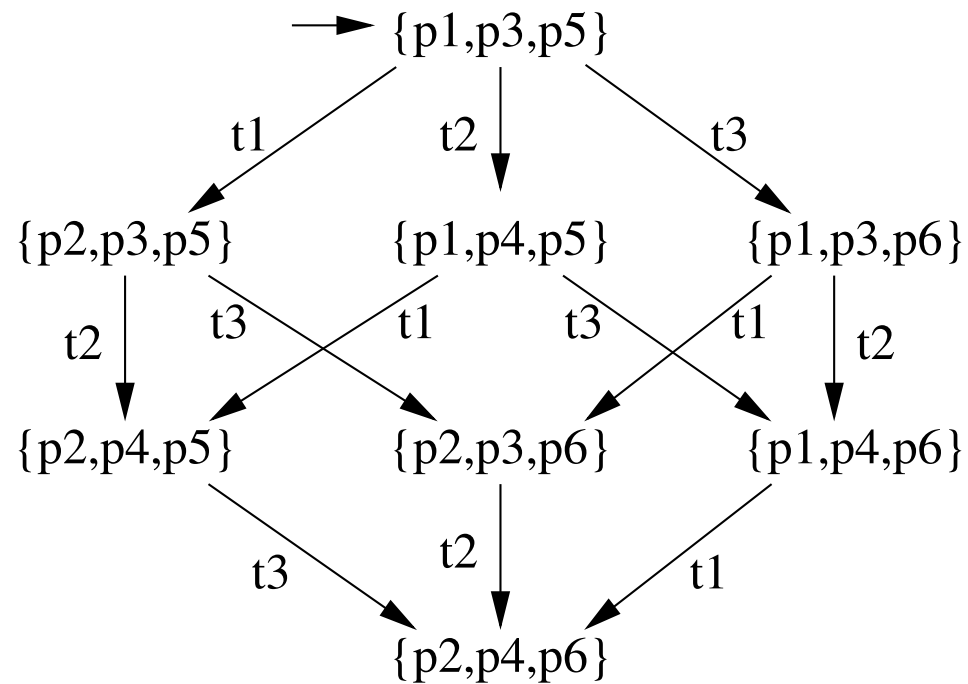
A common reason is **concurrency**.

**Example:** Consider the following Petri net:



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The reachability graph has got  $8 = 2^3$  states and  $6 = 3!$  paths.



With  $n$  components we have  $2^n$  states and  $n!$  paths.

## Reasons for state-space explosion (2)

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A second common reason is **data**.

e.g. programs with few large or many small variables

size of state space: 2 to the number of bits

Counteractions:

**Abstraction**: ignore “unimportant” data

**Compression**: work with *sets* of states; efficient data structures for storing and manipulating sets

**Approximation**: find over- or underapproximations of the reachable states

We will see some examples of these techniques during the course.

# Kripke structures

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Idea: Extract from each state a **valuation**.

$$\mathcal{K} = (S, \rightarrow, r, AP, \nu)$$

$(S, \rightarrow, r)$   $\cong$  underlying **transition system**

$AP$   $\cong$  set of **atomic propositions**

$\nu: S \rightarrow 2^{AP}$   $\cong$  **Interpretation** of atomic propositions (“valuation”)

Remarks:

$2^{AP}$  denotes the *powerset* of  $AP$ .

Valuations are represented here as subsets of  $AP$  rather than functions; the propositions contained in the set are those that are deemed true.



# Example of a Kripke structure

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Transition system  $(S, \rightarrow, r)$  as in Example 1.

Suppose we are interested in the acts of production and consumption.

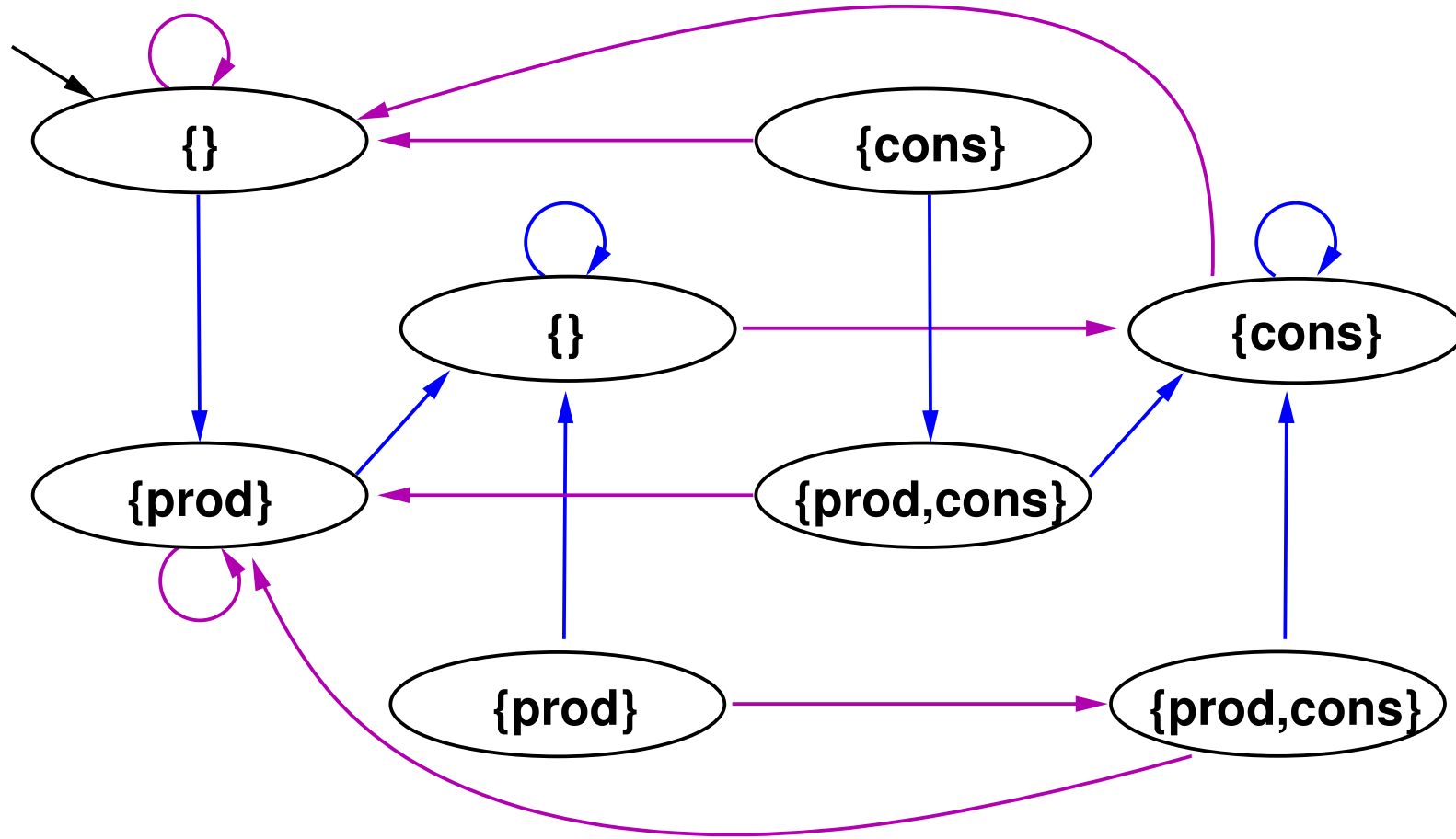
Sei  $AP = \{prod, cons\}$ ;

$$\nu^{-1}(prod) = \{p_0\} \times \{w_1, c_1\} \times \{0, 1\};$$

$$\nu^{-1}(cons) = \{w_0, p_0\} \times \{c_1\} \times \{0, 1\}.$$

---

The valuations in Example 1:



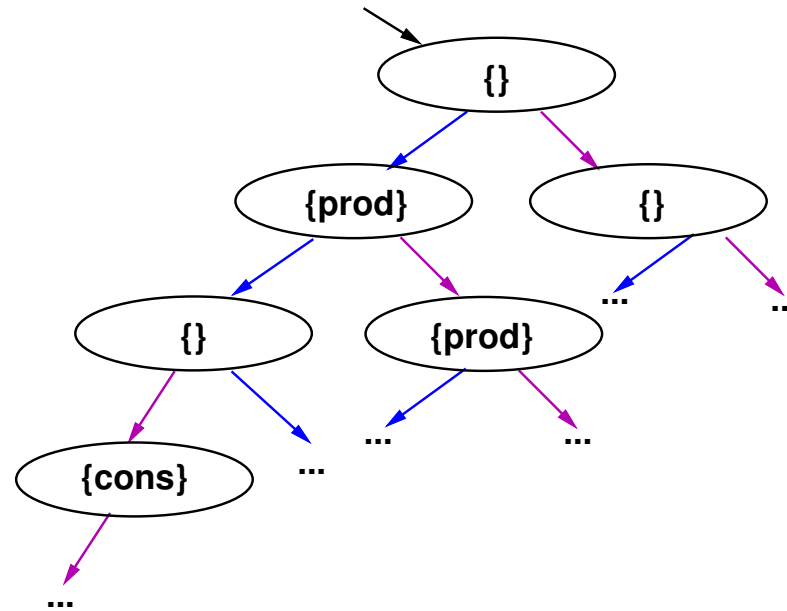
# Sequences and trees of valuations

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In **linear-time logic** we consider the possible valuation sequences:

z.B.  $\emptyset \emptyset \{prod\} \emptyset \{cons\} \dots$  oder  $\emptyset \{prod\} \{prod\} \{prod\} \dots$

In **computation-tree logic** we consider the “tree-wise unfolding” of the Kripke structure:



# Examples of temporal-logic properties

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“It is never possible that *prod* and *cons* hold at the same time.”

Intuitively, this property holds because no state in which both *prod* and *cons* holds is reachable from the beginning, which can be verified by inspecting the sequences and trees.

A property of this form is also called an **invariant**.

“Whenever something is produced it may be consumed afterwards.”

We may take the view that this property does not hold because of the following sequence:  $\emptyset \{prod\} \emptyset \emptyset \emptyset \dots$ . Thus, something is produced but followed by an infinite loop.

A property of this form is also called a **reactivity property**.

# Fairness

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We may also take the view that the second property merely fails because of overly simplistic modelling:

In the counterexample, only one process is acting, making “empty” steps, while the second process does not do anything.

Such a behaviour is usually unrealistic in concurrent systems; even if one process may be faster than another and execute multiple steps, a “fair” scheduler will eventually grant execution time to either process.

We may therefore wish to exclude such unrealistic (“unfair”) executions and only consider “fair” ones. In other words, we work under certain “fairness assumptions”.

Under a reasonable fairness assumption, the second property holds.

# Part 3: Linear-time logic

# Preliminaries

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Linear-time logic in general:

any logic working with sequences of valuations

model: time progresses in discrete steps and in linear fashion, each point in time has exactly one possible future

origins in philosophy/logic

Most prominent species: **LTL**

in use for verification since end of the 1970s

**specification** of correctness properties

# Recap

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Let  $AP$  be a set of atomic propositions.

$2^{AP}$  denotes the powerset of  $AP$ , i.e. its set of subsets.

$(2^{AP})^\omega$  denotes the set of (infinite) sequences of valuations (of  $AP$ ).



# Syntax of LTL

---

Let  $AP$  be a set of atomic propositions. The set of **LTL formulae** over  $AP$  is inductively defined as follows:

If  $p \in AP$  then  $p$  is a formula.

If  $\phi_1, \phi_2$  are formulae then so are

$\neg\phi_1, \quad \phi_1 \vee \phi_2, \quad \mathbf{X} \phi_1, \quad \phi_1 \mathbf{U} \phi_2$

Intuitive meaning:  $\mathbf{X} \hat{=}$  “next”,  $\mathbf{U} \hat{=}$  “until”.

# Remarks

---

This is a minimal syntax that we will use for proofs etc.

For added expressiveness, we will later define some abbreviations based on the minimal syntax.

Comparison of propositional logic (PL) and LTL:

	PL	LTL
Syntax	atomic proposition, logic operators	+ temporal operators
Evaluated on...	valuations	sequences of valuations
Semantics	set of valuations	set of valuation sequences

# Semantics of LTL

---

Let  $\phi$  be an LTL formula and  $\sigma$  a valuation sequence.

We write  $\sigma \models \phi$  for “ $\sigma$  satisfies  $\phi$ .”

$\sigma \models p$	if $p \in AP$ and $p \in \sigma(0)$
$\sigma \models \neg\phi$	if $\sigma \not\models \phi$
$\sigma \models \phi_1 \vee \phi_2$	if $\sigma \models \phi_1$ or $\sigma \models \phi_2$
$\sigma \models \mathbf{X}\phi$	if $\sigma^1 \models \phi$
$\sigma \models \phi_1 \mathbf{U} \phi_2$	if $\exists i: (\sigma^i \models \phi_2 \wedge \forall k < i: \sigma^k \models \phi_1)$

Semantics of  $\phi$ :  $\llbracket \phi \rrbracket = \{ \sigma \mid \sigma \models \phi \}$

# Examples

---

Let  $AP = \{p, q, r\}$ . Find out whether the sequence

$$\sigma = \{p\} \{q\} \{p\}^\omega$$

satisfies the following formulae:

$p$

$q$

$X q$

$X \neg p$

$p U q$

$q U p$

$(p \vee q) U r$

# Extended syntax

---

We will commonly use the following abbreviations:

$$\phi_1 \wedge \phi_2 \equiv \neg(\neg\phi_1 \vee \neg\phi_2)$$

$$\mathbf{F} \phi \equiv \mathbf{true} \mathbf{U} \phi$$

$$\phi_1 \rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$$

$$\mathbf{G} \phi \equiv \neg \mathbf{F} \neg\phi$$

$$\mathbf{true} \equiv p \vee \neg p$$

$$\phi_1 \mathbf{W} \phi_2 \equiv (\phi_1 \mathbf{U} \phi_2) \vee \mathbf{G} \phi_1$$

$$\mathbf{false} \equiv \neg \mathbf{true}$$

$$\phi_1 \mathbf{R} \phi_2 \equiv \neg(\neg\phi_1 \mathbf{U} \neg\phi_2)$$

Meaning:  $\mathbf{F}$   $\hat{=}$  “finally” (eventually),  $\mathbf{G}$   $\hat{=}$  “globally” (always),  
 $\mathbf{W}$   $\hat{=}$  “weak until”,  $\mathbf{R}$   $\hat{=}$  “release”.

# Some example formulae

---

Invariant:  $G \neg(cs_1 \wedge cs_2)$

$cs_1$  and  $cs_2$  are never true at the same time.

Remark: This particular form of invariant is also called **mutex property** (“mutual exclusion”).

Safety:  $(\neg p) W q$

$x$  does not occur before  $y$  has happened.

Remark: It may happen that  $q$  never happens in which case  $p$  also never happens.

Liveness:  $(\neg p) U q$

$p$  does not occur before  $q$  **and**  $q$  eventually happens.

# More examples

---

$\mathbf{GF} p$

$p$  occurs infinitely often.

$\mathbf{FG} p$

At some point  $p$  will continue to hold forever.

$\mathbf{G}(try_1 \rightarrow \mathbf{F} cs_1)$

For mutex algorithms: Whenever process 1 tries to enter its critical section it will eventually succeed.

# Tautology, equivalence

---

Certain concepts from propositional logic can be transferred to LTL.

**Tautology:** A formula  $\phi$  with  $\llbracket \phi \rrbracket = (2^{AP})^\omega$  is called tautology.

**Unsatisfiability:** A formula  $\phi$  with  $\llbracket \phi \rrbracket = \emptyset$  is called unsatisfiable.

**Equivalence:** Two formulae  $\phi_1, \phi_2$  are called equivalent iff  $\llbracket \phi_1 \rrbracket = \llbracket \phi_2 \rrbracket$ .  
Denotation:  $\phi_1 \equiv \phi_2$



# Equivalences: relations between operators

---

$$\mathbf{X}(\phi_1 \vee \phi_2) \equiv \mathbf{X} \phi_1 \vee \mathbf{X} \phi_2$$

$$\mathbf{X}(\phi_1 \wedge \phi_2) \equiv \mathbf{X} \phi_1 \wedge \mathbf{X} \phi_2$$

$$\mathbf{X} \neg \phi \equiv \neg \mathbf{X} \phi$$

$$\mathbf{F}(\phi_1 \vee \phi_2) \equiv \mathbf{F} \phi_1 \vee \mathbf{F} \phi_2$$

$$\neg \mathbf{F} \phi \equiv \mathbf{G} \neg \phi$$

$$\mathbf{G}(\phi_1 \wedge \phi_2) \equiv \mathbf{G} \phi_1 \wedge \mathbf{G} \phi_2$$

$$\neg \mathbf{G} \phi \equiv \mathbf{F} \neg \phi$$

$$(\phi_1 \wedge \phi_2) \mathbf{U} \psi \equiv (\phi_1 \mathbf{U} \psi) \wedge (\phi_2 \mathbf{U} \psi)$$

$$\phi \mathbf{U} (\psi_1 \vee \psi_2) \equiv (\phi \mathbf{U} \psi_1) \vee (\phi \mathbf{U} \psi_2)$$

# Equivalences: idempotence and recursion laws

---

$$\mathbf{F} \phi \equiv \mathbf{F} \mathbf{F} \phi$$

$$\mathbf{G} \phi \equiv \mathbf{G} \mathbf{G} \phi$$

$$\phi \mathbf{U} \psi \equiv \phi \mathbf{U} (\phi \mathbf{U} \psi)$$

$$\mathbf{F} \phi \equiv \phi \vee \mathbf{X} \mathbf{F} \phi$$

$$\mathbf{G} \phi \equiv \phi \wedge \mathbf{X} \mathbf{G} \phi$$

$$\phi \mathbf{U} \psi \equiv \psi \vee (\phi \wedge \mathbf{X}(\phi \mathbf{U} \psi))$$

$$\phi \mathbf{W} \psi \equiv \psi \vee (\phi \wedge \mathbf{X}(\phi \mathbf{W} \psi))$$

# Interpretation of LTL on Kripke structures

---

Let  $\mathcal{K} = (S, \rightarrow, r, AP, \nu)$  be a Kripke structure.

We are interested in the valuation sequences generated by  $\mathcal{K}$ .

Let  $\rho$  in  $S^\omega$  be an infinite path of  $\mathcal{K}$ .

We assign to  $\rho$  an “image”  $\nu(\rho)$  in  $(2^{AP})^\omega$ ; for all  $i \geq 0$  let

$$\nu(\rho)(i) = \nu(\rho(i))$$

i.e.  $\nu(\rho)$  is the corresponding valuation sequence.

Let  $\llbracket \mathcal{K} \rrbracket$  denote the set of all such sequences:

$$\llbracket \mathcal{K} \rrbracket = \{ \nu(\rho) \mid \rho \text{ is an infinite path of } \mathcal{K} \}$$

# The LTL model-checking problem

---

**Problem:** Given a Kripke structure  $\mathcal{K} = (S, \rightarrow, r, AP, \nu)$  and an LTL formula  $\phi$  over  $AP$ , does  $\llbracket \mathcal{K} \rrbracket \subseteq \llbracket \phi \rrbracket$  hold?

**Definition:** If  $\llbracket \mathcal{K} \rrbracket \subseteq \llbracket \phi \rrbracket$  then we write  $\mathcal{K} \models \phi$ .

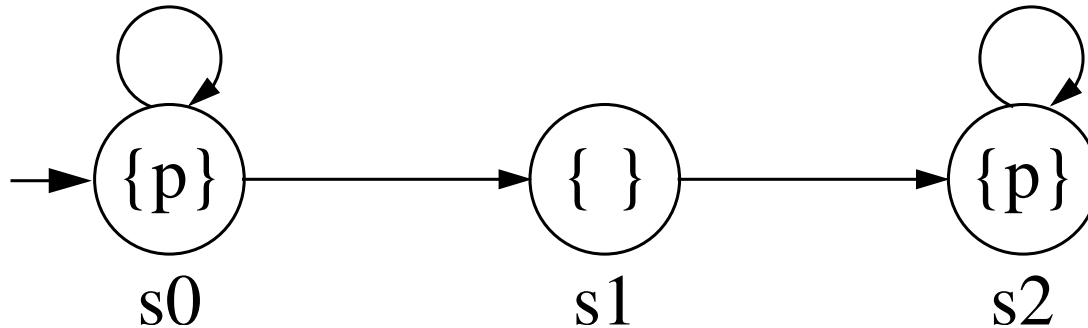
**Interpretation:** **Every** execution of  $\mathcal{K}$  must satisfy  $\phi$  for  $\mathcal{K} \models \phi$  to hold.

**Remark:** We may have  $\mathcal{K} \not\models \phi$  **and**  $\mathcal{K} \not\models \neg\phi$ !

# Example

---

Consider the following Kripke structure  $\mathcal{K}$  with  $AP = \{p\}$ :



There are two classes of infinite paths in  $\mathcal{K}$ :

- (i) Either the system stays in  $s_0$  forever,
- (ii) or it eventually reaches  $s_2$  via  $s_1$  and remains there.

We have:

$\mathcal{K} \models \mathbf{F} \mathbf{G} p$  because all runs eventually end in a state satisfying  $p$ .

$\mathcal{K} \not\models \mathbf{G} p$  because executions of type (ii) contain a non- $p$  state.

# Dealing with deadlocks

---

The definition of the model-checking problem only considers the infinite sequences!

Thus, executions reaching a deadlock (i.e. a state without any successor) will be ignored, with possibly unforeseen consequences:

Suppose  $\mathcal{K}$  contains an error, so that every execution reaches a deadlock.

Then  $\llbracket \mathcal{K} \rrbracket = \emptyset$ , so  $\mathcal{K}$  satisfies *every* formula, according to the definition.

# Possible alternatives

---

Remove deadlocks by design:

- equip deadlock states with a self loop

- Interpretation: system stays in deadlock forever

- adapt formula accordingly, if necessary

Treat deadlocks specially:

- Check for deadlocks before LTL model checking, deal with them separately.

# Tool demonstration: Spin



# Demonstration of Spin

---

**Spin** is a versatile model-checking tool written by **Gerard Holzmann** at **Bell Labs**.

Received the ACM Software System Award in 2002

URL: <http://spinroot.com>

Book: Holzmann, [The Spin Model Checker](#)

# Modelling with Spin

---

System description using **Promela** (Protocol Meta Language)

Suitable for describing **finite** systems

Concurrent processes, synchronous/asynchronous communication, variables, data types

LTL model checking (with fairness and reduction techniques)

# Example: Dekker's Mutex algorithm

---

Model of a protocol for mutual exclusion:

```
bit turn;
```

```
bool flag0, flag1;
```

```
bool crit0, crit1;
```

```
active proctype p0() {
```

```
...
```

```
}
```

```
active proctype p1() {
```

```
...
```

```
}
```

# Dekker: contents of process p0

---

```
active proctype p0() {
again:  flag0 = true;
       do
         :: flag1 ->
           if
             :: turn == 1 ->
               flag0 = false;
               (turn != 1) -> flag0 = true;
             :: else -> skip;
           fi
         :: else -> break;
       od;

       crit0 = true; /* critical section */ crit0 = false;

       turn = 1; flag0 = false;
       goto again;
}
```

Process p1: like p0, but all 0s and 1s exchanged

# What's going on here? (Promela syntax I)

---

Variable declarations:

```
bit turn;  
bool flag0, flag1;  
bool crit0, crit1;
```

`turn` can take values `0` or `1`.

`flag1` can become `true` or `false`.

initial values: by default `0` and `false`, resp.

Other data types: `byte`, user-defined types, ...

# Promela syntax II

---

Process declaration:

```
active proctype p0() {  
  ...  
}
```

`proctype` defines a process *type*. `active` means that one instance of this process type shall be active initially. It is also possible to activate more than one instance initially, e.g.

```
active [2] proctype my_process() {  
  ...  
}
```

Concurrent processes are combined by **interleaving**: In each step of the system one process makes a step while the others remain stationary.

# Promela syntax III

---

labels / assignments / jumps

```
again:  flag0 = true;  
...  
      goto again;
```

empty statement:

```
skip
```

# Promela syntax IV

---

Loop:

```
do
  :: flag1 -> ...
  :: else -> break;
od;
```

`flag1` and `else` are “guards”

Execution branches non-deterministically to some branch whose guard is satisfied.

The `else` branch can only be taken if no other guard is satisfied.

`break` leaves the `do` block.



# Promela syntax V

---

Branching:

```
if
:: turn == 1 -> ...
:: else -> ...;
fi
```

Syntax and semantics as in `do` but without repetition.

```
(turn != 1) -> ...
```

Guarded command: blocks the process until the guard is satisfied.

# Model checking using Spin (Example 1)

---

In the Dekker algorithm the two processes should never be in their critical sections at the same time. This can be expressed by:

$$G \neg(\text{crit0} \wedge \text{crit1})$$

(where the atomic propositions `crit0` and `crit1` mean that the corresponding Boolean variables are true).

LTL syntax in Spin: `[] !(crit0 && crit1)`

# Model checking using Spin (Example 1)

---

In the Dekker algorithm the two processes should never be in their critical sections at the same time. This can be expressed by:

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(where the atomic propositions `crit0` and `crit1` mean that the corresponding Boolean variables are true).

LTL syntax in Spin: `[] !(crit0 && crit1)`

Checking the property with Spin (spinLTL script):

Property satisfied!

## Model checking using Spin (Example 2)

---

In the Dekker algorithm, a process wanting to enter its critical section should eventually succeed.

$$G(\text{flag0} \rightarrow F \text{crit0})$$

(analogously for the other process).

Syntax in Spin: `[] (flag0 -> <> crit0)`

## Model checking using Spin (Example 2)

---

In the Dekker algorithm, a process wanting to enter its critical section should eventually succeed.

$$G(\text{flag0} \rightarrow F \text{crit0})$$

(analogously for the other process).

Syntax in Spin: `[] (flag0 -> <> crit0)`

Checking the property with Spin (spinLTL script):

Property not satisfied!

# Fairness in Spin

---

Process 0 cannot enter its critical section if process 1 gets no share of the computation time to set `flag1` back to `false`.

Such an execution is “unfair”.

Fairness assumption: Consider only those executions in which both processes infinitely often perform a step.

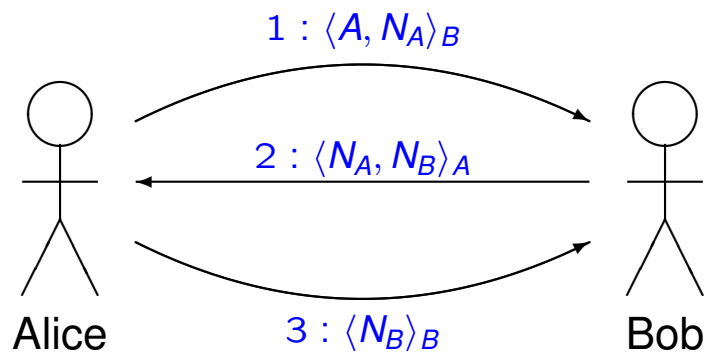
Spin has got a special switch for this (activated by the script `spinFairLTL`).

# Larger example: Needham-Schröder protocol

---

Source: Stephan Merz, Model Checking: A Tutorial Overview, 2001

Goal of the protocol: Alice and Bob try to agree on a “secret”.



- Secret represented by **nonces**  $\langle N_A, N_B \rangle$
- Messages can be intercepted
- Assumption: Alice and Bob can communicate using secure public-key cryptography

Is the protocol secure?

# Analysis of the protocol using Spin

---

## Representation as a finite system

finite number of principals	Alice, Bob, Intruder
finite models of the principals	one (symbolic) nonce per principal intruder can only remember one single message
simple net model	common message channel messages as tuples $\langle \textit{recipient}, \textit{data} \rangle$
cryptography simulated	compare keys rather than numerical computation



# More about Promela

---

Declaration of enumeration types:

```
mtype = {red, green, blue};  
mtype x;
```

The first line declares a bunch of symbolic constants.

The second line declares variable `x`, which can take values `0` (uninitialized), `red`, `green`, `blue`.

Declaration of a record:

```
typedef newtype { bit b; mtype m; }
```

# More about Promela

---

Message channels:

```
chan c = [3] of mtype;
```

`c` is the name of the channel.

The number in brackets gives the capacity; `[0]` means synchronous communication.

`of` is followed by the type of data items that may be sent on the channel.

**Write:** `c!red;`

**Read:** `mtype color; c?color;`

# Promela model: Declarations

---

```
#define success          (statusA == ok && statusB == ok)
#define aliceBob        partnerA == bob
#define bobAlice        partnerB == alice
```

## *Definition of an enumeration type*

```
mtype = {msg1, msg2, msg3, alice, bob, intruder,
         nonceA, nonceB, nonceI, keyA, keyB, keyI, ok};
```

```
typedef Crypt { mtype key, d1, d2; } Record type for messages
chan network = [0] of {mtype, /* message number */
                     mtype, /* recipient */
                     Crypt}; Common message channel
```

```
mtype partnerA, partnerB; State of Alice and Bob
mtype statusA, statusB;
```

```
bool knowNA, knowNB; Nonces known to the intruder
```

# Promela model for Alice

---

```
active proctype Alice() {
  if      choose partner
  :: partnerA = bob; partner_key = keyB;
  :: partnerA = intruder; partner_key = keyI;
  fi;

  send the first msg
  network ! msg1, partnerA, ⟨partner_key, alice, nonceA⟩;

  wait for reply (second msg)
  network ? msg2, alice, data;
  check key and nonce
  (data.key == keyA) && (data.d1 == nonceA);
  partner_nonce = data.d2;
  send the third msg
  network ! msg3, partnerA, ⟨partner_key, partner_nonce⟩;
  statusA = ok;
}
```

The model for Bob is similar

# Promela model for intruder (1)

---

```
active proctype Intruder() {
  do
    receive/intercept msg
  :: network ? msg, _, data ->
    if
      remember msg if undecryptable
    :: intercepted = data;
    :: skip;
  fi;
  if
    evaluate msg if decryptable
  :: (data.key == keyI) ->
    if
      :: (data.d1 == nonceA || data.d2 == nonceA) -> knowNA = true;
      :: else -> skip;
    fi;
    if
      :: (data.d1 == nonceB || data.d2 == nonceB) -> knowNB = true;
      :: else -> skip;
    fi;
  :: else -> skip;
  fi;
  :: ...
}
```

## Promela model for intruder (2)

---

```
:: ...
:: if      send first msg to bob
  :: network ! msg1, bob, intercepted;      repeat intercepted msg
  :: data.key = keyB;                       compose a new msg
    if the intruder can pose as Alice or himself
      :: data.d1 = alice;
      :: data.d1 = intruder;
    fi;
    if uses only known nonces
      :: knowsNA -> data.d2 = nonceA;
      :: knowsNB -> data.d2 = nonceB;
      :: data.d2 = nonceI;
    fi;
    network ! msg1, bob, data;
  fi;
:: ...      similar code for other msgs
od;
}
```

# Analysis of the protocol using Spin

---

Desirable properties:

$$G \left( \text{statusA} = \text{ok} \wedge \text{statusB} = \text{ok} \Rightarrow \right. \\ \left. (\text{partnerA} = \text{bob} \Leftrightarrow \text{partnerB} = \text{alice}) \right)$$

$$G(\text{statusA} = \text{ok} \wedge \text{partnerA} = \text{agentB} \Rightarrow \neg \text{knowsNA})$$

$$G(\text{statusB} = \text{ok} \wedge \text{partnerB} = \text{agentA} \Rightarrow \neg \text{knowsNB})$$

Result of the analysis:

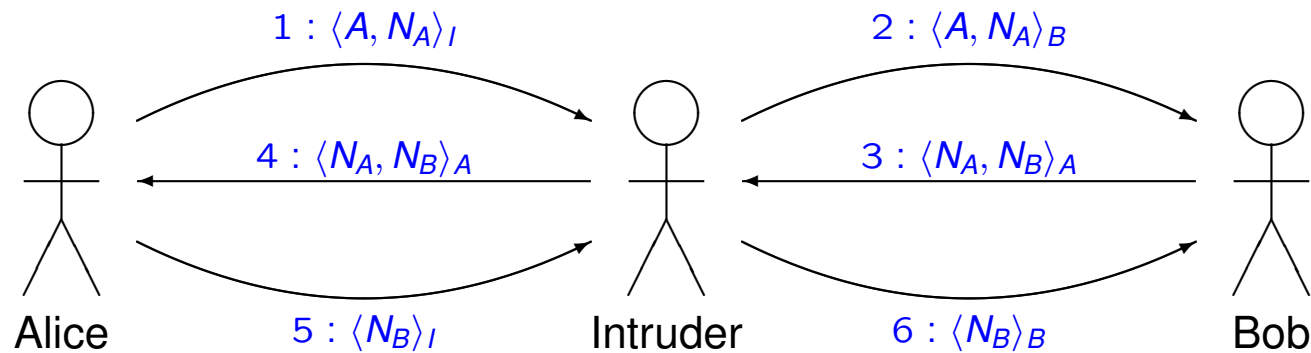
- Property is violated!

# The bug in the protocol

---

Alice opens a connection to Intruder.

Bob mistakenly believes he is talking to Alice.



**This bug was found only 18 years after the protocol was invented!**

[Needham, Schröder (1978), Lowe (1996)]



# Part 4: Büchi automata

# Preview

---

Model-checking problem:  $\llbracket \mathcal{K} \rrbracket \subseteq \llbracket \phi \rrbracket$  – how can we check this **algorithmically**?

(Historically) first approach: Translate  $\mathcal{K}$  into an LTL formula  $\psi_{\mathcal{K}}$ , check whether  $\psi_{\mathcal{K}} \rightarrow \phi$  is a tautology. Problem: very inefficient.

Language-/automata-theoretic approach:  $\llbracket \mathcal{K} \rrbracket$  and  $\llbracket \phi \rrbracket$  are **languages** (of infinite words).

Find a suitable class of automata for representing these languages.

Define suitable operations on these automata for solving the problem.

This is the approach we shall follow.

# Büchi automata

---

A **Büchi automaton** is a tuple

$$\mathcal{B} = (\Sigma, S, s_0, \Delta, F),$$

where:

- $\Sigma$  is a finite **alphabet**;
- $S$  is a finite set of **states**;
- $s_0 \in S$  is an **initial state**;
- $\Delta \subseteq S \times \Sigma \times S$  are **transitions**;
- $F \subseteq S$  are **accepting states**.

Remarks:

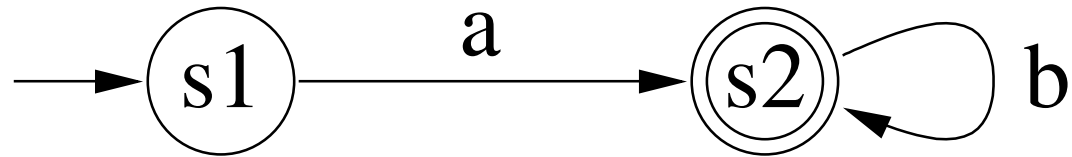
Definition and graphical representation like for finite automata.

However, Büchi automata are supposed to work on *infinite* words, requiring a different acceptance condition.

# Example

---

Graphical representation of a Büchi automaton:



The components of this automaton are  $(\Sigma, S, s_1, \Delta, F)$ , where:

- $\Sigma = \{a, b\}$  (symbols on the edges)
- $S = \{s_1, s_2\}$  (circles)
- $s_1$  (indicated by arrow)
- $\Delta = \{(s_1, a, s_2), (s_2, b, s_2)\}$  (edges)
- $F = \{s_2\}$  (with double circle)

# Language of a Büchi automaton

---

Let  $\mathcal{B} = (\Sigma, S, s_0, \Delta, F)$  be a Büchi automaton.

A **run** of  $\mathcal{B}$  over an infinite word  $\sigma \in \Sigma^\omega$  is an infinite sequence of states  $\rho \in S^\omega$  where  $\rho(0) = s_0$  and  $(\rho(i), \sigma(i), \rho(i+1)) \in \Delta$  for  $i \geq 0$ .

We call  $\rho$  **accepting** iff  $\rho(i) \in F$  for infinitely many values of  $i$ .

I.e.,  $\rho$  infinitely often visits accepting states.

(By the pigeon-hole principle: at least one accepting state is visited infinitely often.)

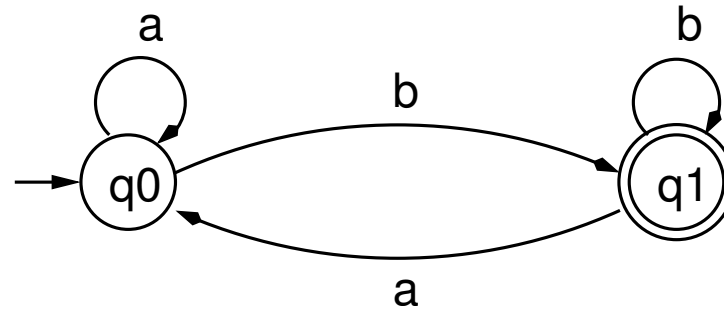
$\sigma \in \Sigma^\omega$  is **accepted** by  $\mathcal{B}$  iff there exists an accepting run over  $\sigma$  in  $\mathcal{B}$ .

The **language of  $\mathcal{B}$** , denoted  $\mathcal{L}(\mathcal{B})$ , is the set of all words accepted by  $\mathcal{B}$ .

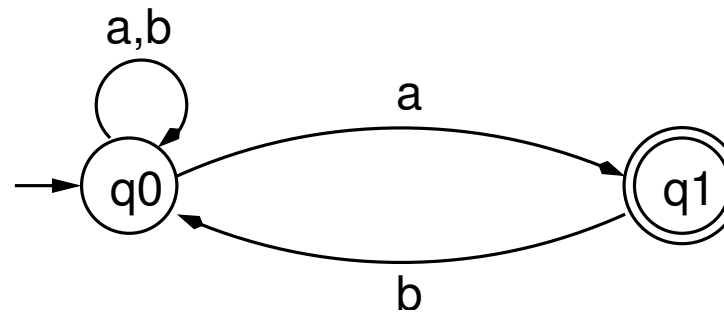
# Büchi automata: examples

---

“infinitely often b”



“infinitely often ab”



# Büchi automata and LTL

---

Let  $AP$  be a set of atomic propositions.

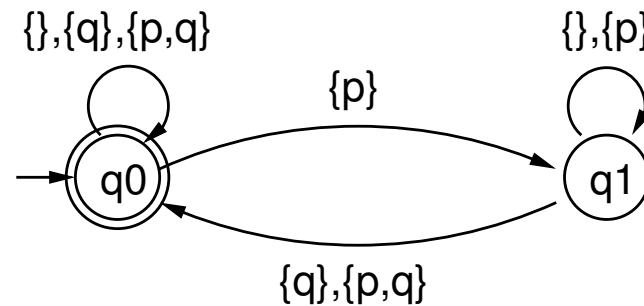
A Büchi automaton with alphabet  $2^{AP}$  accepts a sequence of valuations.

**Claim:** For every LTL formula  $\phi$  there exists a Büchi automaton  $\mathcal{B}$  such that  $\mathcal{L}(\mathcal{B}) = \llbracket \phi \rrbracket$ .

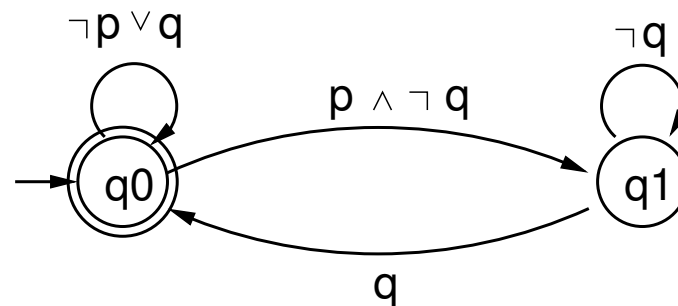
(We shall prove this claim later.)

Examples:  $\mathbf{F} p$ ,  $\mathbf{G} p$ ,  $\mathbf{G} \mathbf{F} p$ ,  $\mathbf{G}(p \rightarrow \mathbf{F} q)$ ,  $\mathbf{F} \mathbf{G} p$

Example automaton for  $G(p \rightarrow F q)$ , with  $AP = \{p, q\}$ .



Alternatively we can label edges with formulae of propositional logic; in this case, a formula  $F$  stands for all elements of  $\llbracket F \rrbracket$ . In this case:





# Operations on Büchi automata

---

The languages accepted by Büchi automata are also called  $\omega$ -regular languages.

Like the usual regular languages,  $\omega$ -regular languages are also closed under Boolean operations.

I.e., if  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are  $\omega$ -regular, then so are

$$\mathcal{L}_1 \cup \mathcal{L}_2, \quad \mathcal{L}_1 \cap \mathcal{L}_2, \quad \mathcal{L}_1^c.$$

We shall now define operations that take Büchi automata accepting some languages  $\mathcal{L}_1$  and  $\mathcal{L}_2$  and produce automata for their union or intersection.

In the following slides we assume  $\mathcal{B}_1 = (\Sigma, S, s_0, \Delta_1, F)$  and  $\mathcal{B}_2 = (\Sigma, T, t_0, \Delta_2, G)$  (with  $S \cap T = \emptyset$ ).

# Union

---

“Juxtapose”  $\mathcal{B}_1$  and  $\mathcal{B}_2$  and add a new initial state.

In other words, the automaton  $\mathcal{B} = (\Sigma, S \cup T \cup \{u\}, u, \Delta_1 \cup \Delta_2 \cup \Delta_u, F \cup G)$  accepts  $\mathcal{L}(\mathcal{B}_1) \cup \mathcal{L}(\mathcal{B}_2)$ , where

$u$  is a “fresh” state ( $u \notin S \cup T$ );

$$\Delta_u = \{ (u, \sigma, s) \mid (s_0, \sigma, s) \in \Delta_1 \} \cup \{ (u, \sigma, t) \mid (t_0, \sigma, t) \in \Delta_2 \}.$$

# Intersection I (a special case)

---

We first consider the case where **all** states in  $\mathcal{B}_2$  are accepting, i.e.  $G = T$ .

**Idea:** Construct a **cross-product automaton** (like for FA), check whether  $F$  is visited infinitely often.

Let  $\mathcal{B} = (\Sigma, S \times T, \langle s_0, t_0 \rangle, \Delta, F \times T)$ , where

$$\Delta = \{ (\langle s, t \rangle, a, \langle s', t' \rangle) \mid a \in \Sigma, (s, a, s') \in \Delta_1, (t, a, t') \in \Delta_2 \}.$$

Then:  $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$ .

## Intersection II (the general case)

---

**Principle:** We again construct a cross-product automaton.

**Problem:** The acceptance condition needs to check whether *both* accepting sets are visited infinitely often.

**Idea:** create **two copies** of the cross product.

- In the first copy we wait for a state from  $F$ .
- In the second copy we wait for a state from  $G$ .
- In both copies, once we've found one of the states we're looking for, we switch to the other copy.

We will choose the acceptance condition in such a way that an accepting run switches back and forth between the copies infinitely often.

---

Let  $\mathcal{B} = (\Sigma, U, u, \Delta, H)$ , where

$$U = S \times T \times \{1, 2\}, \quad u = \langle s_0, t_0, 1 \rangle, \quad H = F \times T \times \{1\}$$

$$(\langle s, t, 1 \rangle, a, \langle s', t', 1 \rangle) \in \Delta \quad \text{iff} \quad (s, a, s') \in \Delta_1, (t, a, t') \in \Delta_2, s \notin F$$

$$(\langle s, t, 1 \rangle, a, \langle s', t', 2 \rangle) \in \Delta \quad \text{iff} \quad (s, a, s') \in \Delta_1, (t, a, t') \in \Delta_2, s \in F$$

$$(\langle s, t, 2 \rangle, a, \langle s', t', 2 \rangle) \in \Delta \quad \text{iff} \quad (s, a, s') \in \Delta_1, (t, a, t') \in \Delta_2, t \notin G$$

$$(\langle s, t, 2 \rangle, a, \langle s', t', 1 \rangle) \in \Delta \quad \text{iff} \quad (s, a, s') \in \Delta_1, (t, a, t') \in \Delta_2, t \in G$$

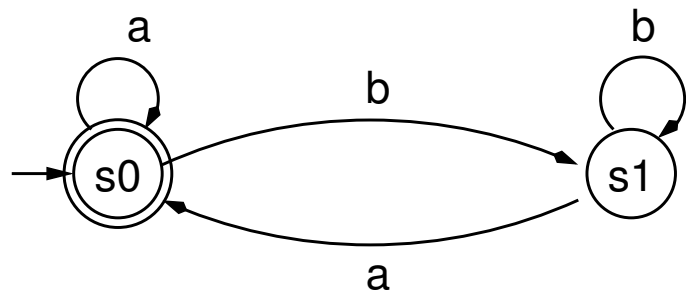
Remarks:

The automaton starts in the first copy.

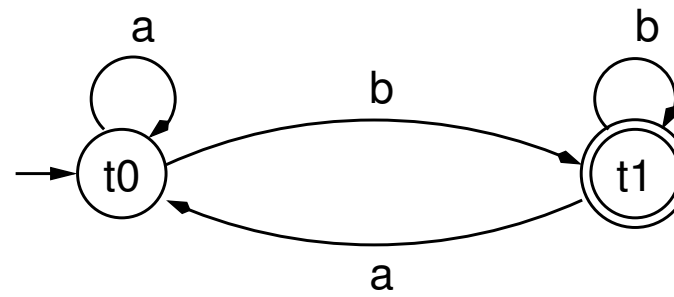
We could have chosen other acceptance conditions such as  $S \times G \times \{2\}$ .

The construction can be generalized to intersecting  $n$  automata.

# Intersection: example

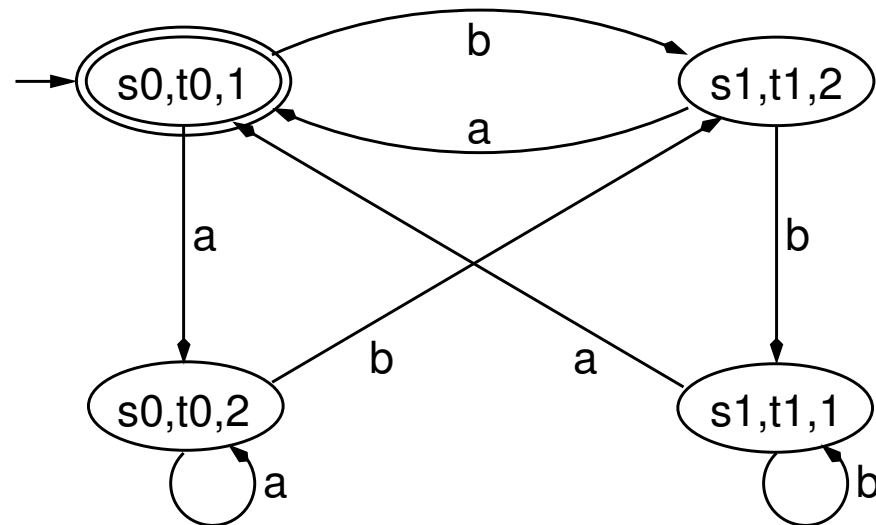


B1



B2

B1 x B2



# Complement

---

Problem: Given  $\mathcal{B}_1$ , construct  $\mathcal{B}$  with  $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{B}_1)^c$ .

Such a construction is possible (but rather complicated). We will not require it for the purpose of this course.

Additional literature:

Wolfgang Thomas, *Automata on Infinite Objects*,  
Chapter 4 in *Handbook of Theoretical Computer Science*,

Igor Walukiewicz, lecture notes on *Automata and Logic*, chapter 3,  
[www.labri.fr/Person/~igw/Papers/igw-eefss01.ps](http://www.labri.fr/Person/~igw/Papers/igw-eefss01.ps)

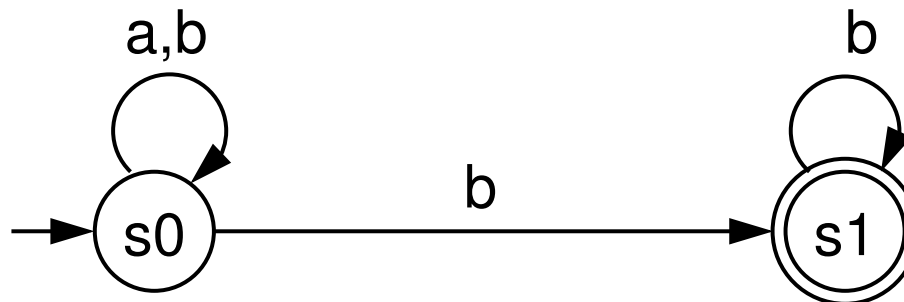
# Deterministic Büchi automata

---

For finite automata (known from *regular language theory*), it is known that every language expressible by a finite automaton can also be expressed by a deterministic automaton, i.e. one where the transition relation  $\Delta$  is a function  $S \times \Sigma \rightarrow S$ .

Such a procedure does not exist for Büchi automata.

In fact, there is no *deterministic* Büchi automaton accepting the same language as the automaton below:



“Only finitely many *a*s.”



---

Proof: Let  $\mathcal{L}$  be the language of infinite words over  $\{a, b\}$  containing only finitely many  $a$ s. Assume that a deterministic Büchi automaton  $\mathcal{B}$  with  $\mathcal{L}(\mathcal{B}) = \mathcal{L}$  exists, and let  $n$  be the number of states in  $\mathcal{B}$ .

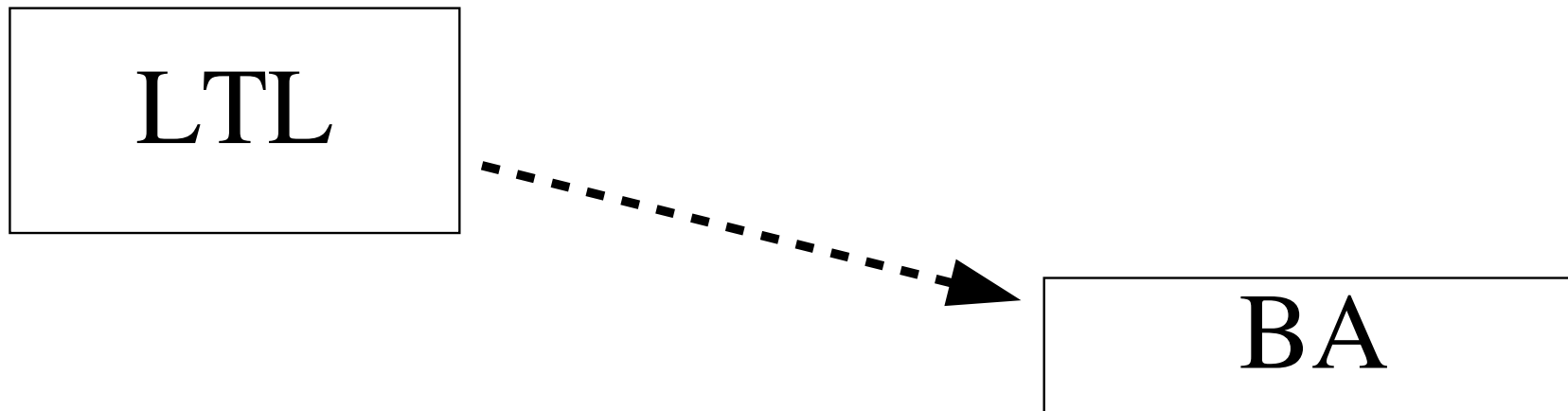
We have  $b^\omega \in \mathcal{L}$ , so let  $\alpha_1$  be the (unique) accepting run for  $b^\omega$ . Suppose that an accepting state is first reached after  $n_1$  letters, i.e.  $s_1 := \alpha_1(n_1)$  is the first accepting state in  $\alpha_1$ .

We now regard the word  $b^{n_1}ab^\omega$ , which is still in  $\mathcal{L}$ , therefore accepted by some run  $\alpha_2$ . Since  $\mathcal{B}$  is deterministic,  $\alpha_1$  and  $\alpha_2$  must agree on the first  $n_1$  states. Now, watch for the second occurrence of an accepting state in  $\alpha_2$ , i.e. let  $s_2 := \alpha_2(n_1 + 1 + n_2)$  be an accepting state for  $n_2$  minimal. Then,  $s_1 \neq s_2$  because otherwise there would be a loop around an accepting state containing a transition with an  $a$ .

We now repeat the argument for  $b^{n_1}ab^{n_2}ab^\omega$ , derive the existence of a third distinct state, etc. After doing this  $n + 1$  times, we conclude that  $\mathcal{B}$  must have more than  $n$  distinct states, a contradiction.

# Preview

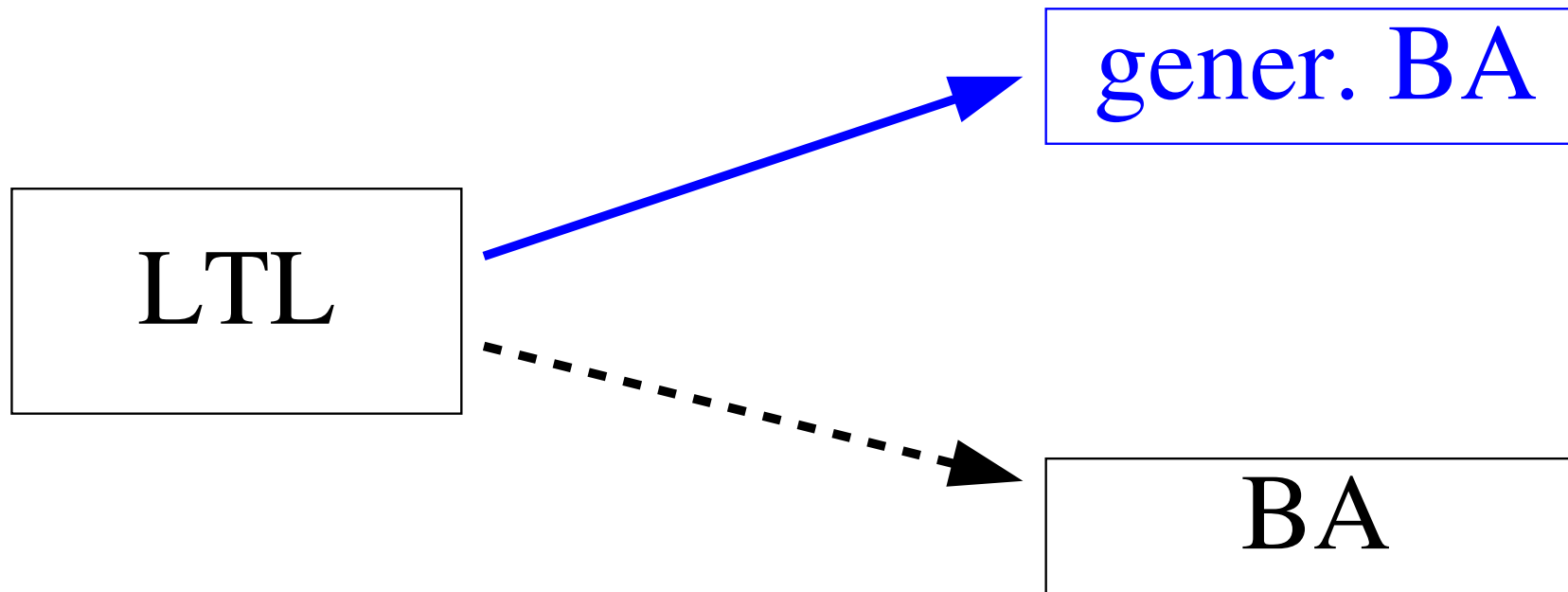
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We desire to translate LTL formulae into Büchi automata.

# Preview

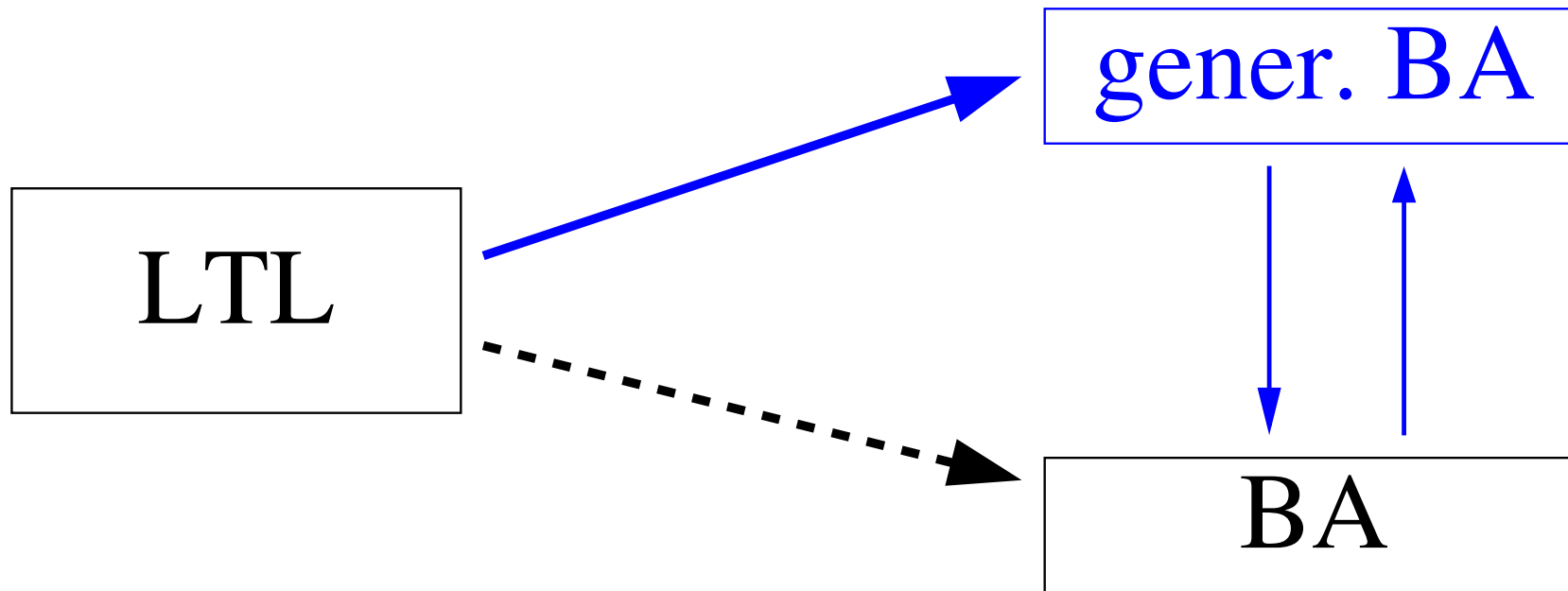
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Detour: We translate them into so-called *generalized* Büchi automata (GBA).

# Preview

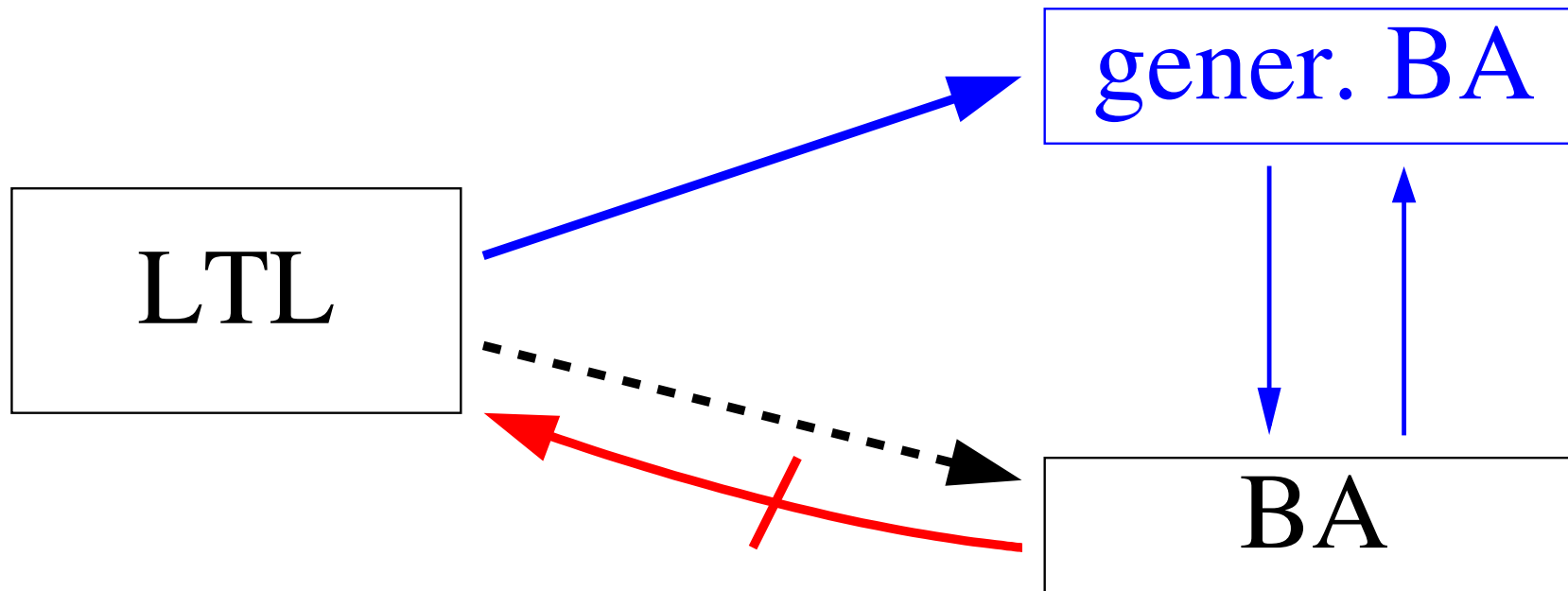
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GBA accept the same class of languages as BA.

# Preview

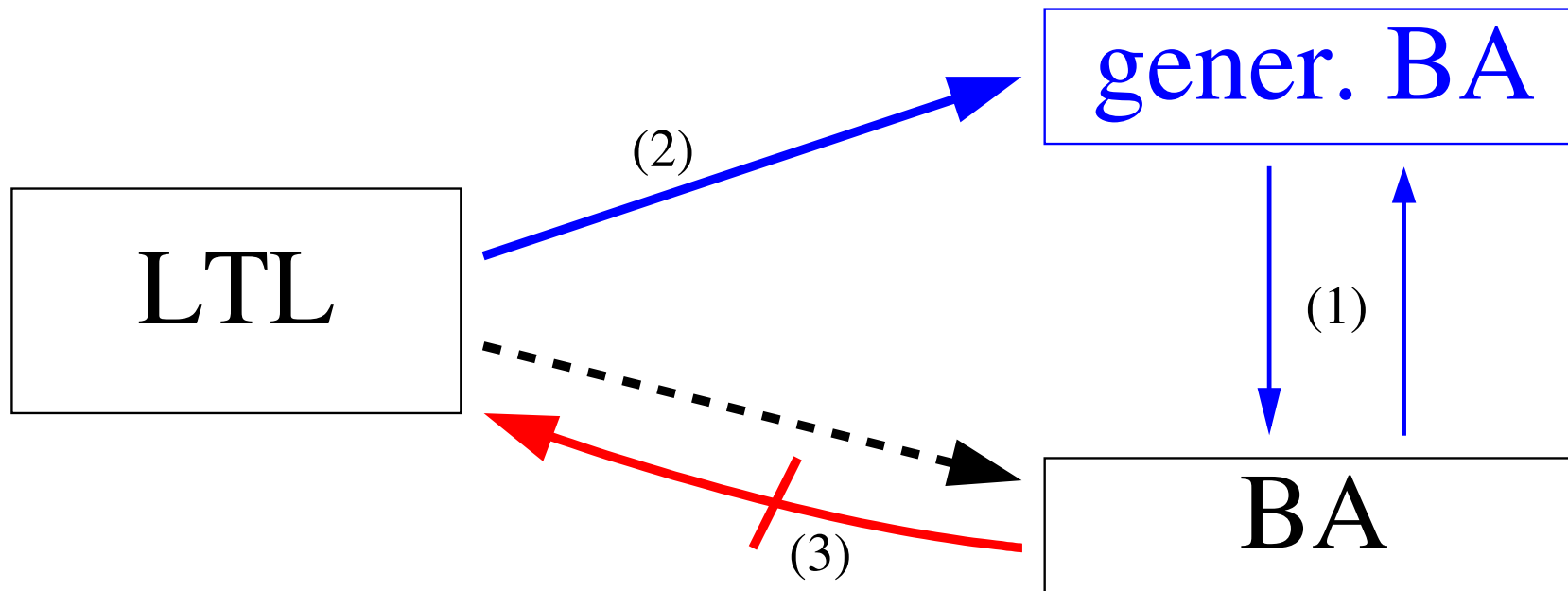
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Translation from BA to LTL not possible in general.

# Preview

---



We shall proceed in the order indicated above.

# Generalized Büchi automata

---

A **generalized Büchi automaton** (GBA) is a tuple  $\mathcal{G} = (\Sigma, S, s_0, \Delta, \mathcal{F})$ .

There is only one difference w.r.t. normal BA:

The acceptance condition  $\mathcal{F} \subseteq 2^S$  is a *set of sets of states*.

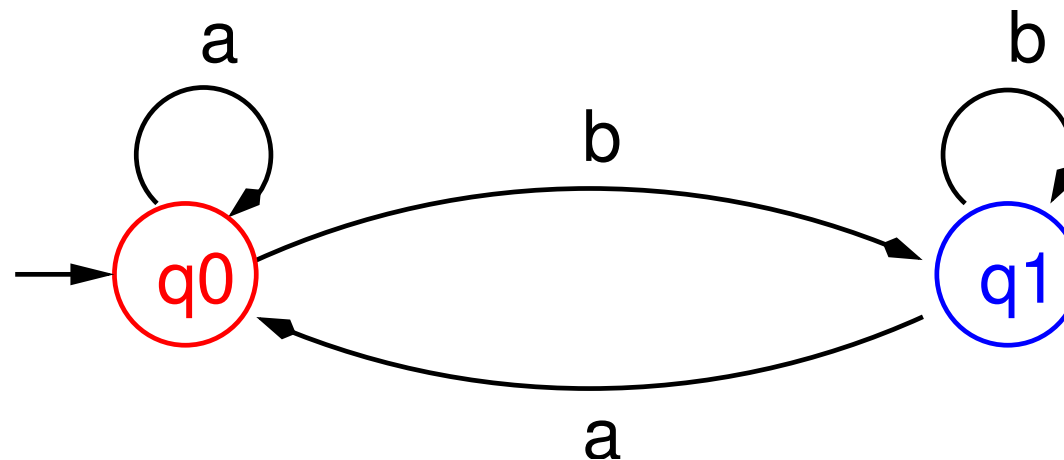
E.g., let  $\mathcal{F} = \{F_1, \dots, F_n\}$ . A run  $\rho$  of  $\mathcal{G}$  is called accepting iff for every  $F_i$  ( $i = 1, \dots, n$ ),  $\rho$  visits infinitely many states of  $F_i$ .

Put differently: many acceptance conditions at once.

# GBA: Example

---

For the GBA shown below, let  $\mathcal{F} = \{ \{q_0\}, \{q_1\} \}$ .



Language of the automaton: “infinitely often *a* and infinitely often *b*”

Note: In general, the acceptance conditions need not be pairwise disjoint.

Advantage: GBA may be more succinct than BA.



# Translations $BA \leftrightarrow GBA$

---

GBA accept the same class of languages as BA.

I.e., for every BA there is a GBA accepting the same language, and vice versa.

Part 1 of the claim ( $BA \rightarrow GBA$ ):

Let  $\mathcal{B} = (\Sigma, S, s_0, \Delta, F)$  be a BA.

Then  $\mathcal{G} = (\Sigma, S, s_0, \Delta, \{F\})$  is a GBA with  $\mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{B})$ .

---

Part 2 of the claim (GBA  $\rightarrow$  BA):

Let  $\mathcal{G} = (\Sigma, S, s_0, \Delta, \{F_1, \dots, F_n\})$  be a GBA.

We construct  $\mathcal{B} = (\Sigma, S', s'_0, \Delta', F)$  as follows:

$$S' = S \times \{1, \dots, n\}$$

$$s'_0 = (s_0, 1)$$

$$F = F_1 \times \{1\}$$

$$((s, i), a, (s', k)) \in \Delta' \text{ iff } 1 \leq i \leq n, (s, a, s') \in \Delta$$

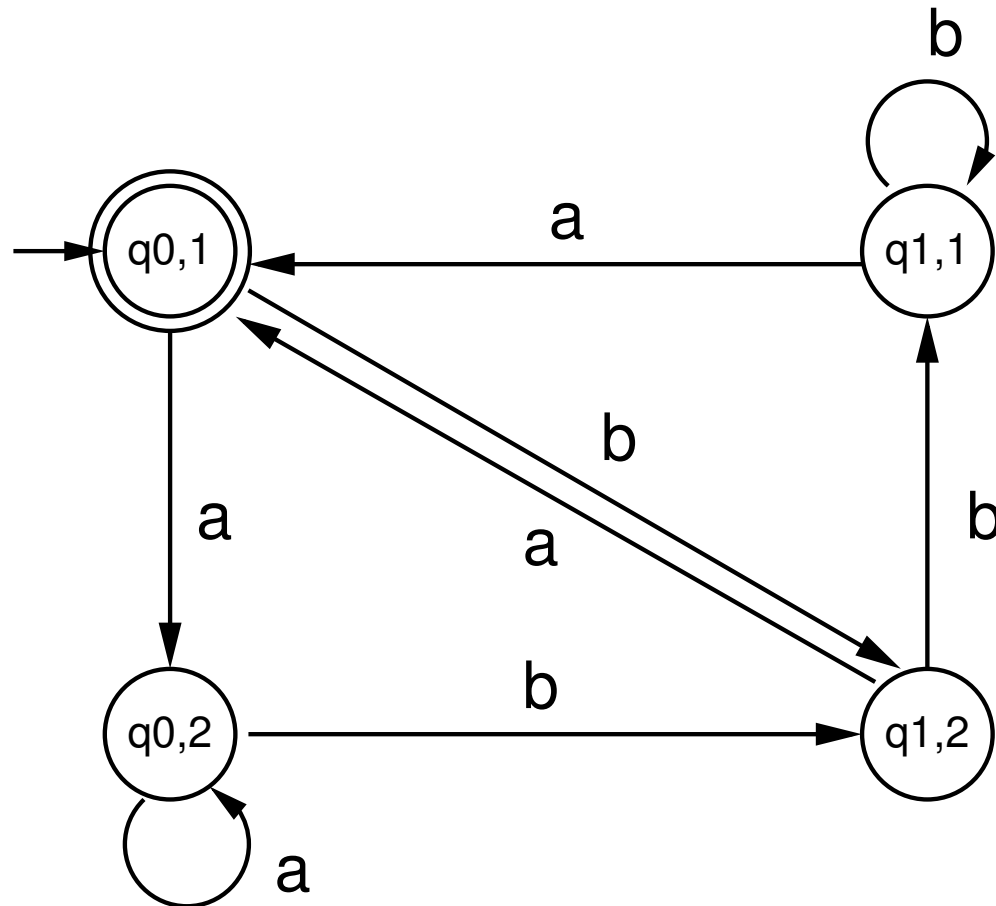
$$\text{and } k = \begin{cases} i & \text{if } s \notin F_i \\ (i \bmod n) + 1 & \text{if } s \in F_i \end{cases}$$

Then we have  $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{G})$ . (Idea:  $n$ -fold intersection)

## GBA $\rightarrow$ BA: example

---

The BA corresponding to the previous GBA (“infinitely often *a* and infinitely often *b*”) is as follows:



## Remark: Multiple initial states

---

Our definitions of BA and GBA require exactly one initial state.

For the translation  $LTL \rightarrow BA$  it will be convenient to use GBA with multiple initial states.

Intended meaning: A word is regarded as accepted if it is accepted starting from *any* initial state.

Obviously, every (G)BA with multiple initial states can easily be converted into a (G)BA with just one initial state.

# Part 5: LTL and Büchi automata

# Overview

---

In this part, we shall solve the following problem:

Given an LTL formula  $\phi$  over  $AP$ , we shall construct a GBA  $\mathcal{G}$  (with multiple initial states) such that  $\mathcal{L}(\mathcal{G}) = \llbracket \phi \rrbracket$ .

( $\mathcal{G}$  can then be converted to a normal BA.)

## Remarks:

Analogous operation for regular languages: reg. expression  $\rightarrow$  NFA

The crucial difference: it is not possible to provide an LTL  $\rightarrow$  BA translation **in modular fashion**.

The automaton may have to check multiple subformulae *at the same time* (e.g.:  $(\mathbf{G F } p) \rightarrow (\mathbf{G}(q \rightarrow \mathbf{F } r))$  or  $(p \mathbf{U } q) \mathbf{U } r$ ).

---

## More remarks:

The construction shown in the following is comparatively simplistic.

It will produce rather suboptimal automata (size *always* exponential in  $|\phi|$ ).

Obviously, this is quite inefficient, and not meant to be done by pen and paper, only as a “proof of concept”.

There are far better translation procedures but the underlying theory is rather beyond the scope of the course.

Interesting, on-going research area!

# Structure of the construction

---

1. We first convert  $\phi$  into a certain **normal form**.
2. **States** will be “responsible” for some set of subformulae.
3. The **transition relation** will ensure that “simple” subformulae such as  $p$  or  $X p$  are satisfied.
4. The **acceptance condition** will ensure that **U**-subformulae are satisfied.



# Negation normal form

---

Let  $AP$  be a set of atomic propositions. The set of **NNF formulae** over  $AP$  is inductively defined as follows:

If  $p \in AP$  then  $p$  and  $\neg p$  are NNF formulae.

(Remark: Negations occur *exclusively* in front of atomic propositions.)

If  $\phi_1$  and  $\phi_2$  are NNF formulae then so are

$$\phi_1 \vee \phi_2, \quad \phi_1 \wedge \phi_2, \quad \mathbf{X} \phi_1, \quad \phi_1 \mathbf{U} \phi_2, \quad \phi_1 \mathbf{R} \phi_2.$$

**Claim:** For every LTL formula  $\phi$  there is an equivalent NNF formula:

$$\begin{aligned} \neg(\phi_1 \mathbf{R} \phi_2) &\equiv \neg\phi_1 \mathbf{U} \neg\phi_2 & \neg(\phi_1 \mathbf{U} \phi_2) &\equiv \neg\phi_1 \mathbf{R} \neg\phi_2 \\ \neg(\phi_1 \wedge \phi_2) &\equiv \neg\phi_1 \vee \neg\phi_2 & \neg(\phi_1 \vee \phi_2) &\equiv \neg\phi_1 \wedge \neg\phi_2 \\ \neg \mathbf{X} \phi &\equiv \mathbf{X} \neg\phi & \neg\neg\phi &\equiv \phi \end{aligned}$$

# NNF: Example

---

Translation into an NNF formula:

$$\begin{aligned} G(p \rightarrow F q) &\equiv \neg F \neg(p \rightarrow F q) \\ &\equiv \neg(\text{true U } \neg(p \rightarrow F q)) \\ &\equiv \neg\text{true R } (p \rightarrow F q) \\ &\equiv \text{false R } (\neg p \vee F q) \\ &\equiv \text{false R } (\neg p \vee (\text{true U } q)) \end{aligned}$$

**Remark:** Because of this, we shall henceforth assume that the LTL formula in the translation procedure is given in NNF.

# Subformulae

---

Let  $\phi$  be an NNF formula. The set  $Sub(\phi)$  is the smallest set satisfying:

$\phi \in Sub(\phi)$ ;

**true**  $\in Sub(\phi)$ ;

if  $\phi_1 \in Sub(\phi)$  then  $\neg\phi_1 \in Sub(\phi)$ , and vice versa;

if **X**  $\phi_1 \in Sub(\phi)$  then  $\phi_1 \in Sub(\phi)$ ;

if  $\phi_1 \vee \phi_2 \in Sub(\phi)$  then  $\phi_1, \phi_2 \in Sub(\phi)$ ;

if  $\phi_1 \wedge \phi_2 \in Sub(\phi)$  then  $\phi_1, \phi_2 \in Sub(\phi)$ ;

if  $\phi_1 \mathbf{U} \phi_2 \in Sub(\phi)$  then  $\phi_1, \phi_2 \in Sub(\phi)$ ;

if  $\phi_1 \mathbf{R} \phi_2 \in Sub(\phi)$  then  $\phi_1, \phi_2 \in Sub(\phi)$ .

**Note:** We have  $|Sub(\phi)| = \mathcal{O}(|\phi|)$  (one subformula per syntactic element).

# Consistent sets

---

Recall item 2 of the construction:

Every state will be labelled with a subset of  $Sub(\phi)$ .

Idea: A state labelled by set  $M$  will accept a sequence iff it satisfies every single subformula contained in  $M$  and violates every single subformula contained in  $Sub(\phi) \setminus M$ .

For this reason, we will a priori exclude some sets  $M$  which would obviously lead to empty languages.

The other states will be called **consistent**.

---

**Definition:** We call a set  $M \subset Sub(\phi)$  *consistent* if it satisfies the following conditions:

$true \in M$

if  $\phi_1 \in Sub(\phi)$  then  $\neg\phi_1 \in M$  gdw.  $\phi_1 \notin M$ ;

if  $\phi_1 \wedge \phi_2 \in Sub(\phi)$  then  $\phi_1 \wedge \phi_2 \in M$  iff  $\phi_1 \in M$  and  $\phi_2 \in M$ ;

if  $\phi_1 \vee \phi_2 \in Sub(\phi)$  then  $\phi_1 \vee \phi_2 \in M$  iff  $\phi_1 \in M$  or  $\phi_2 \in M$ .

By  $CS(\phi)$  we denote the set of all consistent subsets of  $Sub(\phi)$ .

# Translation (1)

---

Let  $\phi$  be an NNF formula and  $\mathcal{G} = (\Sigma, \mathcal{S}, S_0, \Delta, \mathcal{F})$  be a GBA such that:

$$\Sigma = 2^{AP}$$

(i.e. the valuations over  $AP$ )

$$\mathcal{S} = CS(\phi)$$

(i.e. every state is a consistent set)

$$S_0 = \{M \in \mathcal{S} \mid \phi \in M\}$$

(i.e. the initial states admit sequences satisfying  $\phi$ )

$\Delta$  and  $\mathcal{F}$ : see next slide

## Translation (2)

---

**Transitions:**  $(M, \sigma, M') \in \Delta$  iff  $\sigma = M \cap AP$  and:

- if  $\mathbf{X} \phi_1 \in \text{Sub}(\phi)$  then  $\mathbf{X} \phi_1 \in M$  iff  $\phi_1 \in M'$ ;
- if  $\phi_1 \mathbf{U} \phi_2 \in \text{Sub}(\phi)$  then  $\phi_1 \mathbf{U} \phi_2 \in M$   
iff  $\phi_2 \in M$  or  $(\phi_1 \in M$  and  $\phi_1 \mathbf{U} \phi_2 \in M')$ ;
- if  $\phi_1 \mathbf{R} \phi_2 \in \text{Sub}(\phi)$  then  $\phi_1 \mathbf{R} \phi_2 \in M$   
iff  $\phi_1 \wedge \phi_2 \in M$  or  $(\phi_2 \in M$  and  $\phi_1 \mathbf{R} \phi_2 \in M')$ .

**Acceptance condition:**

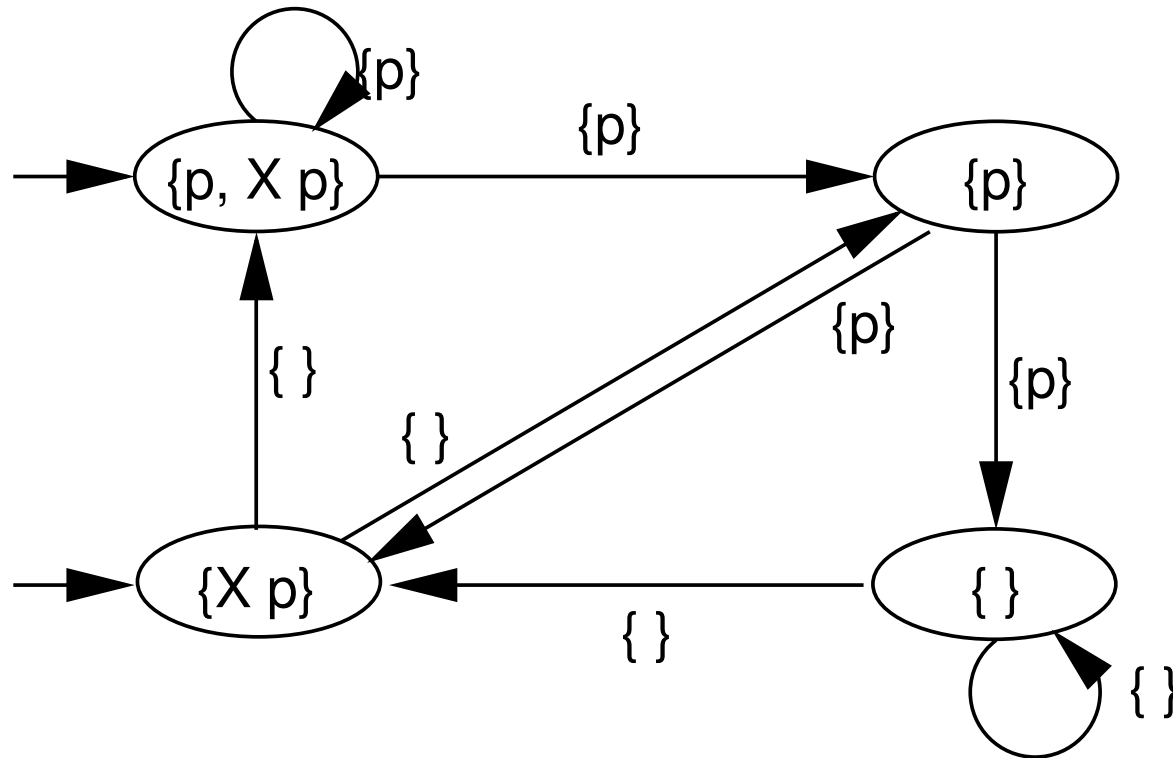
$\mathcal{F}$  contains a set  $F_\psi$  for every subformula  $\psi$  of the form  $\phi_1 \mathbf{U} \phi_2$ , where

$$F_\psi = \{ M \in CS(\phi) \mid \phi_2 \in M \text{ or } \neg(\phi_1 \mathbf{U} \phi_2) \in M \}.$$

# Translation: Example 1

---

$$\phi = \mathbf{X} p$$



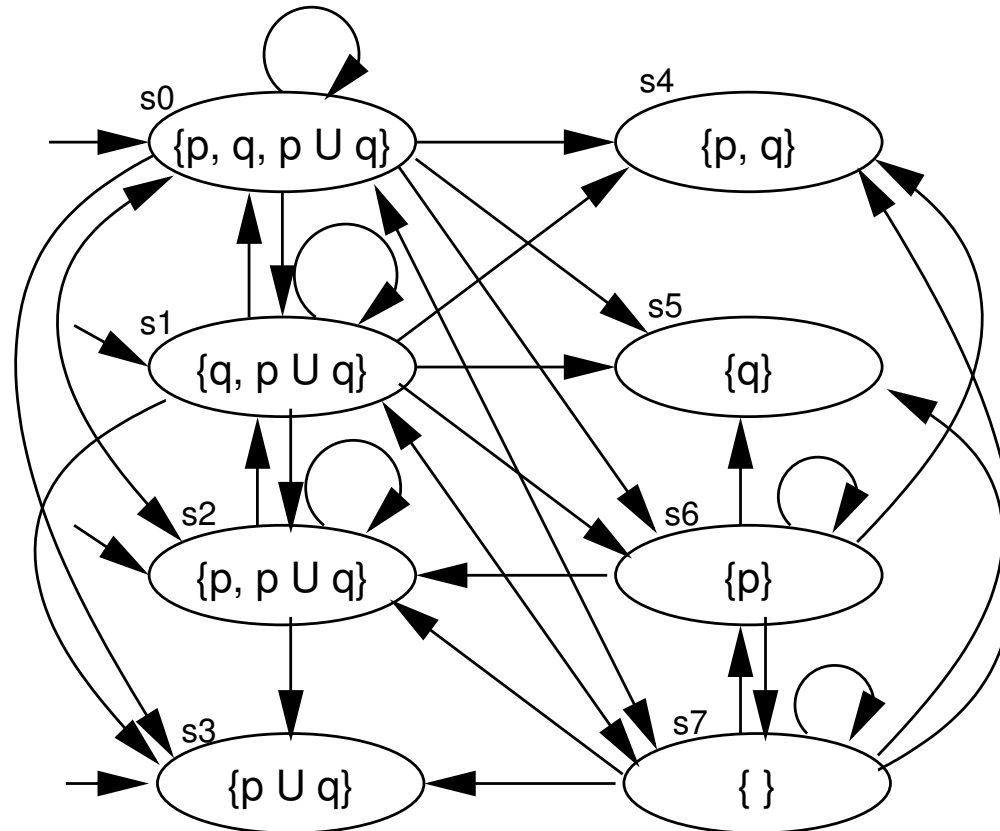
This GBA has got two initial states and the acceptance condition  $\mathcal{F} = \emptyset$ , i.e. every infinite run is accepting. (Negated Formulas omitted from state labels.)



# Translation: Example 2

---

$$\phi \equiv p \text{ U } q$$



GBA with  $\mathcal{F} = \{\{s_0, s_1, s_4, s_5, s_6, s_7\}\}$ , transition labels also omitted.

# Proof of correctness

---

We want to prove the following:

$$\sigma \in \mathcal{L}(\mathcal{G}) \quad \text{gdw.} \quad \sigma \in \llbracket \phi \rrbracket$$

To this aim, we shall prove the following stronger property:

Let  $\alpha$  be a sequence of consistent sets (i.e., states of  $\mathcal{G}$ ) and let  $\sigma$  be a sequence of valuations over  $AP$ .

$\alpha$  is an accepting run of  $\mathcal{G}$  over  $\sigma$   
iff  $\sigma^i \in \llbracket \psi \rrbracket$  for all  $i \geq 0$  and  $\psi \in \alpha(i)$ .

The desired proof then follows from the choice of initial states.

## Correctness (2)

---

Remark: By construction, we have  $\sigma(i) = \alpha(i) \cap AP$  for all  $i \geq 0$ .

Proof via structural induction over  $\psi$ :

for  $\psi = p$  and  $\psi = \neg p$  if  $p \in AP$ :

obvious since  $\sigma^i \in \llbracket p \rrbracket$  iff  $p \in \sigma(i)$  iff  $p \in \alpha(i)$ .

for  $\psi_1 \vee \psi_2$  and  $\psi_1 \wedge \psi_2$ : follows from consistency of  $\alpha(i)$  and from the induction hypothesis for  $\psi_1$  and  $\psi_2$ , resp.

for  $\mathbf{X} \psi_1$ : follows from the construction of  $\Delta$  and induction hypothesis for  $\psi_1$ .

## Correctness (3)

---

for  $\psi = \psi_1 \mathbf{R} \psi_2$ :

Follows from the construction of  $\Delta$ , the recursion equation for  $\mathbf{R}$  and the induction hypothesis.

for  $\psi = \psi_1 \mathbf{U} \psi_2$ :

Analogous to  $\mathbf{R}$ , but additionally we must ensure that  $\psi_2 \in \alpha(k)$  for some  $k \geq i$ . Assume that this is not the case, then we have  $\psi_1 \mathbf{U} \psi_2 \in \alpha(k)$  for all  $k \geq i$ . However, none of these states is in  $F_\psi$ , therefore  $\alpha$  cannot be accepting, which is a contradiction.

# Complexity of the translation

---

The translation procedure produces an automaton of size  $O(2^{|\phi|})$ , for a formula  $\phi$ .

**Question:** Is there a better translation procedure?

---

**Answer 1:** No (not in general). There exist formulae for which any Büchi automaton has necessarily exponential size.

**Example:** The following LTL formula over  $\{p_0, \dots, p_{n-1}\}$  simulates an  $n$ -bit counter.

$$G(p_0 \nleftrightarrow X p_0) \wedge \bigwedge_{i=1}^{n-1} G \left( (p_i \nleftrightarrow X p_i) \leftrightarrow (p_{i-1} \wedge \neg X p_{i-1}) \right)$$

The formula has size  $\mathcal{O}(n)$ . Obviously, any automaton for this formula must have at least  $2^n$  states.

---

**Answer 2:** Yes (sometimes). There are translation procedures that produce smaller automata *for most cases*.

Some tools:

Spin (Aufruf: `spin -f 'p U q'`)

LTL2BA (web applet)

Literature:

Gerth, Peled, Vardi, Wolper: *Simple On-the-fly Automatic Verification of Linear Temporal Logic*, 1996

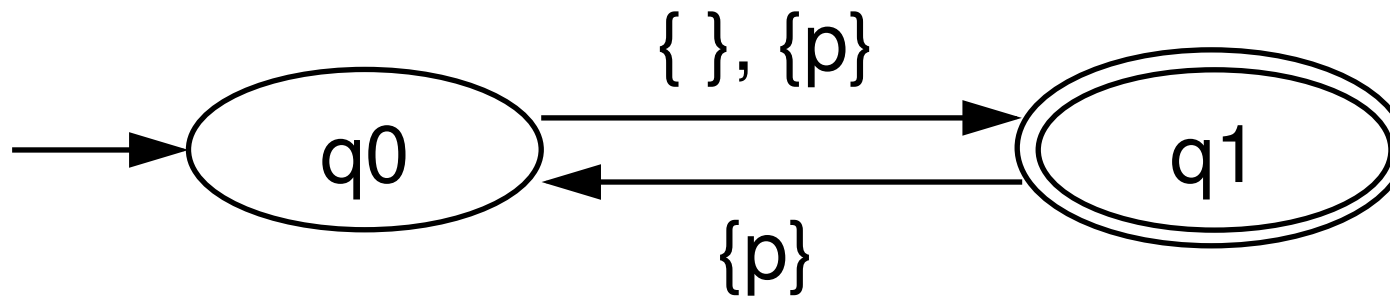
Oddoux, Gastin: *Fast LTL to Büchi Automata Translation*, 2001

# Translation BA $\rightarrow$ LTL

---

The reverse translation (BA  $\rightarrow$  LTL) is not possible in general.

I.e., there are Büchi automata  $\mathcal{B}$  such that there is no formula  $\phi$  with  $\mathcal{L}(\mathcal{B}) = \llbracket \phi \rrbracket$  (Wolper, 1983).



The property “ $p$  holds in every second step” is not expressible in LTL (proof: next slide).



# Proof (BA $\not\rightarrow$ LTL)

---

We first show a more general lemma:

Let  $\phi$  be an arbitrary LTL formula over  $AP$  and  $n$  the number of  $\mathbf{X}$  operators in  $\phi$ . We regard the sequences

$$\sigma_i = \{p\}^i \emptyset \{p\}^\omega$$

for  $i \geq 0$ . For all pairs  $i, k > n$  we have:  $\sigma_i \models \phi$  iff  $\sigma_k \models \phi$ .

Proof by structural induction over  $\phi$ :

If  $\phi = p$ , for  $p \in AP$ , then  $n = 0$  and  $i, k \geq 1$ .

Thus,  $\sigma_i \models p$  and  $\sigma_k \models p$ .

For the other cases, the induction hypothesis assumes that the property holds for  $\phi_1$  und  $\phi_2$ , i.e. if  $\phi_1, \phi_2$  contain  $n_1$  and  $n_2$  occurrences of  $\mathbf{X}$ , resp., then for all  $i_1, k_1 > n_1$  we have  $\sigma_{i_1} \models \phi_1$  iff  $\sigma_{k_1} \models \phi_1$ , and analogously for  $\phi_2$ .

---

If  $\phi = \neg\phi_1$ , then the proof follows directly from the induction hypothesis.

For  $\phi = \phi_1 \vee \phi_2$ : same

If  $\phi = \mathbf{X}\phi_1$ , then  $n_1 = n - 1$ . Since  $i - 1, k - 1 > n - 1 = n_1$ , the induction hypothesis implies:  $\sigma_i^1 = \sigma_{i-1} \models \phi_1$  iff  $\sigma_k^1 = \sigma_{k-1} \models \phi_1$ , which implies the proof.

For  $\phi = \phi_1 \mathbf{U} \phi_2$ : Let  $m > n$ . We have:

$$\phi_1 \mathbf{U} \phi_2 \equiv \phi_2 \vee (\phi_1 \wedge \mathbf{X}(\phi_1 \mathbf{U} \phi_2))$$

Applying this law recursively we obtain:

$$\sigma_m \models \phi \quad \text{gdw.} \quad \sigma_m \models \phi_2 \vee (\sigma_m \models \phi_1 \wedge (\sigma_{m-1} \models \phi_2 \vee (\dots \\ (\sigma_{n+1} \models \phi_1 \wedge \sigma_n \models \phi_1 \mathbf{U} \phi_2))))$$

---

According to the induction hypothesis, we can replace indices bigger than  $n$  equivalently by  $n + 1$ :

$$\sigma_m \models \phi \quad \text{gdw.} \quad \sigma_{n+1} \models \phi_2 \vee (\sigma_{n+1} \models \phi_1 \wedge (\sigma_{n+1} \models \phi_2 \vee (\dots (\sigma_{n+1} \models \phi_1 \wedge \sigma_n \models \phi_1 \mathbf{U} \phi_2))))$$

This can be simplified to the following:

$$\sigma_m \models \phi \quad \text{gdw.} \quad \sigma_{n+1} \models \phi_2 \vee (\sigma_{n+1} \models \phi_1 \wedge \sigma_n \models \phi_1 \mathbf{U} \phi_2)$$

Thus, the validity of  $\sigma_m \models \phi$  is completely independent of  $m$ , leading to the desired property for  $i$  and  $k$ , which concludes the proof of the lemma.

---

Let us now assume that there exists an LTL formula  $\phi$  expressing the property of the aforementioned BA (“ $p$  holds in every second step”). Let  $n$  be the number of occurrences of  $X$  in  $\phi$ .

Let us consider the sequences  $\sigma_{n+1}$  and  $\sigma_{n+2}$ .

If  $n$  is even then  $\sigma_{n+1} \not\models \phi$  and  $\sigma_{n+2} \models \phi$ . If  $n$  is odd, then vice versa.

However, the previous lemma tells us that this is impossible: either  $\sigma_{n+1}$  and  $\sigma_{n+2}$  both satisfy  $\phi$ , or none of them does. Therefore, such a formula  $\phi$  cannot exist.

# The model-checking problem for LTL (preview)

---

**Problem:** Given a Kripke structure  $\mathcal{K} = (S, \rightarrow, r, AP, \nu)$  and an LTL formula  $\phi$  over  $AP$ , we ask whether  $\mathcal{K} \models \phi$ .

**Solution:** (sketch)

We re-interpret  $\mathcal{K}$  as a Büchi automaton  $\mathcal{B}_{\mathcal{K}}$ :

$$\mathcal{B}_{\mathcal{K}} = (2^{AP}, S, r, \Delta, S), \text{ where } \Delta = \{ (s, \nu(s), t) \mid s \rightarrow t \}$$

Obviously,  $\llbracket \mathcal{K} \rrbracket = \mathcal{L}(\mathcal{B}_{\mathcal{K}})$ .

Moreover, we translate  $\neg\phi$  into a Büchi automaton  $\mathcal{B}_{\neg\phi}$ .

---

We have:

$$\begin{aligned} & \mathcal{K} \models \phi \\ \iff & \llbracket \mathcal{K} \rrbracket \subseteq \llbracket \phi \rrbracket \\ \iff & \llbracket \mathcal{K} \rrbracket \cap \llbracket \neg\phi \rrbracket = \emptyset \\ \iff & \mathcal{L}(\mathcal{B}_{\mathcal{K}}) \cap \mathcal{L}(\mathcal{B}_{\neg\phi}) = \emptyset \end{aligned}$$

Therefore:

We construct Büchi automata  $\mathcal{B}_{\mathcal{K}}$  and  $\mathcal{B}_{\neg\phi}$ .

We intersect both automata (using the special-case construction).

Thus, the model-checking problem reduces to the problem of deciding whether the product automaton accepts the empty language.

# Part 6: Efficient Emptiness Test for Büchi Automata

# Overview

---

As we have seen, the model-checking problem reduces to checking whether the language of a certain Büchi automaton  $\mathcal{B}$  is *empty*.

Reminder:  $\mathcal{B}$  arises from the intersection of a Kripke structure  $\mathcal{K}$  with a BA for the *negation* of  $\phi$ .

If  $\mathcal{B}$  accepts some word, we call such a word a **counterexample**.

$\mathcal{K} \models \phi$  iff  $\mathcal{B}$  accepts the empty language.



---

Typical instances:

Size of  $\mathcal{K}$ : between several hundreds to millions of states.

Size of  $\mathcal{B}_{\neg\phi}$ : usually just a couple of states

Typical setting (e.g., in Spin):

$\mathcal{K}$  indirectly given in some description language (C, Java / in Spin: Promela);  
model-checking tools will generate  $\mathcal{K}$  internally.

$\mathcal{B}_{\neg\phi}$  generated from  $\phi$  before start of emptiness check.

---

Typical setting:

$\mathcal{B}$  generated “on-the-fly” from (the description of)  $\mathcal{K}$  and from  $\mathcal{B}_{\neg\phi}$  and tested for emptiness *at the same time*.

As a consequence, the size of  $\mathcal{K}$  (and of  $\mathcal{B}$ ) is not known initially!

At the beginning, only the initial state is known, and we have a function  $\text{succ}: \mathcal{S} \rightarrow 2^{\mathcal{S}}$  for computing the immediate successors of a given state (the function implements the semantics of the description).

# Memory requirements

---

Transitions not stored explicitly, will be explored “on demand” by calling `succ` (calls to `succ` will be comparatively costly).

Hash table for explored states.

Information stored for each state:

**Descriptor:** program counter, variable values, active processes, etc  
(often dozens or hundreds of bytes)

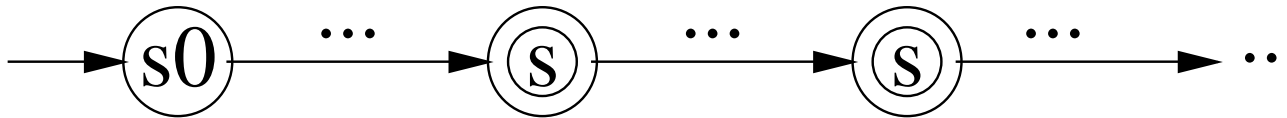
**Auxiliary information:** Data needed by the emptiness test  
(a couple of bytes)

# Simple solution I: Check for Lassos

---

Let  $\mathcal{B} = (\Sigma, S, s_0, \delta, F)$  be a Büchi automaton.

$\mathcal{L}(\mathcal{B}) \neq \emptyset$  iff there is  $s \in F$  such that  $s_0 \xrightarrow{*} s \xrightarrow{+} s$



Naïve solution:

Check for each  $s \in F$  whether there is a cycle around  $s$ ; let  $F_\circ \subseteq F$  denote the set of states with this property.

Check whether  $s_0$  can reach some state in  $F_\circ$ .

Time requirement: Each search takes linear time in the size of  $\mathcal{B}$ , altogether quadratic run-time  $\rightarrow$  unacceptable for millions of states.

# Strongly connected components

---

$C \subseteq S$  is called a **strongly connected component** (SCC) iff

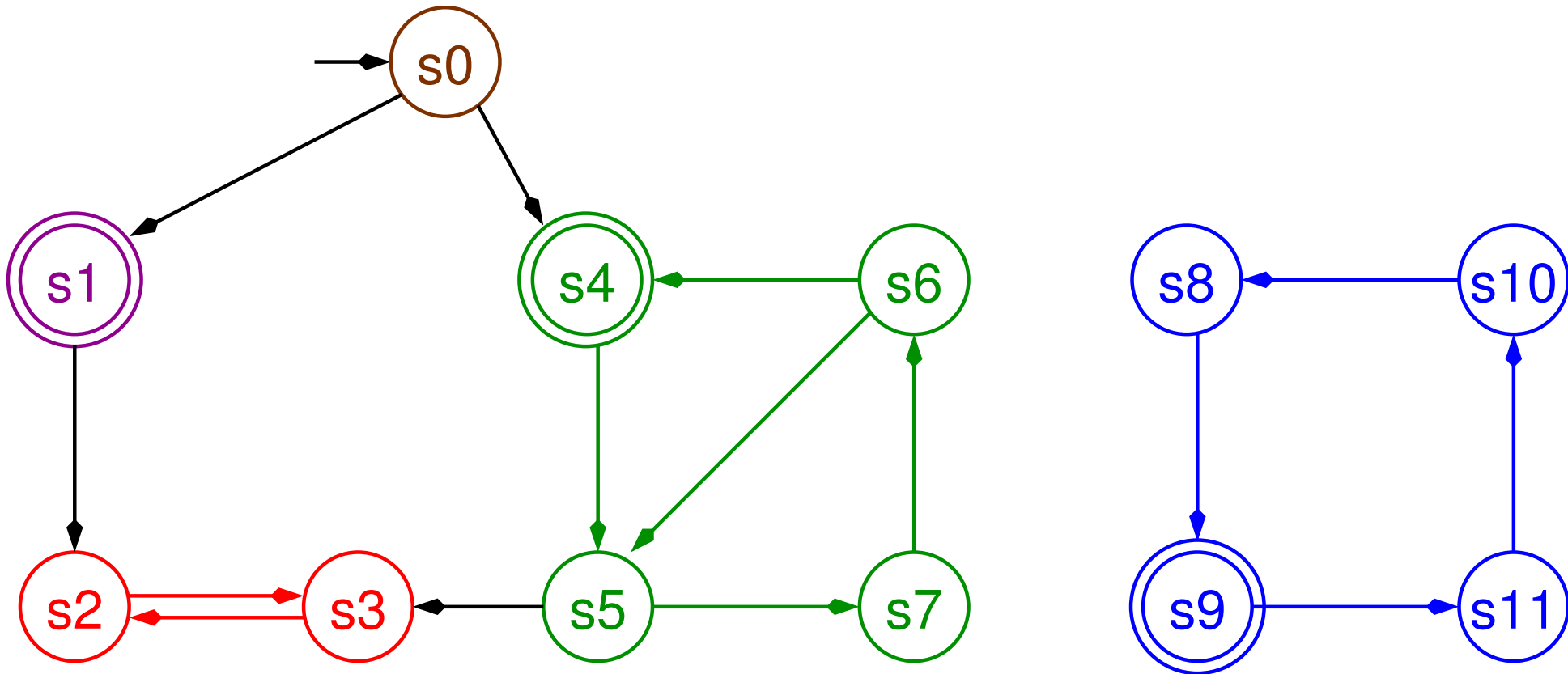
$s \rightarrow^* s'$  for all  $s, s' \in C$ ;

$C$  is maximal w.r.t. the above property, i.e. there is no proper superset of  $C$  satisfying the above.

An SCC  $C$  is called **trivial** if  $|C| = 1$  and for the unique state  $s \in C$  we have  $s \not\rightarrow s$  (single state without loop).

# Example: SCCs

---



The SCCs  $\{s_0\}$  and  $\{s_1\}$  are trivial.

## Simple algorithm II: SCCs

---

Observation:  $\mathcal{L}(\mathcal{B}) \neq \emptyset$  iff  $\mathcal{B}$  has a non-trivial SCC that is reachable from  $s_0$  and contains an accepting state.

Simple algorithm: for every accepting state  $s$

compute the set  $V_s$  of the predecessors of  $s$ ;

compute the set  $N_s$  of the successors of  $s$ ;

$V_s \cap N_s$  is the SCC containing  $s$ ;

test whether  $V_s \cap N_s \supset \{s\}$  or  $s \rightarrow s$ .

Running time: again quadratic

# Efficient solution

---

In the following, we shall discuss a solution whose run-time is **linear** in  $|\mathcal{B}|$  (i.e. proportional to  $|\mathcal{S}| + |\delta|$ ).

The solution is based on **depth-first search** (DFS) and on partitioning  $\mathcal{B}$  into its SCCs.

Literature: [Tarjan 1972], Couvreur 1999, Gabow 2000



## Depth-first search (basic version)

---

```
nr = 0;
hash = {};
dfs(s0);
exit;

dfs(s) {
    add s to hash;
    nr = nr+1;
    s.num = nr;

    for (t in succ(s)) {
        // deal with transition s -> t
        if (t not yet in hash) { dfs(t); }
    }
}
```

# Memory usage

---

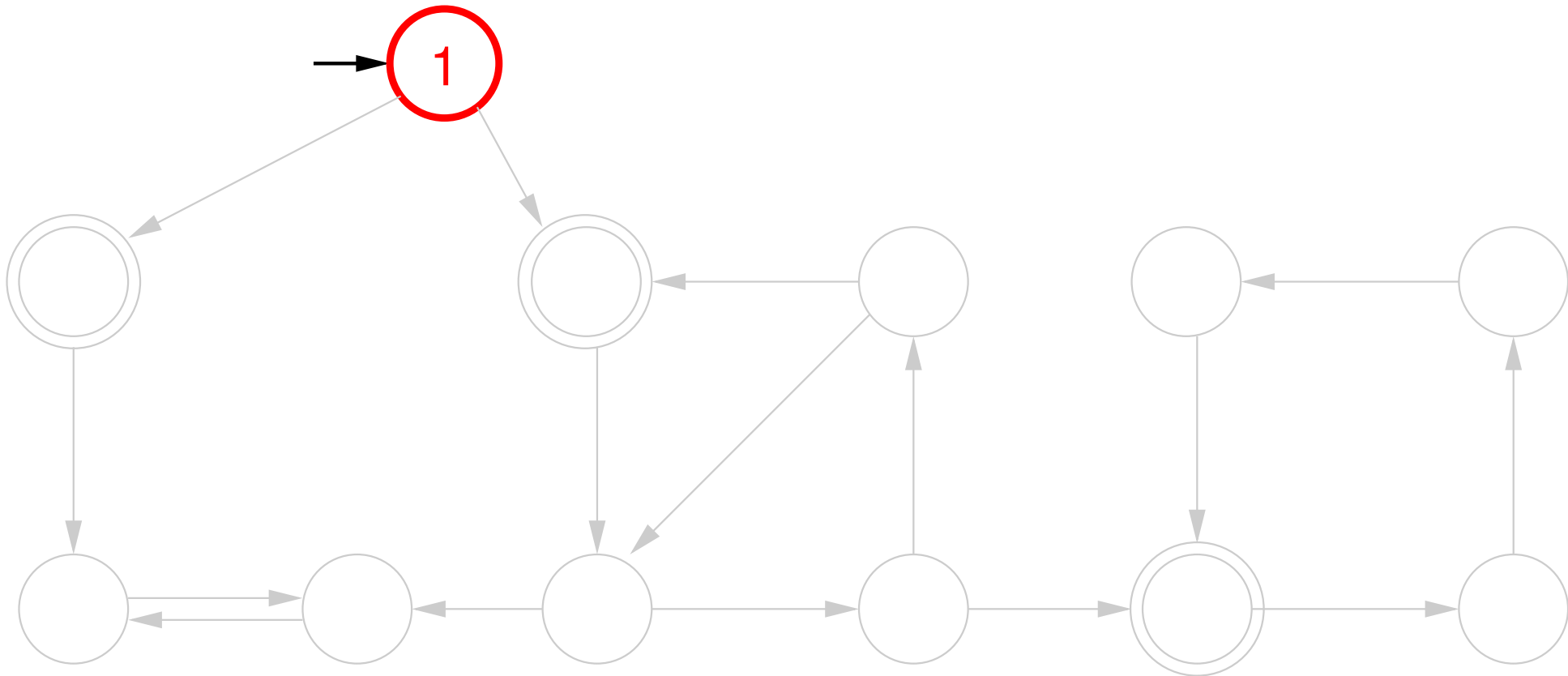
**Global variables:** counter  $nr$ , hash table for states

**Auxiliary information:** “DFS number”  $s.num$

**search path:** Stack for memorizing the “unfinished” calls to  $dfs$

# Example: Depth-first search

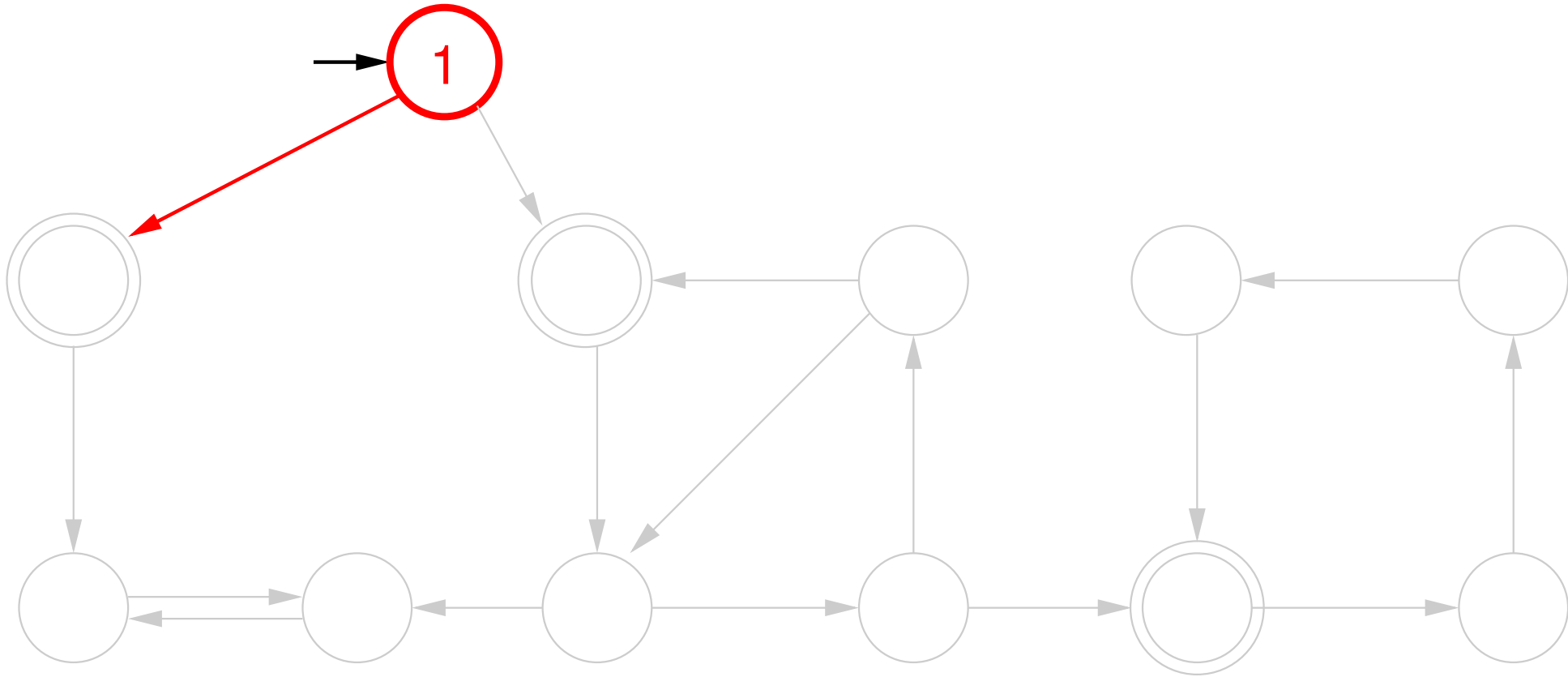
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Search path shown in red, other visited states black, states not yet seen grey.

# Example: Depth-first search

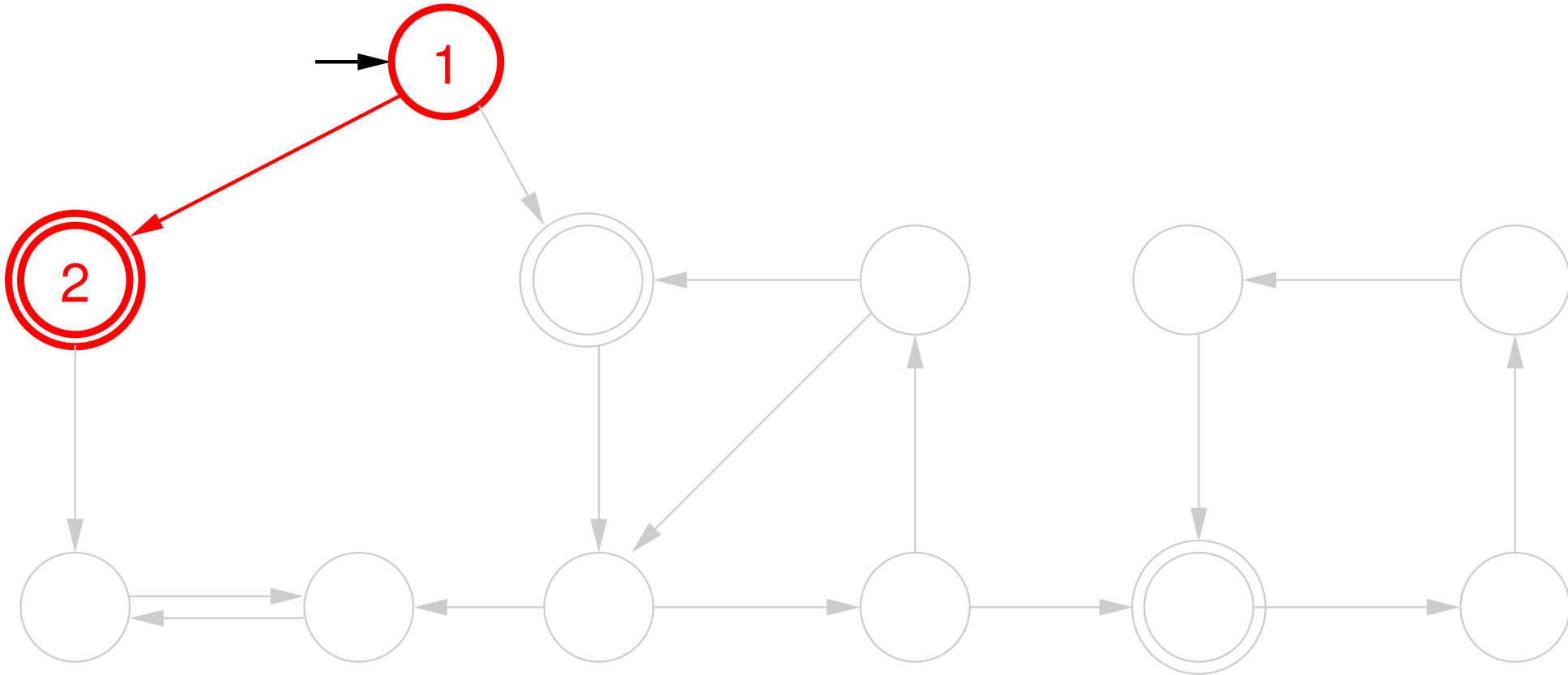
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DFS starts at initial state and explores some immediate successor.

# Example: Depth-first search

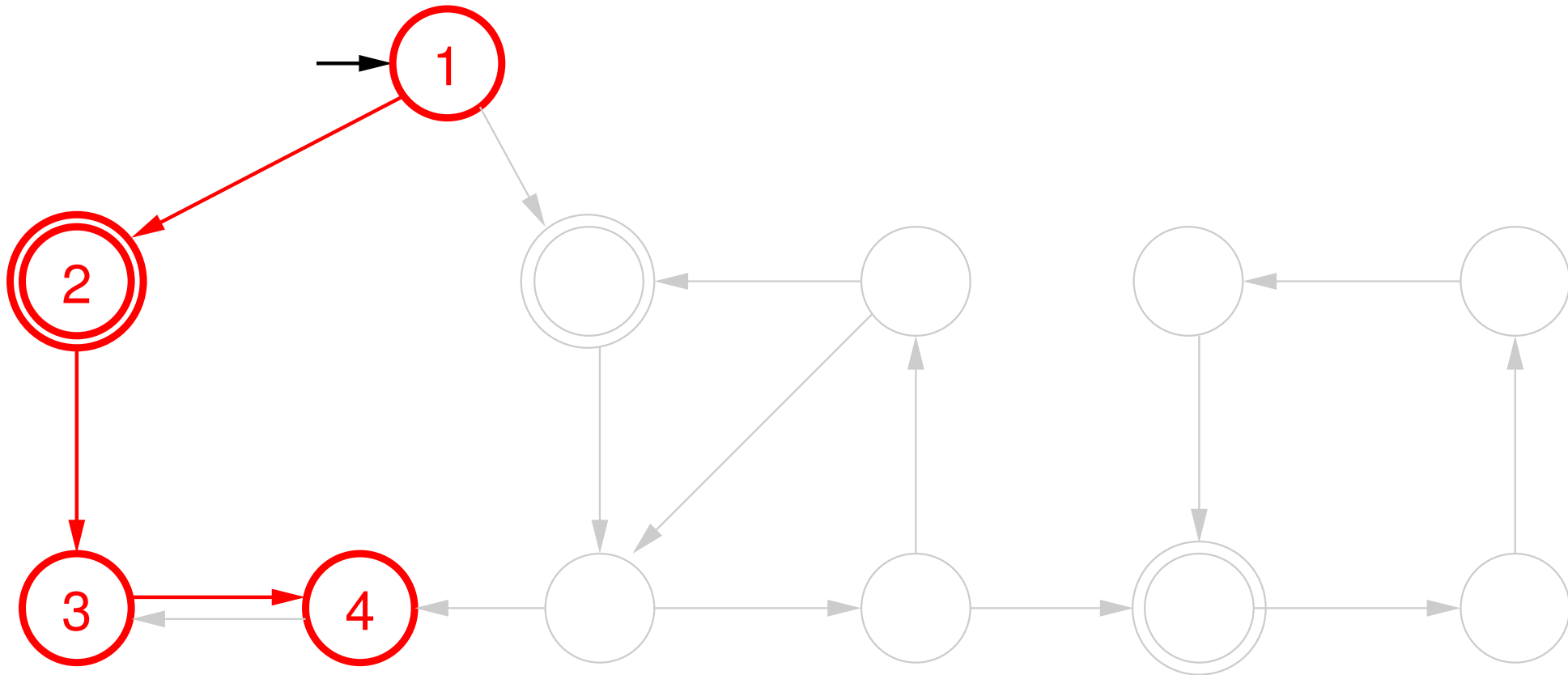
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Successor state not yet visited; recursive call, assigned to number 2.

# Example: Depth-first search

---

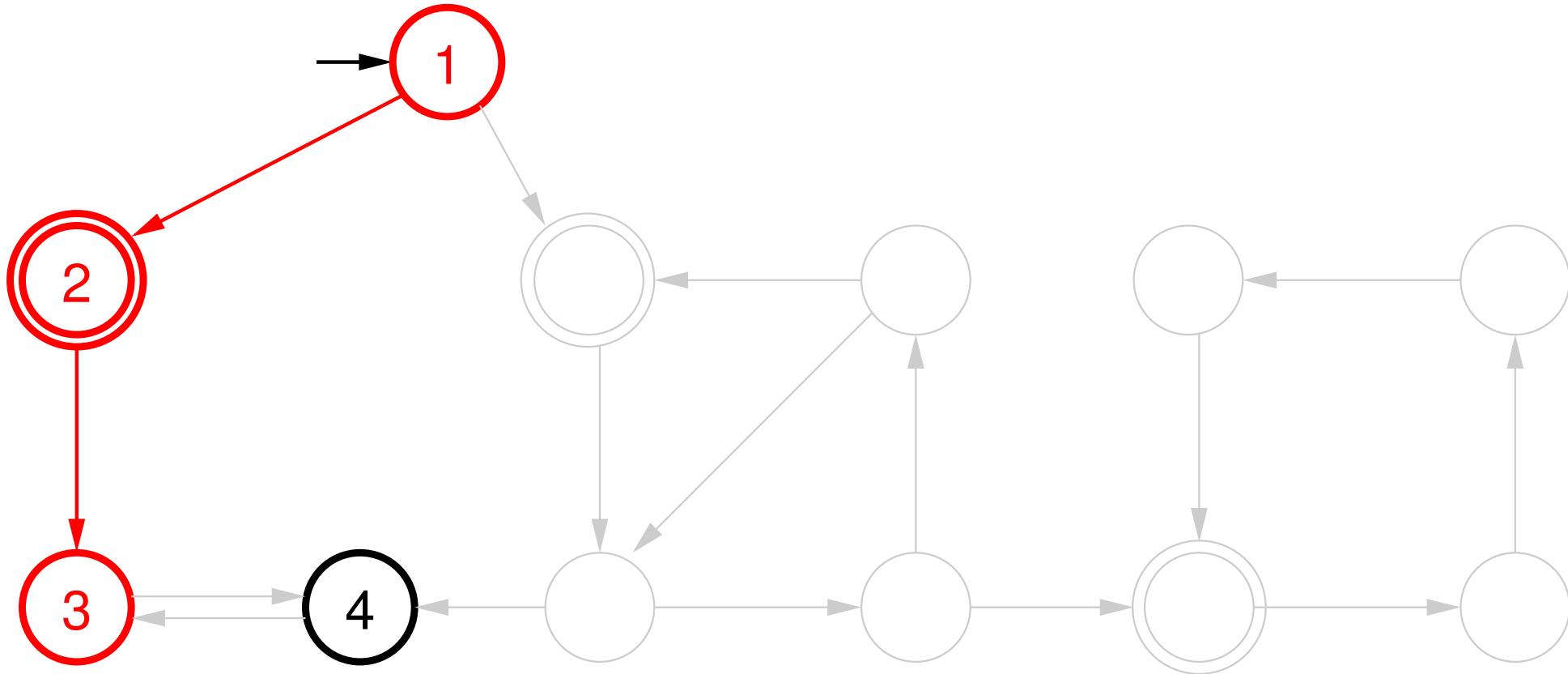


More unvisited states are being explored...



# Example: Depth-first search

---

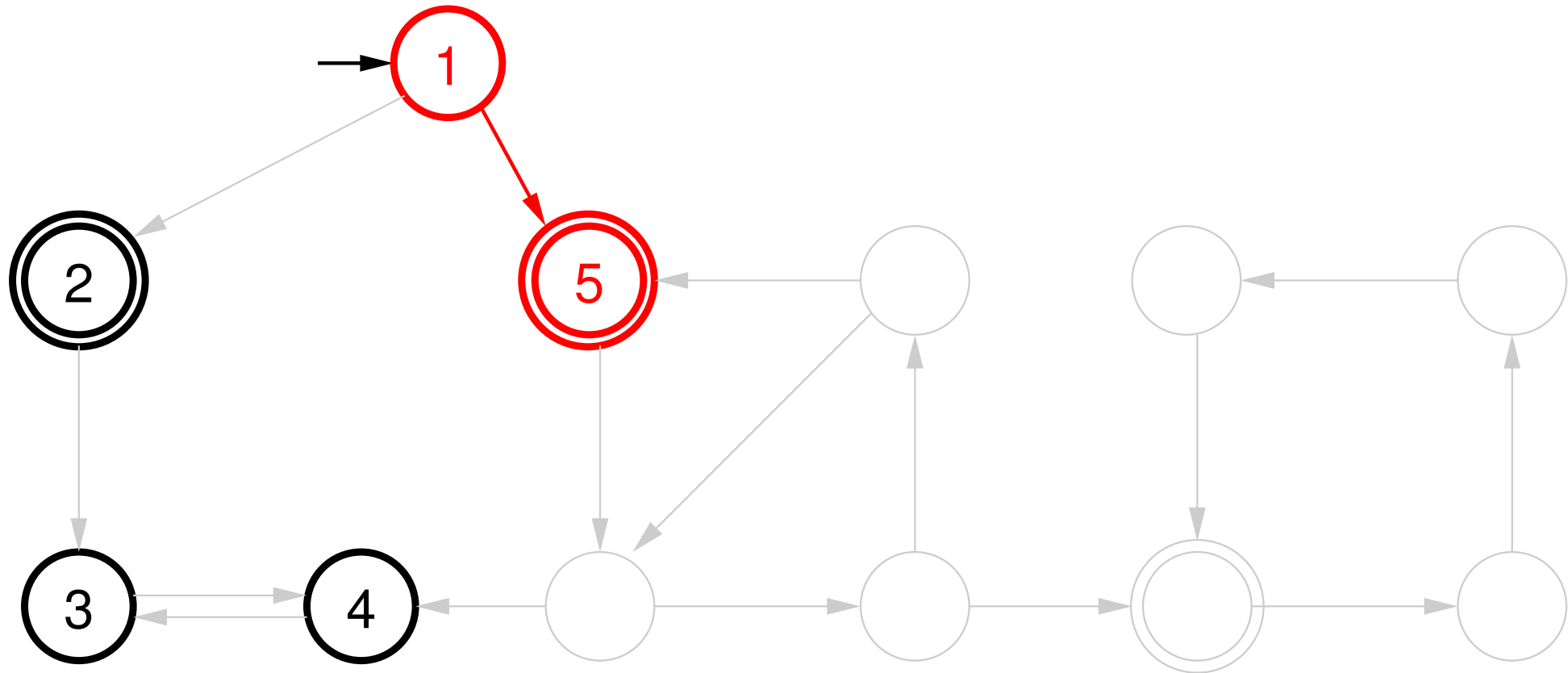


All immediate successors of 4 have been explored; backtrack.



## Example: Depth-first search

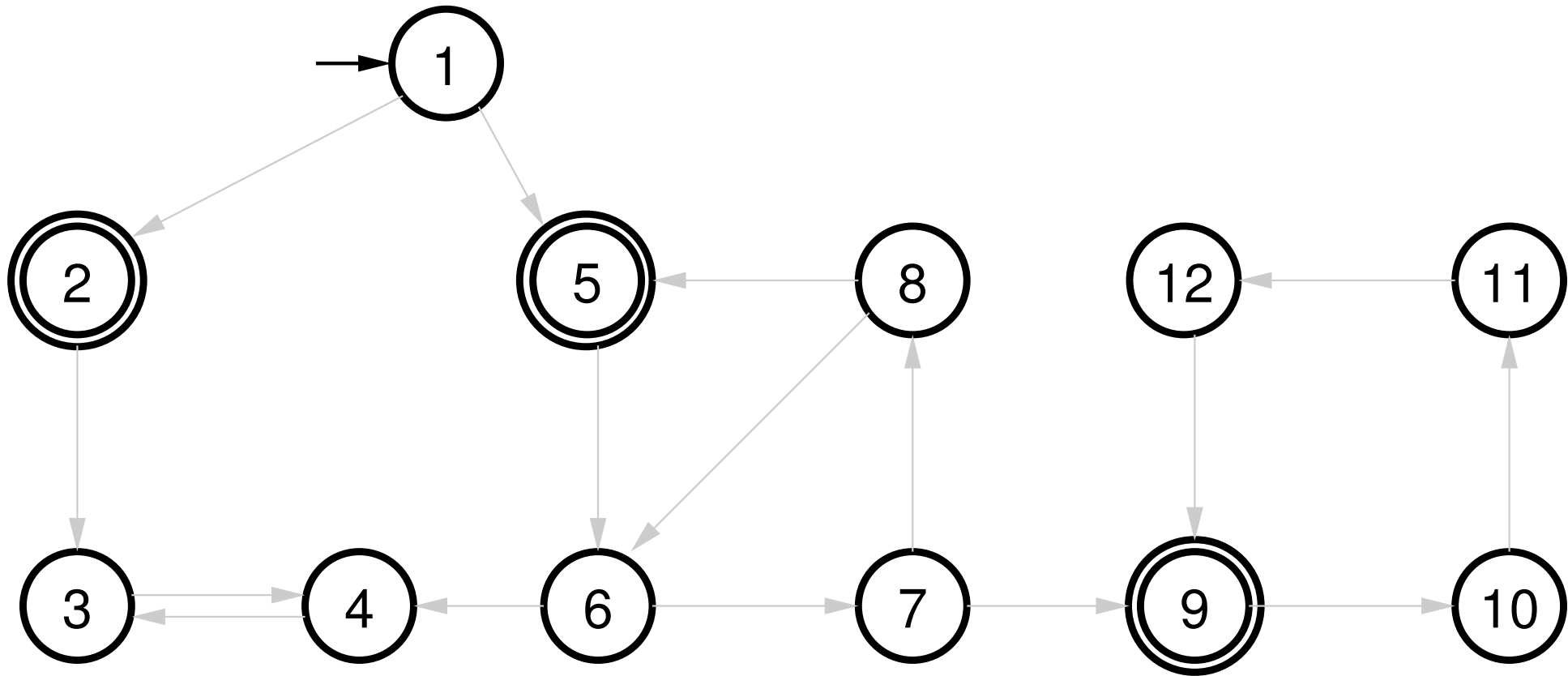
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Backtracking proceeds to state 1, next successor gets number 5.

# Example: Depth-first search

---



Possible numbering at the end of DFS.

# Properties of the search path

---

- (1) Let  $s_0 s_1 \dots s_n$  be the search path at some point during DFS.  
Then we have  $s_i.num < s_j.num$  iff  $i < j$ .  
Moreover,  $s_i \rightarrow^* s_j$  if  $i < j$ .

Proof: follows from the logic of the program and the order of recursive calls.

# Search order

---

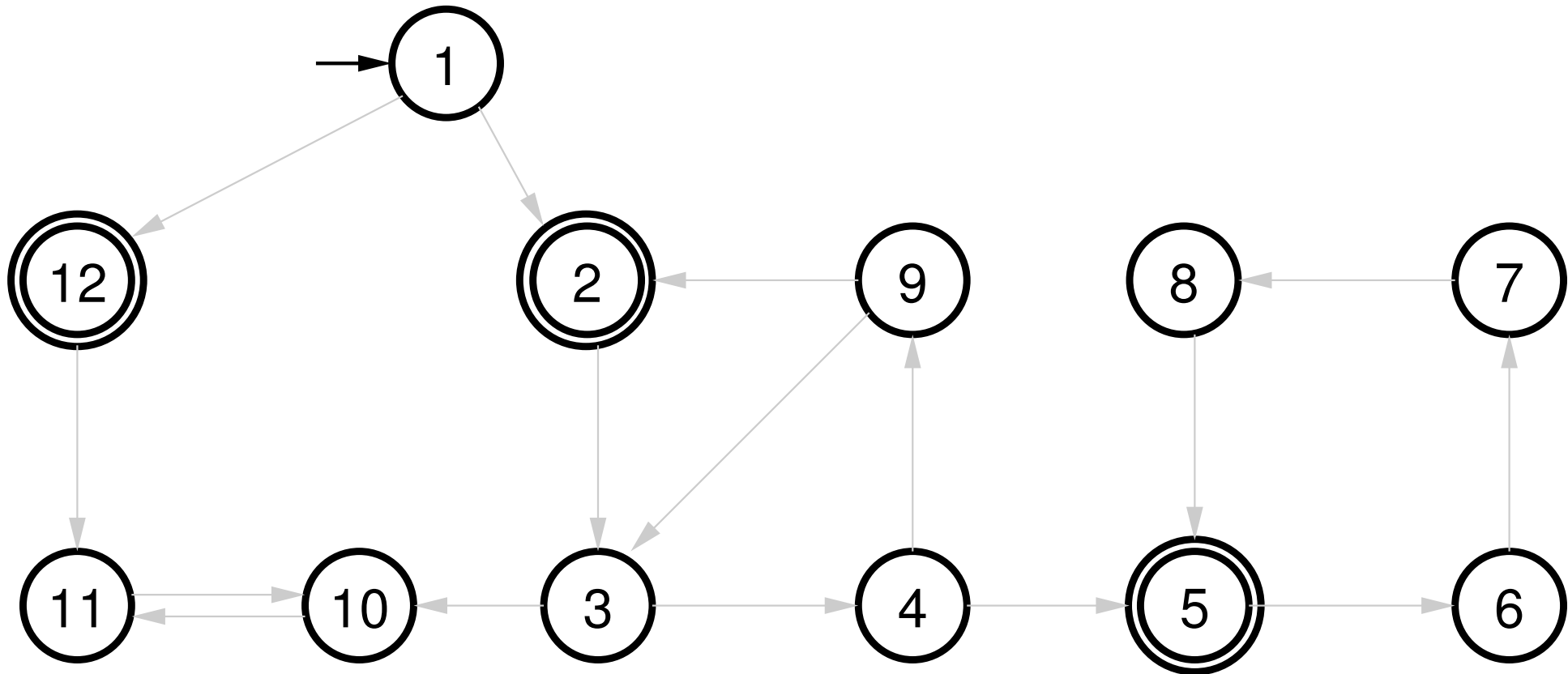
If a state has got multiple immediate successors, they must be explored in *some* order.

The DFS numbering therefore depends on the order in which these successors are explored; multiple different numberings are possible.

The search order may influence how quickly a counterexample is found (if one exists)!

## Example: Search order

---



Possible alternative numbering for a different search order.

# Search order

---

If a state has got multiple immediate successors, they must be explored in *some* order.

The DFS numbering therefore depends on the order in which these successors are explored; multiple different numberings are possible.

The search order may influence how quickly a counterexample is found (if one exists)!

Assumption: search-order non-deterministic (or fixed from outside)

Possible extension: “intelligent” search order exploiting additional knowledge about the model to find counterexamples more quickly.

# Roots

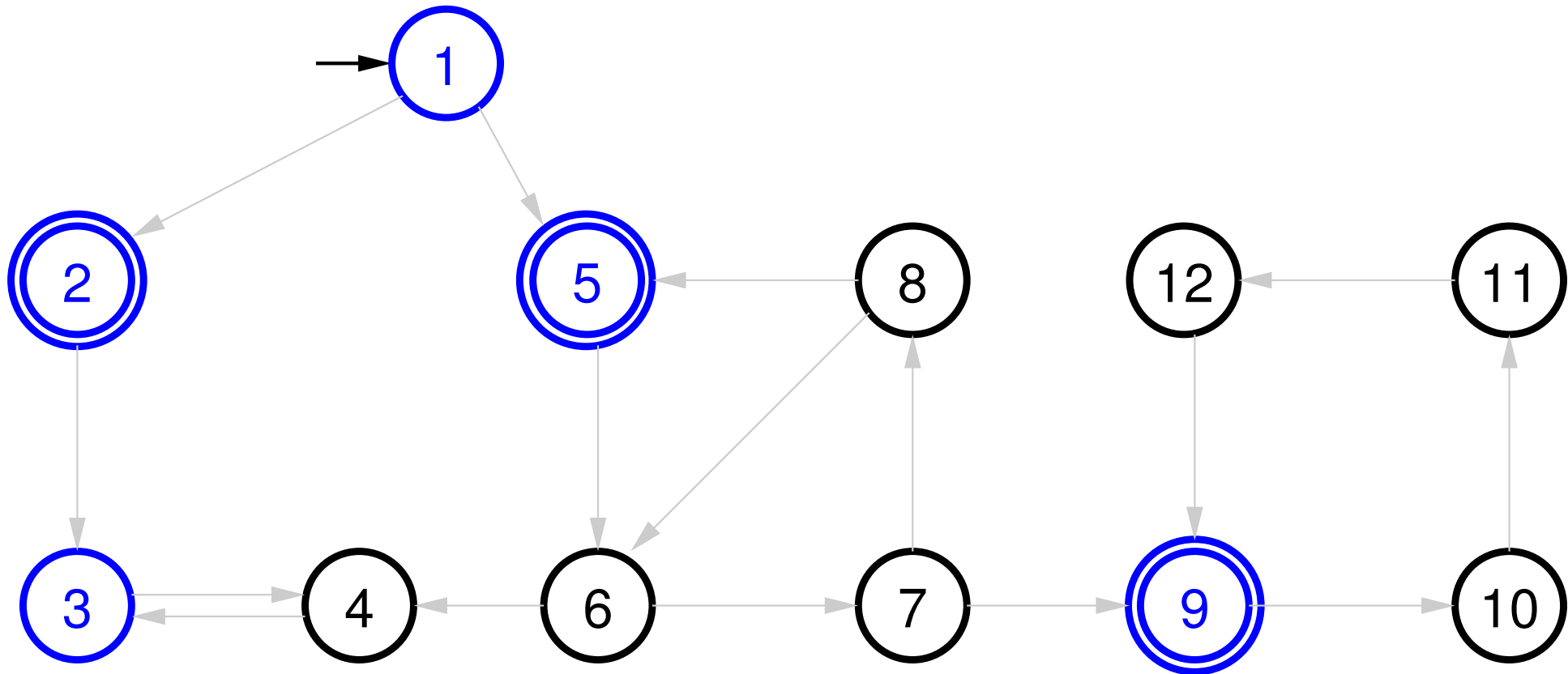
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The unique (w.r.t. a fixed search order) state of an SCC that is visited first during DFS is called its **root**.

**Remark:** Different search orders may lead to different states designated as roots.

# Example: Search order

---



Roots shown in blue when using the previous search order.



# Properties of roots

---

(2) A root has the **smallest** DFS number within its SCC.

Proof: obvious

(3) Within each SCC, the root is the **last** state from which DFS backtracks, and, at that point, the SCC has been explored completely (i.e., all states and edges have been considered).

Proof: Suppose the DFS first reaches a root  $r$ . At that point, no other state of the SCC has been visited so far, and all are reachable from  $r$ . Therefore, the DFS will visit all those states (and possibly others) and backtrack from them before it can backtrack from  $r$ .

# Explored/active Subgraph

---

At each point during the DFS, let us distinguish two specific subgraphs of  $\mathcal{B}$ .

The **explored graph** of  $\mathcal{B}$  denotes the subgraph containing all visited states and explored transitions.

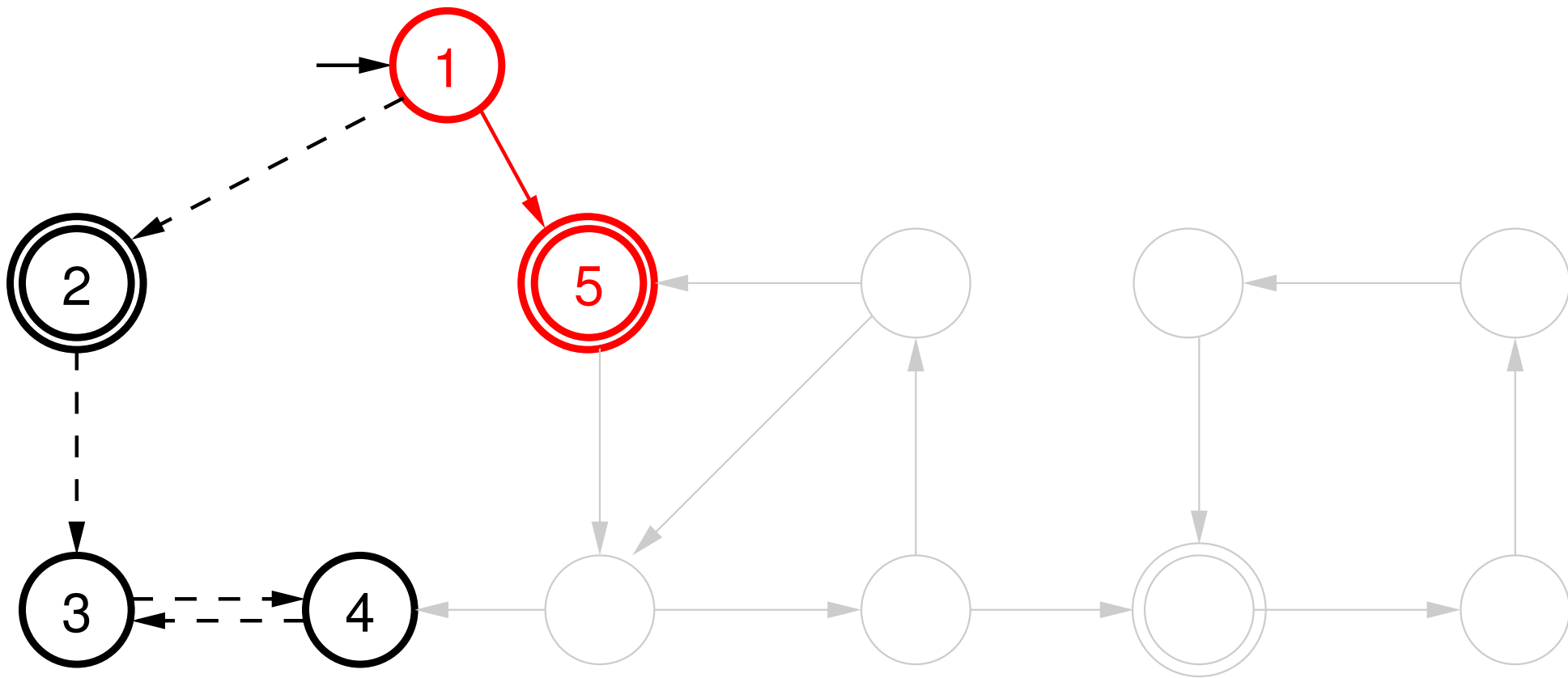
We call an SCC of the explored graph(!) **active**, if the search path contains at least one of its states (whose DFS call has not yet terminated).

A state is called **active** if it is part of an active SCC (it is not necessary for the state itself to be on the search path).

The **active graph** is the subgraph of the explored graph induced by the active states.

## Example: Explored/active subgraph

---



Here: explored graph shown in **red** and black, active SCCs: {1} and {5}, inactive SCCs {2} and {3, 4}.

# Properties of the active graph

---

(4) An SCC becomes inactive when we backtrack from its root.

Proof: follows from (3).

(5) An inactive SCC of the explored graph is also an SCC of  $\mathcal{B}$ .

Proof: Follows immediately from (3) and (4).

(6) The roots of the active graph are a subsequence of the search path.

Proof: Follows from (4) because the root of an active SCC must be on the search path.

- 
- (7) Let  $s$  be an active state and  $t$  (where  $t.num \leq s.num$ ) the root of its SCC in the active graph. Then there is no active root  $u$  with  $t.num < u.num < s.num$ .

Proof: Assume that such an active root  $u$  exists. Since  $u$  is active, it is on the search stack, just like  $t$ , see (4). Then, because of (1), we have  $t \rightarrow^* u$ . As  $\text{dfs}(u)$  has not yet terminated and  $u.num < s.num$ ,  $s$  must have been reached from  $u$ , i.e.  $u \rightarrow^* s$ . Because  $s, t$  are in the same SCC,  $s \rightarrow^* t$  holds. But then,  $t, u$  are in the same SCC and cannot both be its root.

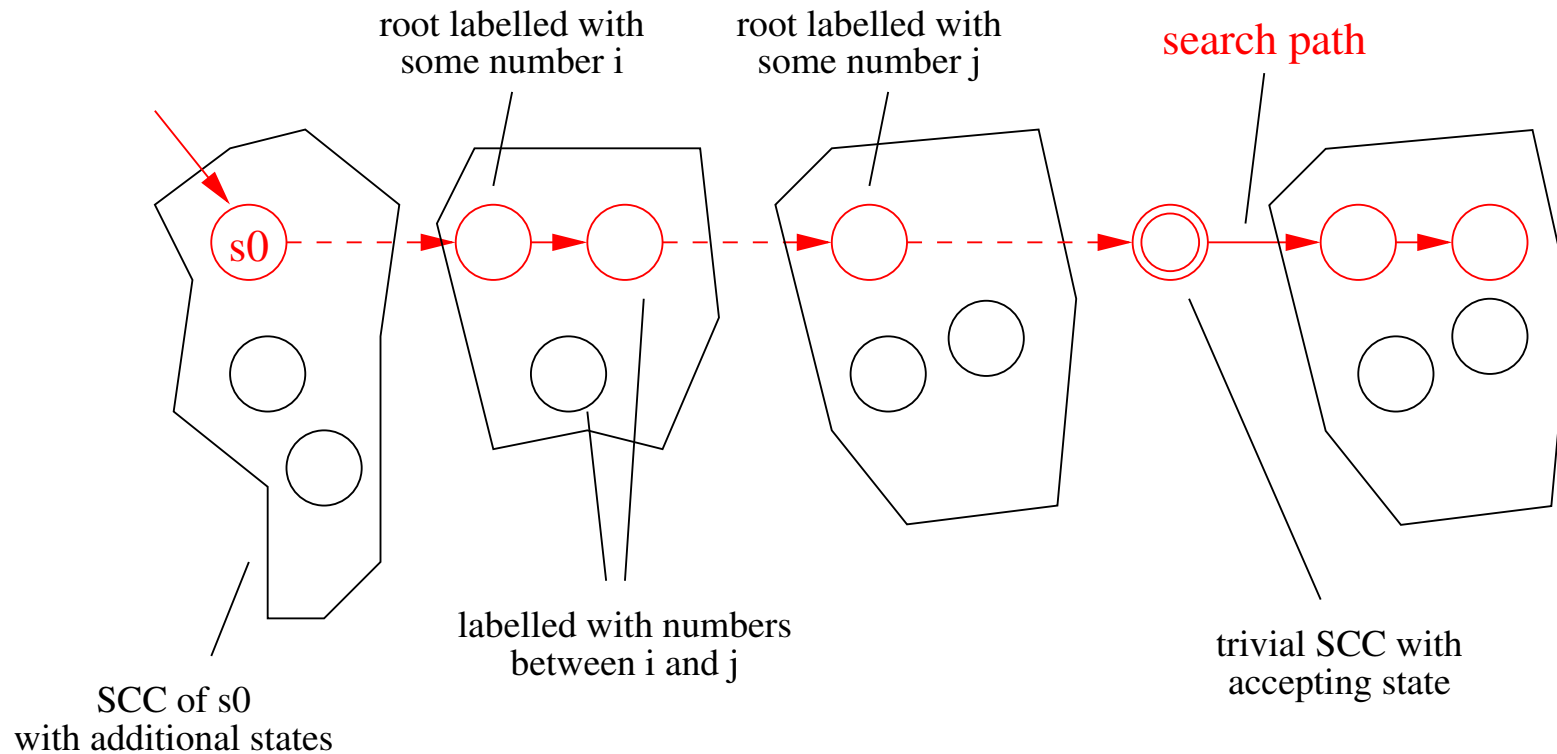
- (8) Let  $s$  and  $t$  be two active states with  $s.num \leq t.num$ . Then  $s \rightarrow^* t$ .

Let  $s', t'$  be the (active) roots for  $s$  and  $t$ , resp. Because of (7) we have  $s'.num \leq t'.num$ , thus because of (1)  $s' \rightarrow^* t'$ , and therefore  $s \rightarrow^* t$ .

# Visualization

---

From the properties we've just proved, it follows that the active graph and its SCCs are always of the following form, at any time during DFS:



# Properties of our emptiness-checking algorithm

---

Run-time linear in  $|S| + |\delta|$ .

Explores  $\mathcal{B}$  using DFS; reports a counterexample *as soon as the explored graph contains one*. (\*)

For every explored state  $s$  the algorithm computes  $\text{succ}(s)$  only once.

→ saves time because  $\text{succ}$  is the most expensive operation in practice.

# Additional memory usage

---

Stack  $W$  with elements of the form  $(s, C)$ , where

$s$  is the root of an active SCC;

$C$  is the set of state in the SCC of  $s$ .

( $C$  may be implemented as a linked list, one additional pointer for each state.)

One bit per state indicating whether a state is active or not.



# How the algorithm works

---

Actions of the algorithm:

Initialization

Discovering new edges (to old or new states)

Backtracking

With each action, we

update the contents of  $W$  and the “active” bits;

check whether the explored graph contains a counterexample.

# Initialization

---

Explored graph consists just of the initial state  $s_0$ , no edges.

One single element in  $W$ : the tuple  $(s_0, \{s_0\})$

$s_0$  is active.

# Dealing with new edges

---

Suppose we discover an edge  $s \rightarrow t$ . We distinguish five cases:

Case 1:  $t$  was never seen before:

The explored graph is extended by the state  $t$  and the edge  $s \rightarrow t$ .

$t$  is active and forms a trivial SCC within the active graph.

Extend  $W$  by  $(t, \{t\})$ .

Recursively start DFS on  $t$ .

## Dealing with new edges

---

Suppose we discover an edge  $s \rightarrow t$ . We distinguish five cases:

Case 2:  $t$  has been visited before and is inactive.

If  $t$  is inactive, then its SCC has been completely explored, see (3) and (4). Therefore,  $s, t$  must belong to different SCCs, in particular,  $t \rightarrow^* s$  cannot hold. Therefore, the edge  $s \rightarrow t$  cannot be part of a lasso, and we can ignore it.

No recursive call,  $W$  and the “active” bits remain unchanged.

# Dealing with new edges

---

Suppose we discover an edge  $s \rightarrow t$ . We distinguish five cases:

Case 3:  $t$  was visited before and is active, and  $t.num > s.num$ .

From (8) we already know that  $s \rightarrow^* t$  holds, therefore the SCCs of the active graph do not change, and no new counterexample can be generated in this way. Thus, we ignore the edge.

No recursive call,  $W$  and the “active” bits remain unchanged.

# Dealing with new edges

---

Suppose we discover an edge  $s \rightarrow t$ . We distinguish five cases:

Case 4:  $t$  was visited before and  $t.num = s.num$ .

Then  $s = t$ .

A counterexample has been discovered iff  $s$  is accepting.

Otherwise: no recursive call,  $W$  and the “active” bits remain unchanged.

## Dealing with new edges

---

Suppose we discover an edge  $s \rightarrow t$ . We distinguish five cases:

Case 5:  $t$  was seen before and is active,  $t.num < s.num$ .

Then because of (8) we have  $t \rightarrow^* s$ . Thus,  $s, t$  belong to the same SCC. Let  $u$ , with  $u.num \leq t.num$ , be the root of the SCC to which  $t$  belongs. Since  $s$  is the latest element on the search path, it follows from (1) that all SCCs stored on  $W$  from  $u$  downwards must be merged into one SCC.

We find  $u$  by removing elements from  $W$  until we find a root whose number is no larger than  $t.num$ , compare (7).

A new counterexample is generated iff one of the merged SCCs was hitherto trivial. Therefore, while removing elements from  $W$  we simply check whether any of the roots is an accepting state.

# Backtracking

---

Suppose that all elements in  $\text{succ}(s)$  have been explored.

Case 1:  $s$  is a root.

Then  $s$  and its entire SCC become inactive, see (4).

Moreover, we remove the topmost element from  $W$ .

Case 2:  $s$  is not a root.

Then the root of its SCC is still active.

$W$  and the “active” bits remain unchanged.



# Et voilà...

---

```
nr = 0; hash = {}; W = {}; dfs(s0); exit;
```

```
dfs(s) {  
  add s to hash; s.active = true;  
  nr = nr+1; s.num = nr;  
  push (s, {s}) onto W;  
  for (t in succ(s)) {  
    if (t not yet in hash) { dfs(t); }  
    else if (t.active) {  
      D = {};  
      repeat  
        pop (u,C) from W;  
        if u is accepting { report success; halt; }  
        merge C into D;  
      until u.num <= t.num;  
      push (u,D) onto W;  
    } }  
  if s is the top root in W {  
    pop (s,C) from W;  
    for all t in C { t.active = false; }  
  }  
}
```

## Remarks on the algorithm

---

Cases 3 to 5 for handling edges are dealt with uniformly in the repeat-until loop.

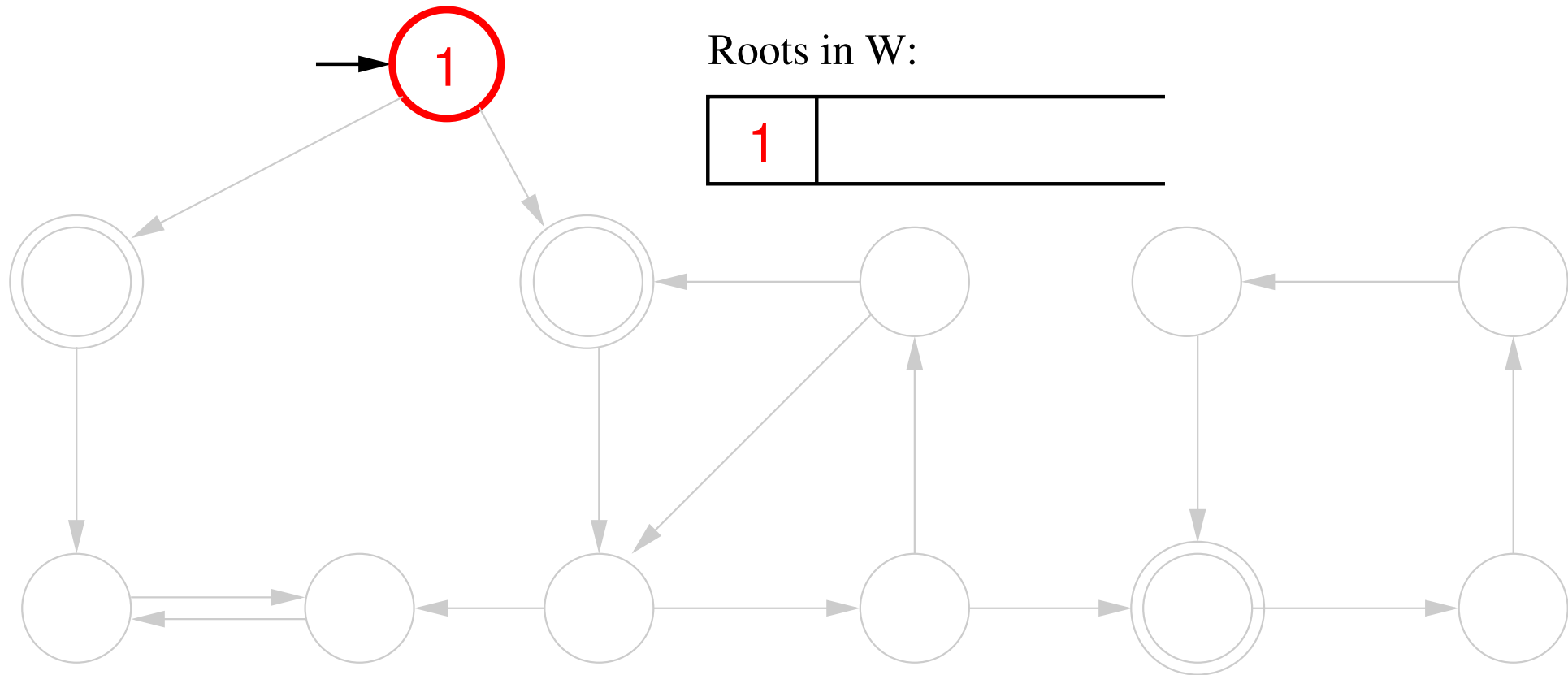
The statement `report success` symbolizes the discovery of a counterexample.

If `dfs(s0)` terminates, no counterexample exists.

Run-time linear in number of states plus number of transitions.

# Example: Execution of the algorithm

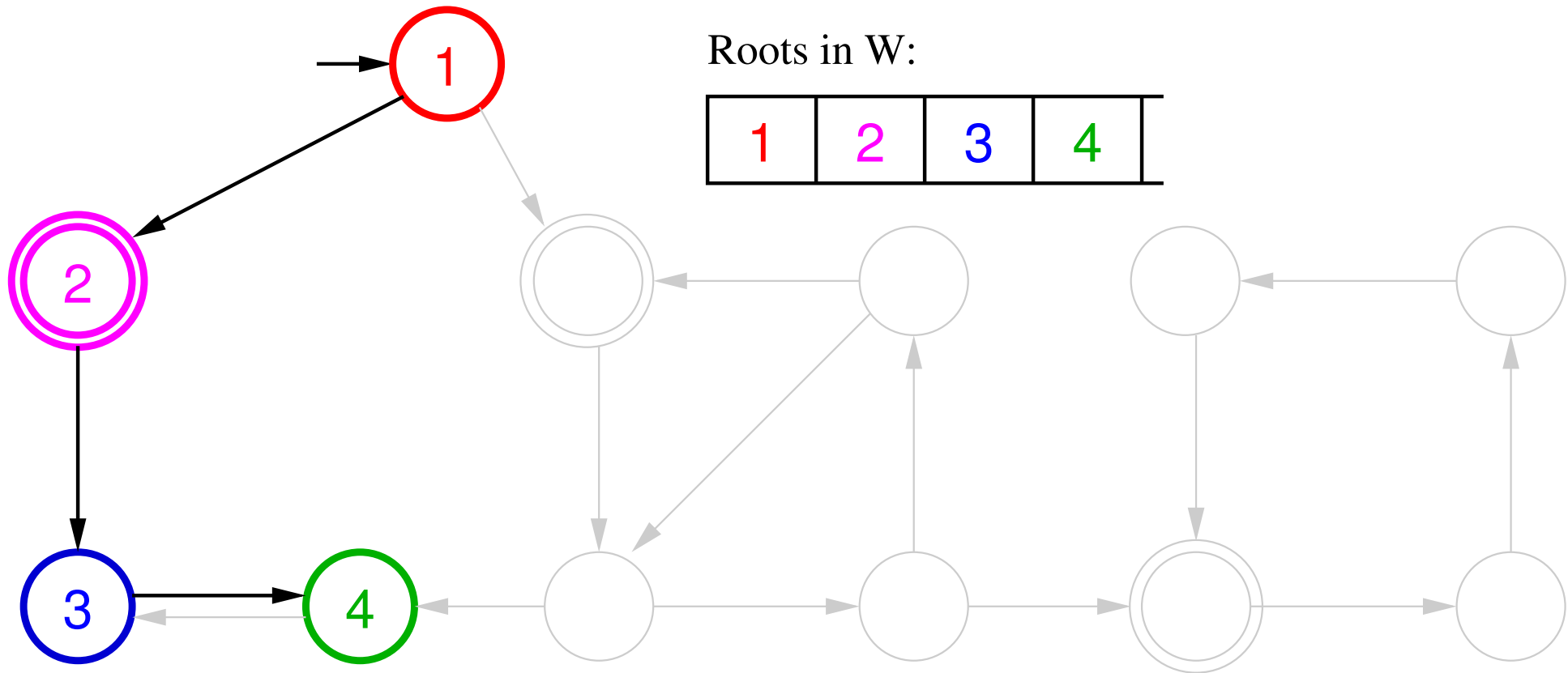
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Situation at the beginning, only  $s_0$  explored.

# Example: Execution of the algorithm

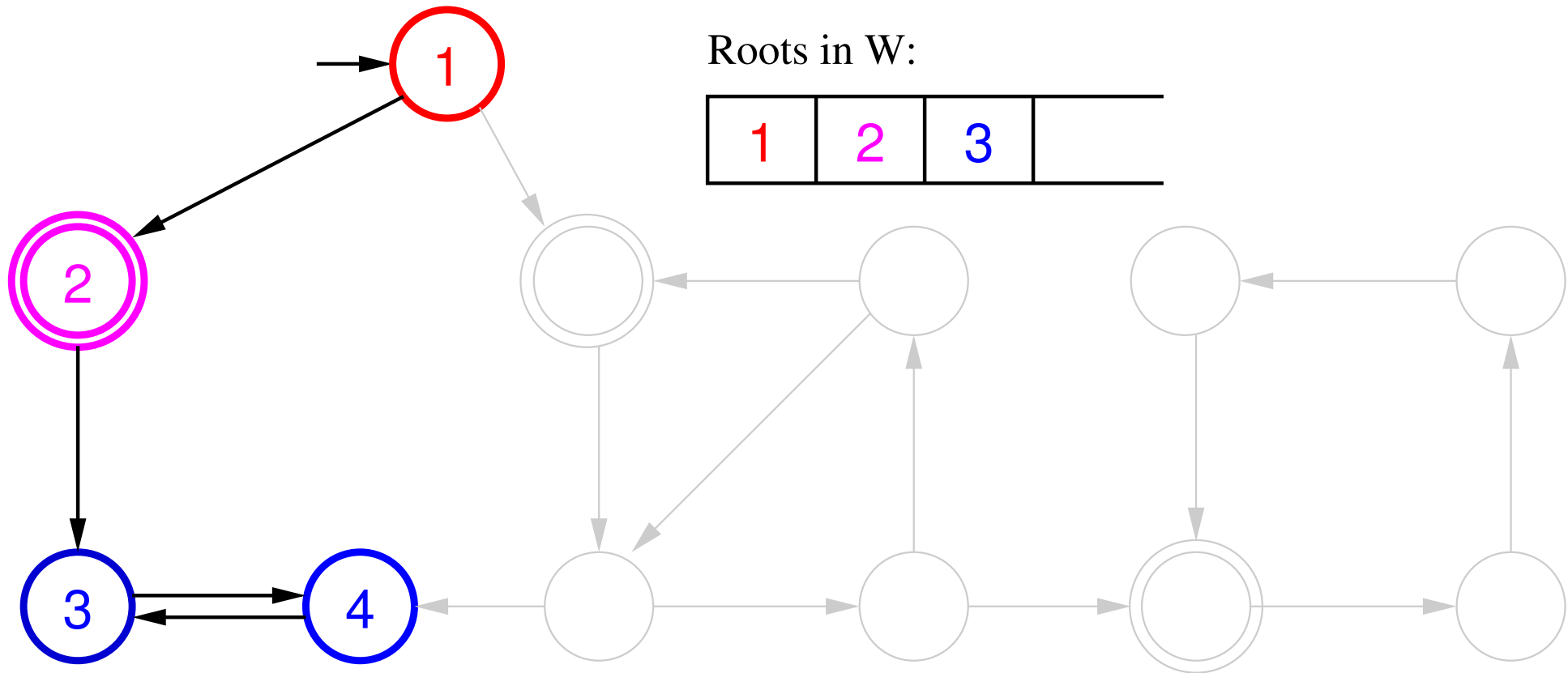
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Situation after discovering three edges.

# Example: Execution of the algorithm

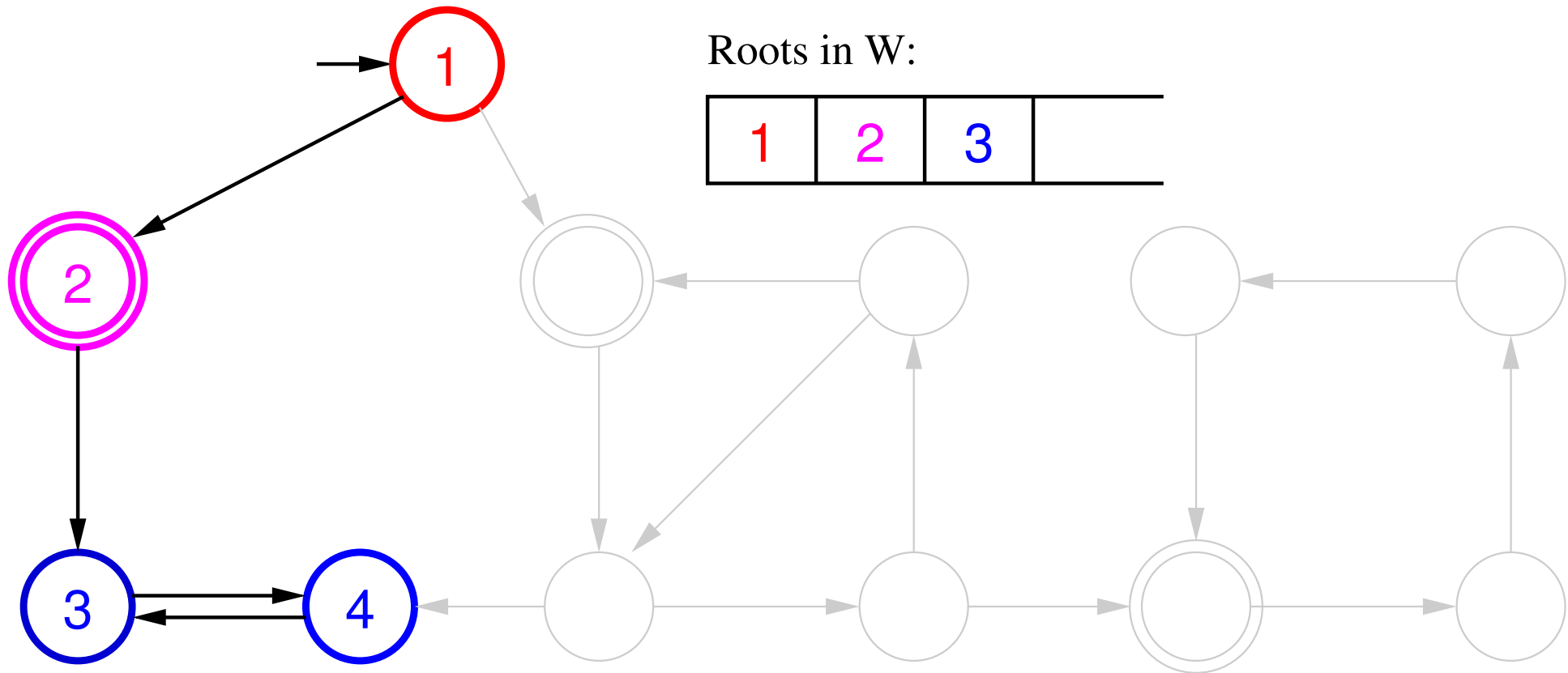
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Edge  $4 \rightarrow 3$  leads to merger of two SCCs.

# Example: Execution of the algorithm

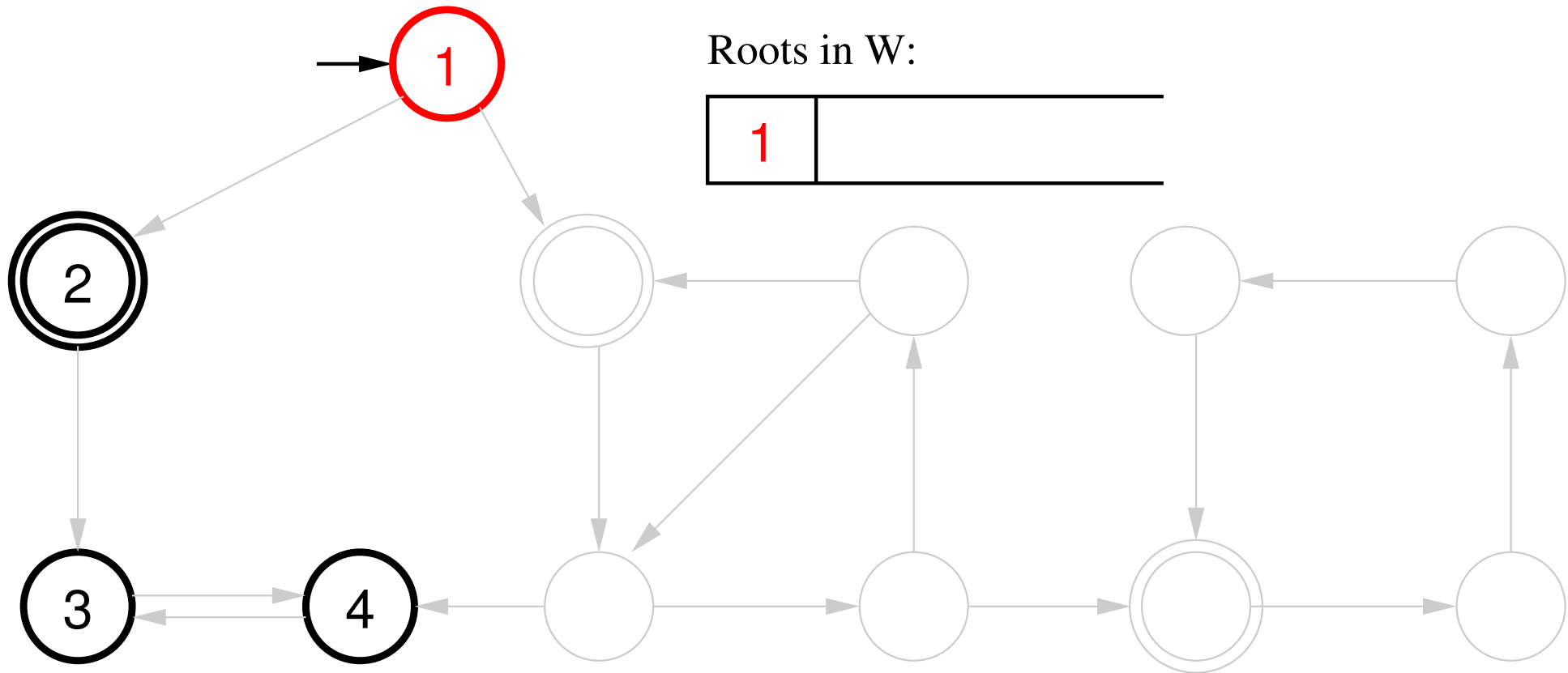
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(set component of each entry in  $W$  indicated by colours)

# Example: Execution of the algorithm

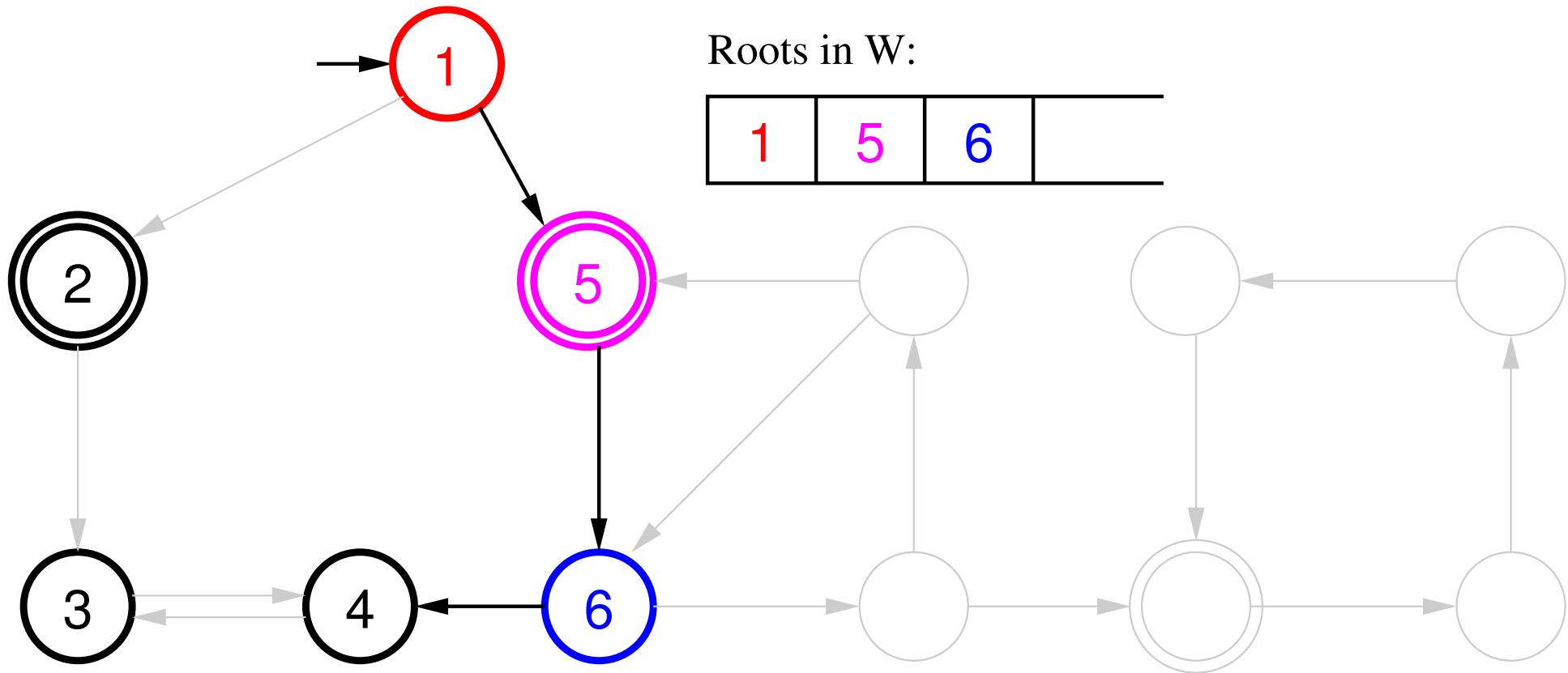
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Backtracking makes 2, 3, and 4 inactive (shown in black).

# Example: Execution of the algorithm

---

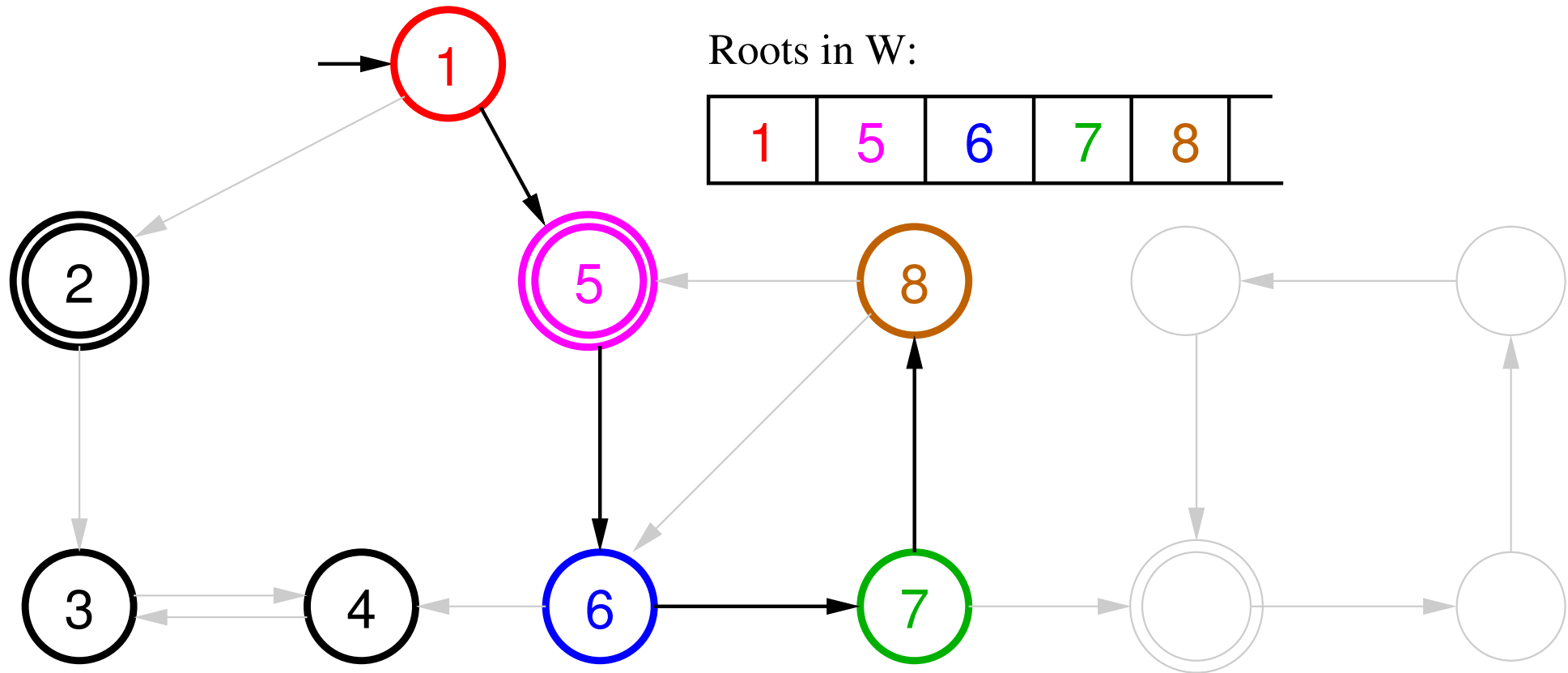


Edge  $6 \rightarrow 4$  is an example of Case 2 and may be ignored.



# Example: Execution of the algorithm

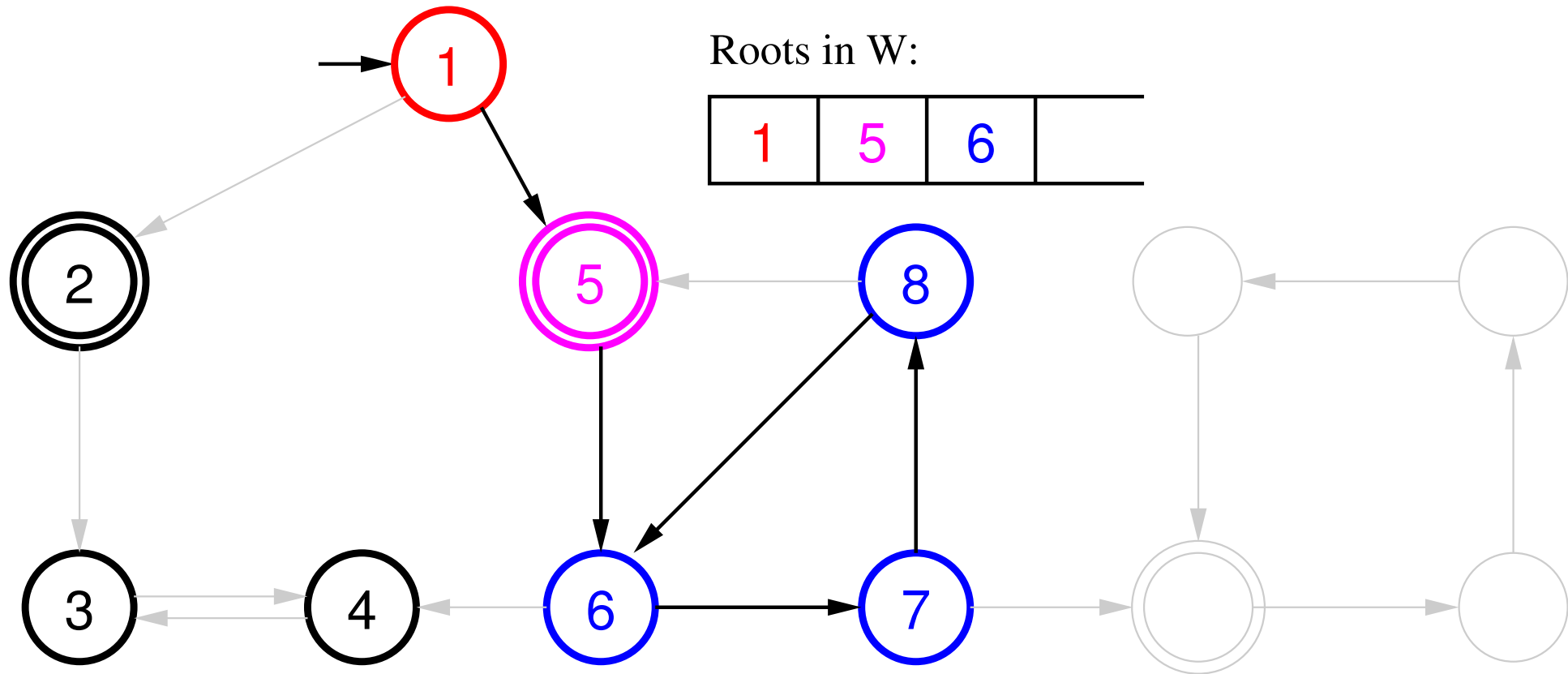
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Situation when reaching 8.

# Example: Execution of the algorithm

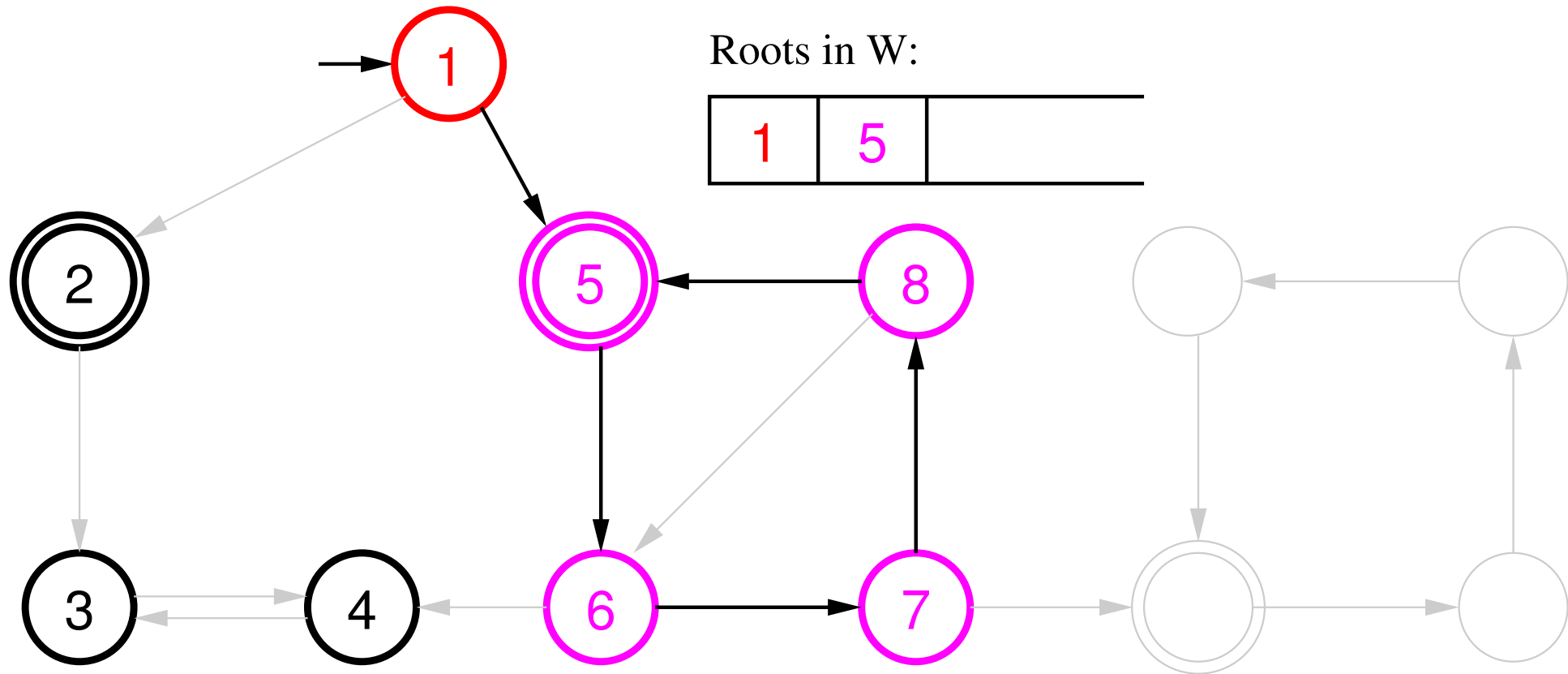
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Edge  $8 \rightarrow 6$  leads to a merger.

# Example: Execution of the algorithm

---



Edge  $8 \rightarrow 5$ : Counterexample discovered because root 5 is accepting.

# Extension to generalized Büchi automata

---

Let  $\mathcal{G}$  be a GBA with  $n$  acceptance sets  $F_1, \dots, F_n$ .

$\mathcal{L}(\mathcal{G})$  is non-empty iff there exists a non-trivial SCC intersecting each set  $F_i$  ( $1 \leq i \leq n$ ).

Let us label each state  $s$  with the index set of the acceptance sets it is contained in, denoted  $M_s$ . (E.g., if  $s$  in  $F_1$  and in  $F_3$ , but in no other acceptance set, then  $M_s = \{1, 3\}$ .)

We extend  $W$  by a third component, an index set, i.e. a subset of  $\{1, \dots, n\}$ .

---

During the algorithm, we uphold the following invariant: if  $W$  has an entry  $(s, C, M)$ , then  $M = \bigcup_{t \in C} M_t$ .

When two SCCs are merged, we take the union of the index sets.

A counterexample is discovered if this leads to an index set  $\{1, \dots, n\}$ .

If  $n$  is “small”, the required operations can be implemented using bit vectors (constant time).

# Modification for computing SCCs

---

The algorithm can also be used to partition the BA (or, in fact, any directed graph) into its SCCs.

For this, we simply omit the acceptance test when merging active SCCs.

The algorithm may output a complete SCC as soon as one backtracks from its root.

# Part 7: Partial-order reduction

# The story so far

---

Given:

a system  $S$  described as Promela model, C program, Petri net, ...

$\Rightarrow$  obtain a Kripke structure  $\mathcal{K}$

specification as LTL formula  $\phi$

$\Rightarrow$  obtain a Büchi automaton  $\mathcal{B}$

Approach:

Construct the product of  $\mathcal{K}$  and  $\mathcal{B}$  and analyze it “on the fly”

run-time linear in  $|\mathcal{K}| \cdot |\mathcal{B}|$



# State-space explosion

---

Size of  $\mathcal{B}$ :

worst-case exponential in  $|\phi|$ , but mostly harmless

unavoidable (in general)

Size of  $\mathcal{K}$ :

often exponential in  $|S|$

“State-space explosion” caused by concurrency, data, ...

In the following, we will consider an improvement that tackles the effect of **concurrency**.

# Example: Leader-election protocol

---

The following protocol is due to [Dolev, Klawe, Rodeh \(1982\)](#). A model of it is contained in the Spin distribution.

The protocol consists of  $n$  participants (where  $n$  is a parameter). The participants are connected by a ring of unidirectional message channels. Communication is asynchronous, and the channels are reliable. Each participant has a unique ID (e.g., some random number).

**Goal:** The participants communicate to elect a “leader” (i.e., some distinguished participant). The protocol shown here ensures low communication overhead ( $\mathcal{O}(n \log n)$  messages).

# Leader-election protocol

---

Participants are either **active** or **inactive**. Initially, all participants are *active*.

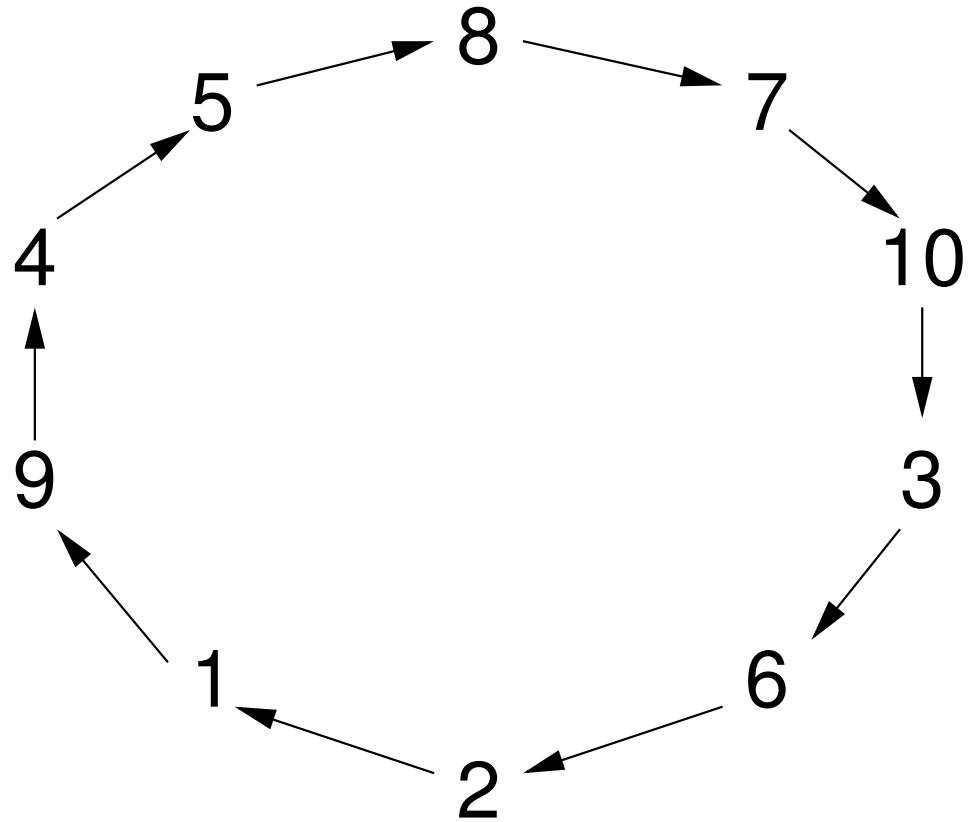
The protocol proceeds in **rounds**. In each round, at least half of the participants will become inactive. (As a consequence, there are at most  $O(\log n)$  rounds.

In each round every active participant receives the numbers of the two nearest active participants (in incoming direction). A participant remains active only if the value of the nearest neighbour is the largest of the three. In this case, the participant adopts this largest number as its own.

The last remaining active participant is declared the leader.

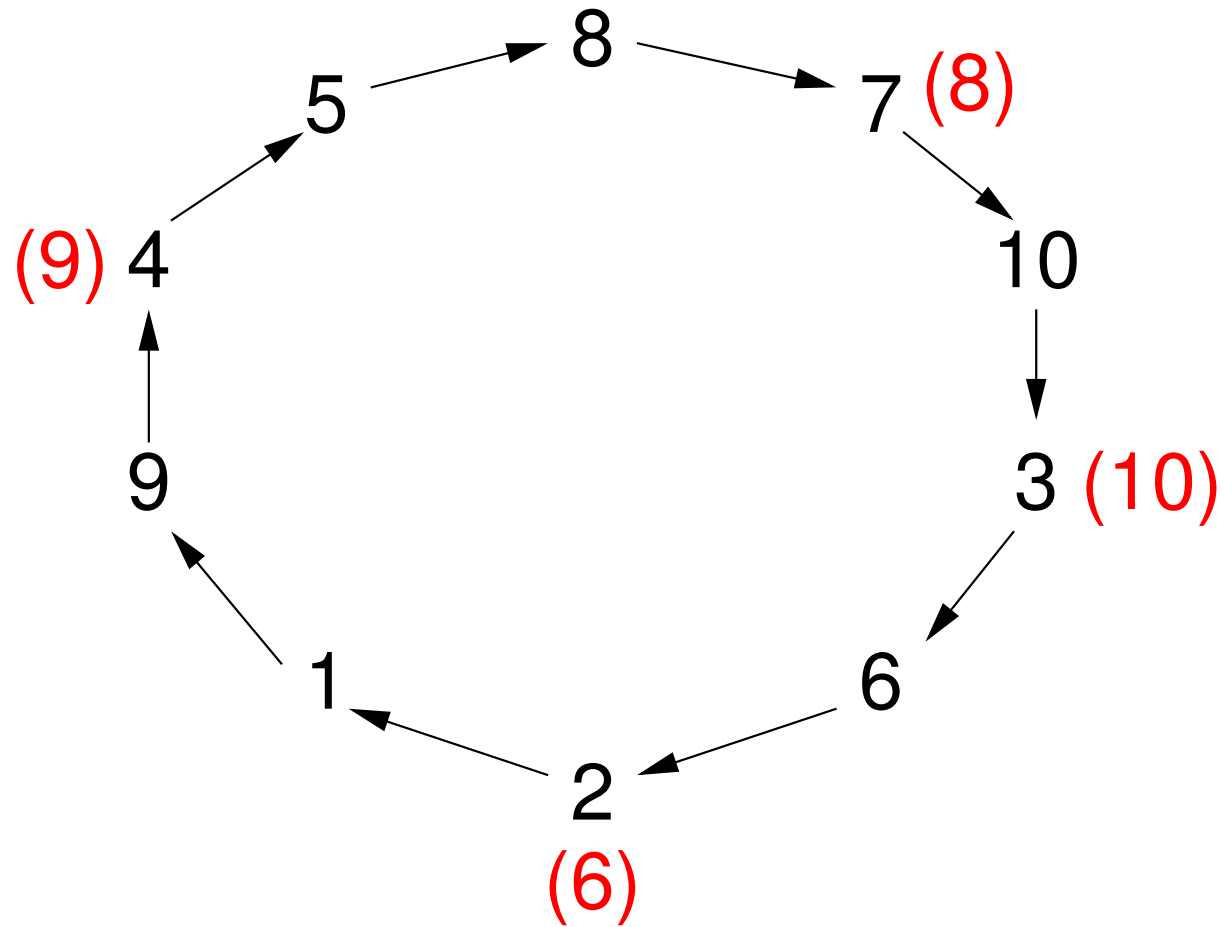
# Leader Election: Example

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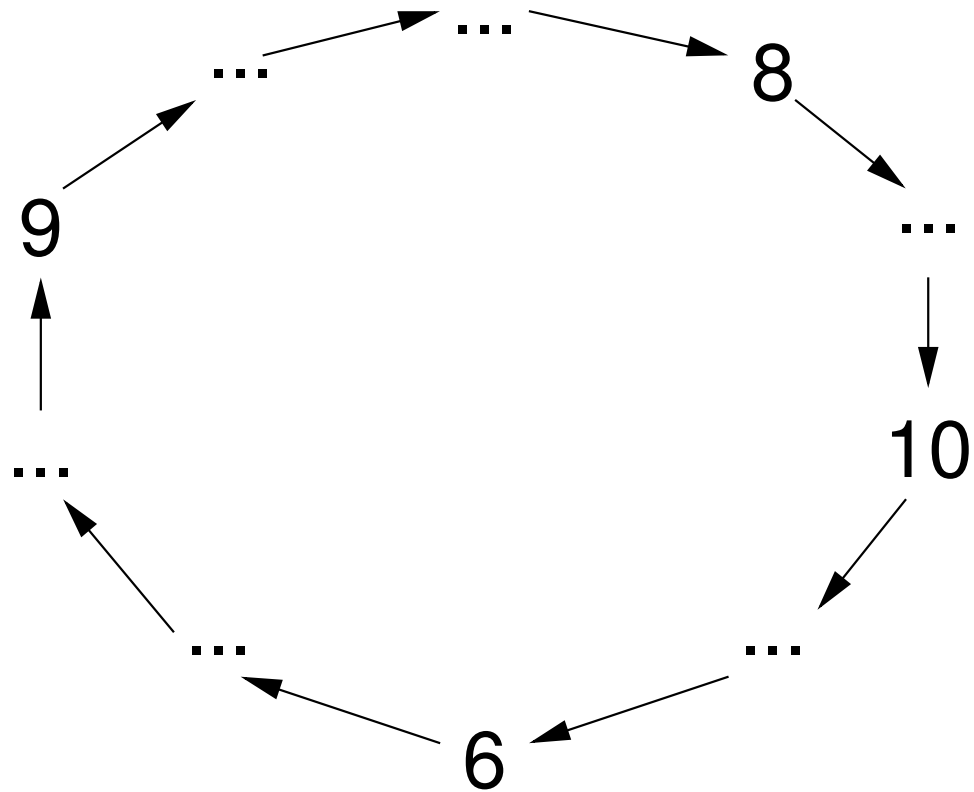
# Leader Election: First round

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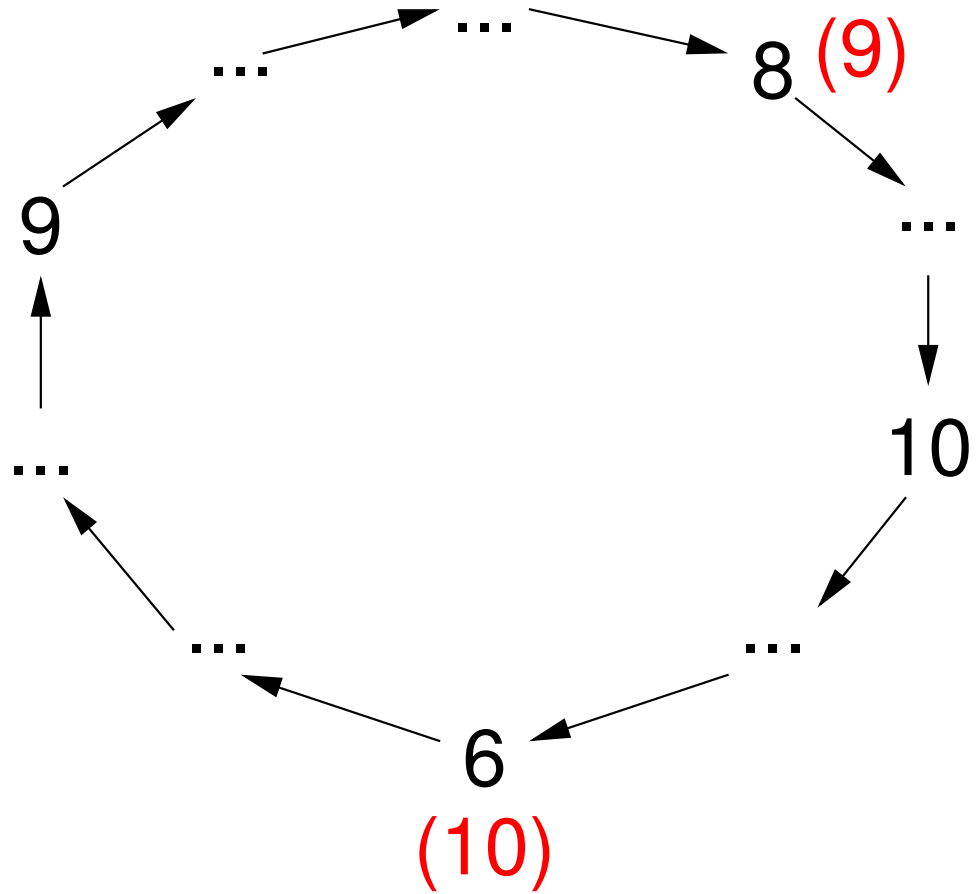
# Leader Election: Result of the first round

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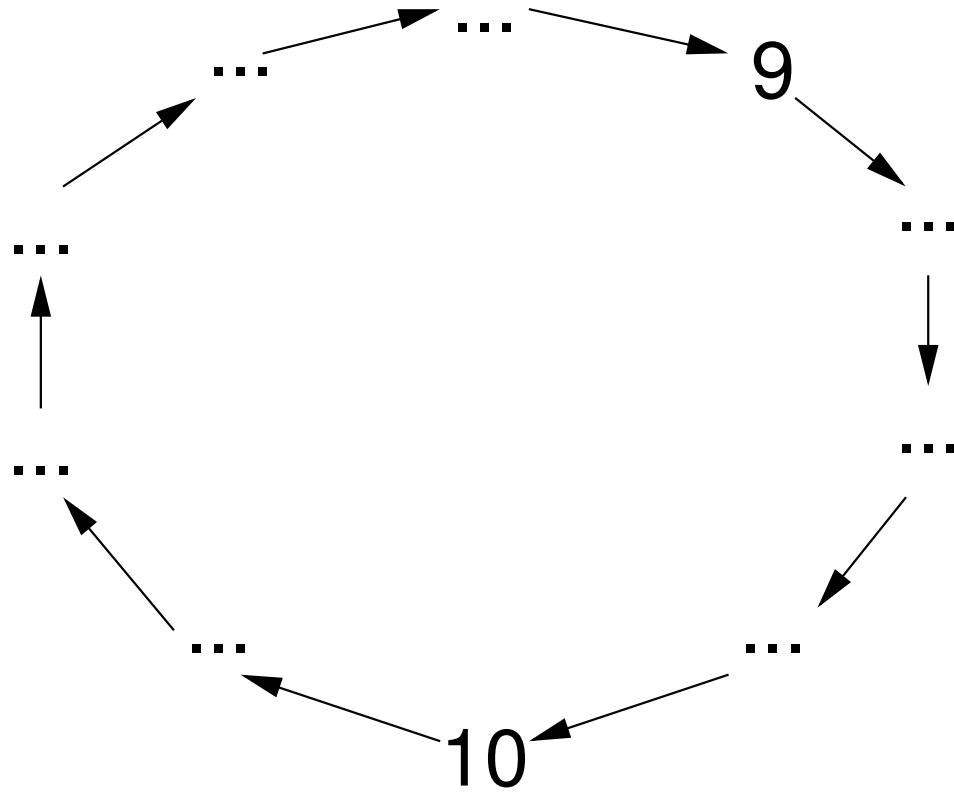
# Leader Election: Second round

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# Leader Election: Result of the second round

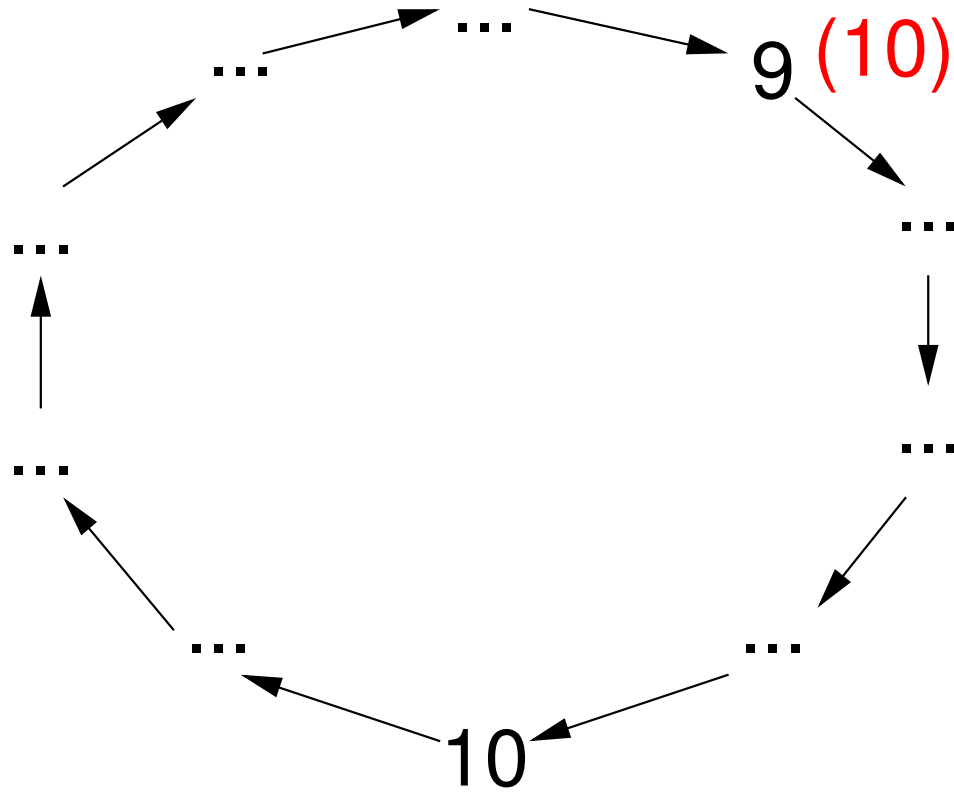
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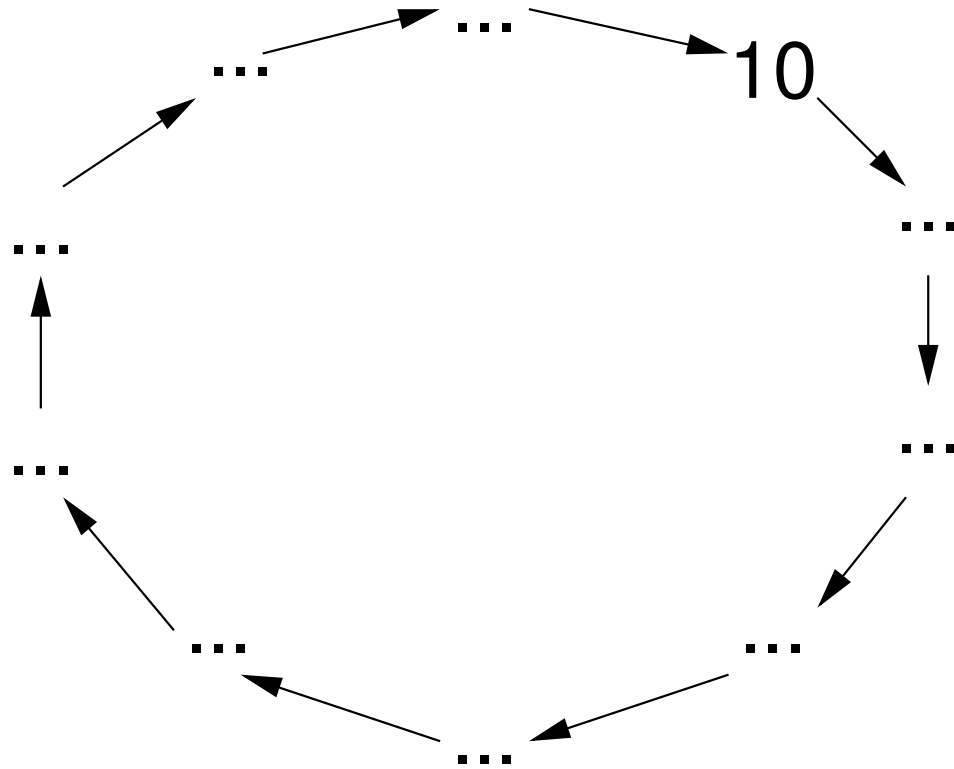
# Leader Election: Third round

---



# Leader Election: Final result

---



# Leader-Election protocol

---

Motivation: Low message overhead ( $\mathcal{O}(n \log n)$  messages).  
(Most naïve approaches require quadratically many messages.)

We generate the state space of this example

- with the methods we've seen so far
- using Spin

We will see that the methods we see so far will explore a number of states exponential in  $n$ . It will run out of memory for fewer than 10 participants.

In contrast, Spin generates very few states (linearly many in  $n$ ). This is due to the way Spin handles concurrent processes. Let us take a closer look at what's happening.

# Example

---

Pseudocode program with two concurrent processes:

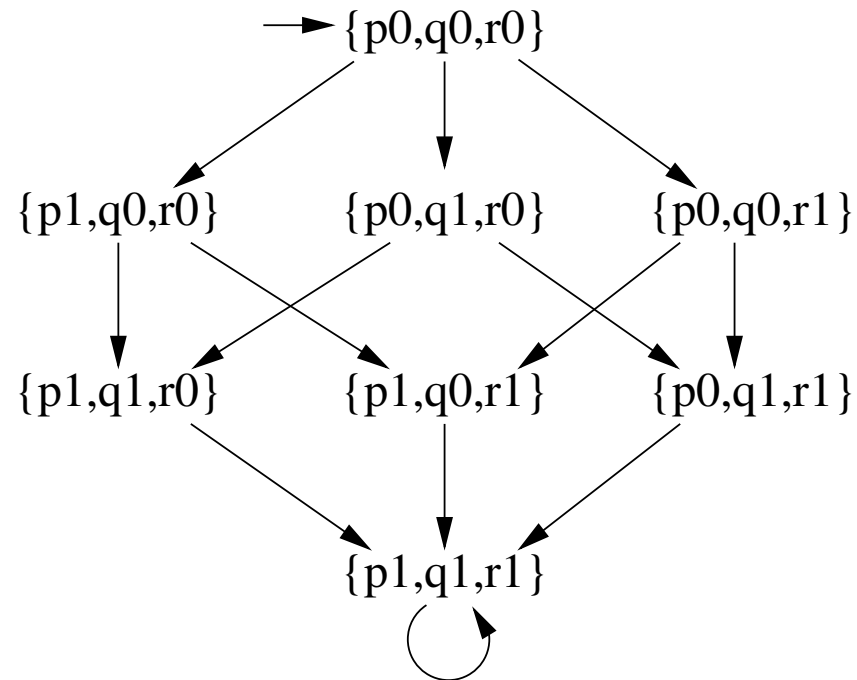
```
int x,y,z init 0;  
cobegin { P || Q || R } coend
```

```
P =   p0: x := 1;      Q =   q0: y := 1;      R =   r0: z := 1;  
      p1: end         q1: end         r1: end
```

---

The transition system has got  $8 = 2^3$  states and  $6 = 3!$  possible paths.

We omit the values of the variables in the states

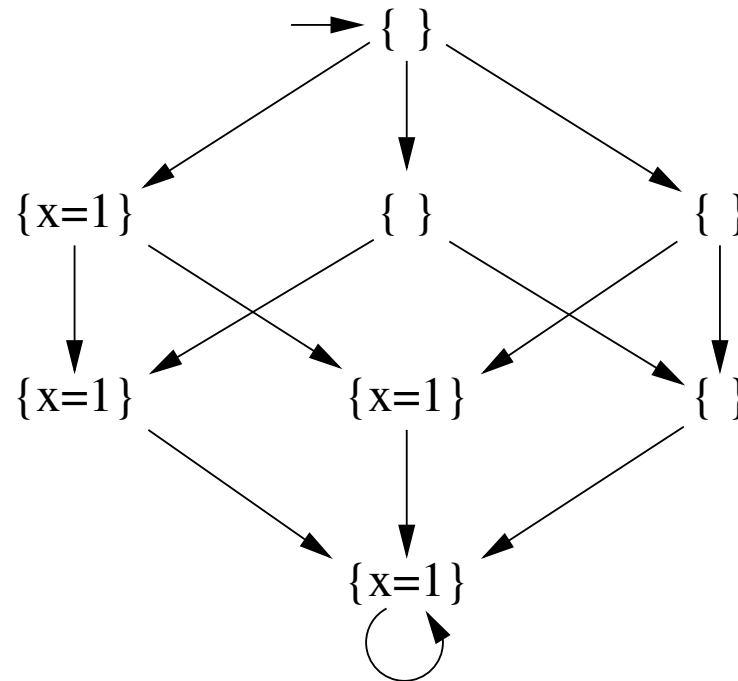


For  $n$  components we have  $2^n$  states and  $n!$  paths.

---

Consider properties like  $F(x = 1)$  or  $GF(x = 1)$ .

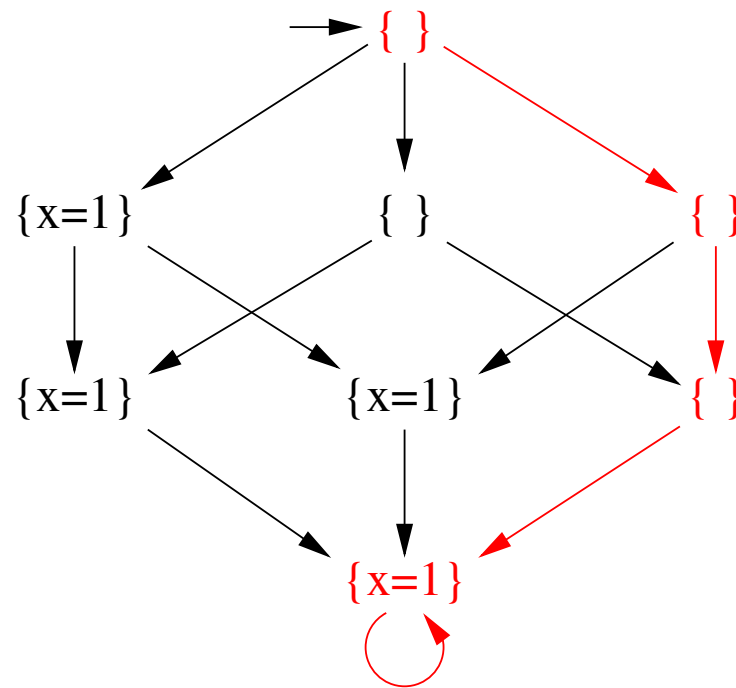
The Kripke structure for  $AP = \{x = 1\}$  is:



Idea: **reduce** size by considering only one path

---

Caution: Obviously, this is only possible if the paths are “equivalent”.



I.e., in the eliminated states nothing “interesting” happens.

# Partial-order techniques

---

**Partial-order techniques** aim to reduce state-space explosion due to **concurrency**.

One tries to exploit *independences* between transitions, e.g.

Assignments of variables that do not depend upon each other: .

$$x := z + 5 \quad || \quad y := w + z$$

Send and receive on FIFO channels that are neither empty nor full.

**Idea:** avoid exploring all **interleavings** of independent transitions

correctness depends on whether the property of interest does not distinguish between different such interleavings

may reduce the state space by an exponential factor

Methods: **ample sets**, **stubborn sets**, **persistent sets**, **sleep sets**



# On-the-fly Model Checking

---

**Important:** It would be pointless to construct  $\mathcal{K}$  first and then reduce its size.

(does not save space, we can analyze  $\mathcal{K}$  during construction anyway)

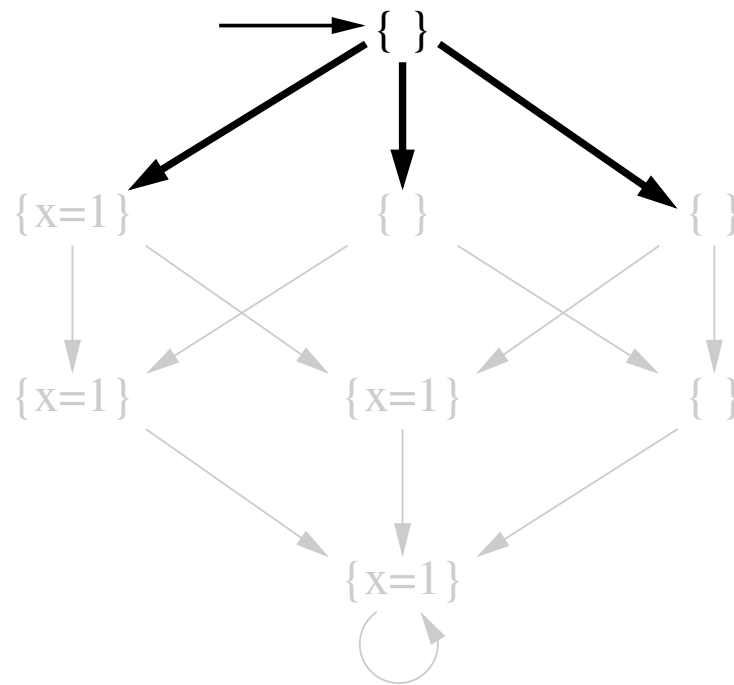
**Thus:** The reduction must be done “on-the-fly”, i.e. while  $\mathcal{K}$  is being constructed (from a compact description such as a Promela model) and during analysis.

⇒ combination with depth-first search

# Reduction and DFS

---

We must decide which paths to explore *at this moment*.

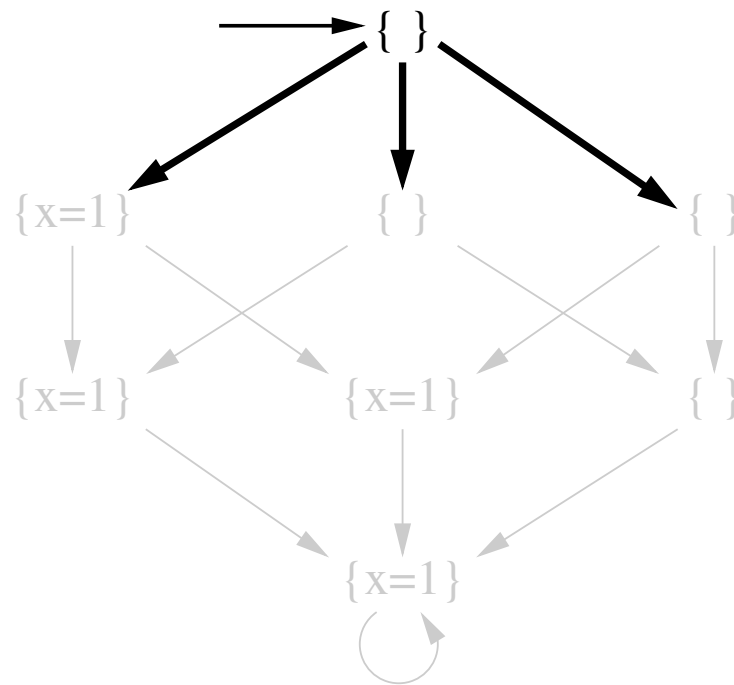


I.e. before having constructed (or “seen”) the rest!

# Reduction and DFS

---

We must decide which paths to explore *at this moment*.



→ only possible with additional information!

# Additional information

---

Transitions labelled with **actions**.

extracted from the underlying description of  $\mathcal{K}$ , e.g. the statements of a Promela model etc

Information about **independence** between actions

Do two actions influence each other?

Information about **visibility** of actions

Can an action influence the validity of any atomic proposition?

# Labelled Kripke structures

---

We extend our model with **actions**:

$$\mathcal{K} = (S, A, \rightarrow, r, AP, \nu)$$

$S, r, AP, \nu$  as before,  $A$  is a set of **actions**, and  $\rightarrow \subseteq S \times A \times S$ .

We assume forthwith that transitions are **deterministic**, i.e. for each  $s \in S$  and  $a \in A$  there is at most one  $s' \in S$  such that  $(s, a, s') \in \rightarrow$ .

$en(s) := \{a \mid \exists s' : (s, a, s') \in \rightarrow\}$  are called the **enabled actions** in  $s$ .

# Independence

---

$I \subseteq A \times A$  is called **independence relation** für  $\mathcal{K}$  if:

for all  $a \in A$  we have  $(a, a) \notin I$  (irreflexivity);

for all  $(a, b) \in I$  we have  $(b, a) \in I$  (symmetry);

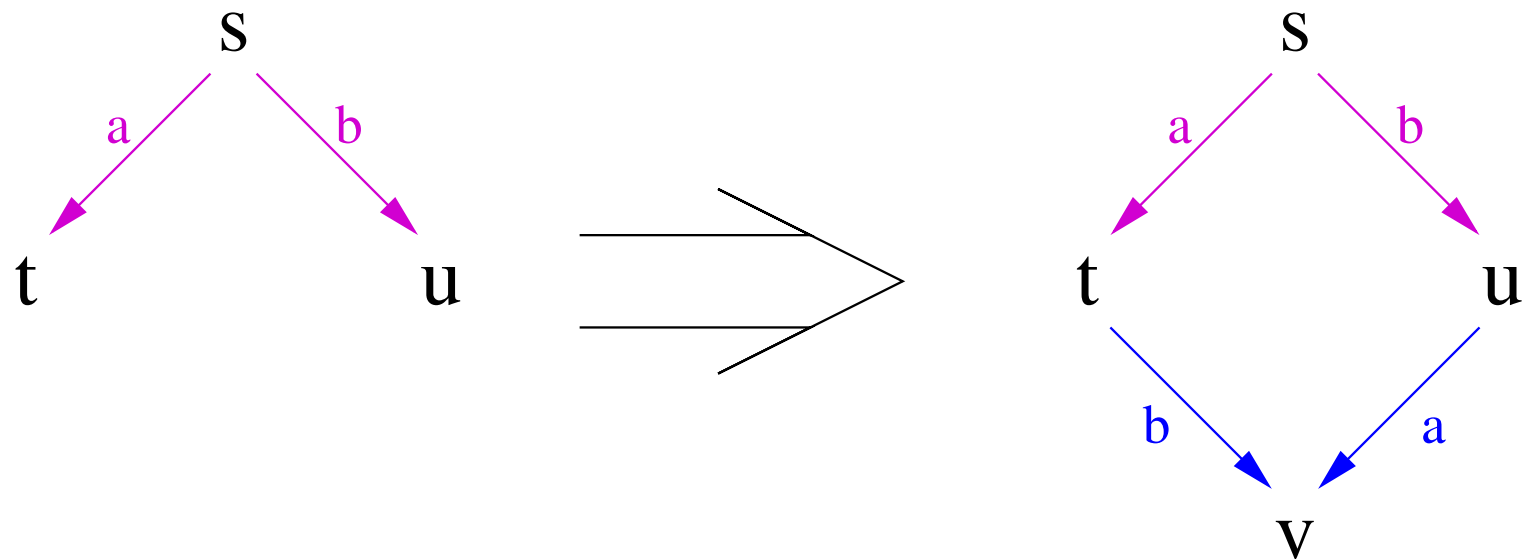
for all  $(a, b) \in I$  and all  $s \in S$  we have:

if  $a, b \in en(s)$ ,  $s \xrightarrow{a} t$ , and  $s \xrightarrow{b} u$ ,

then there exists  $v$  such that  $a \in en(u)$ ,  $b \in en(t)$ ,  $t \xrightarrow{b} v$  and  $u \xrightarrow{a} v$ .

# Independence

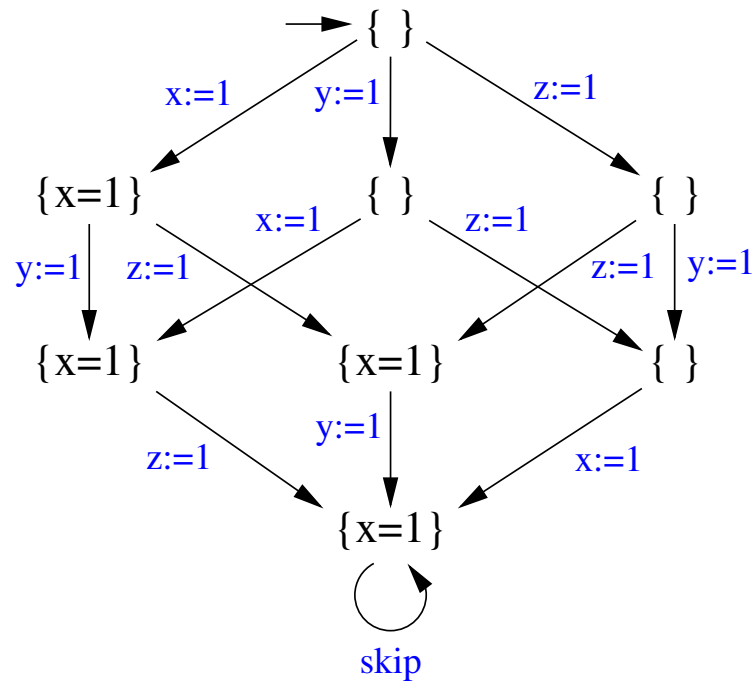
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# Independence: Example

---

In the example all pairs of distinct actions are independent.



Remark: In general, an independence relation may not be transitive!



# (In-)Visibility

---

$U \subseteq A$  is called an **invisibility set**, if all  $a \in U$  have the following property:

for all  $(s, a, s') \in \rightarrow$  we have:  $\nu(s) = \nu(s')$ .

I.e., no action in  $U$  ever changes the validity of an atomic proposition.

In the example:  $\{y := 1, z := 1\}$  (or every subset of it) is an invisibility set.

Motivation: Interleavings of visible actions may not be eliminated because they might influence the validity of LTL properties.

# Remarks

---

Sources for  $I$  and  $U$ : “external” knowledge about the model and the actions possible in it

e.g. instruction only touches local variables, . . .

will *not* be obtained from first constructing all of  $\mathcal{K}$ !

Every (symmetric) subset of an independence relation remains an independence relation, every subset of an invisibility set remains an invisibility set.

→ conservative approximation possible

But: The bigger  $I$  and  $U$  are, the more information we have at hand to improve the reduction.

---

In the following, we assume some fixed independence relation  $I$  and invisibility set  $U$ .

We call  $a$  and  $b$  **independent** if  $(a, b) \in I$ , and **dependent** otherwise.

We call  $a$  **invisible** if  $a \in U$ , and **visible** otherwise.

In the example we take:  $I = \{x := 1, y := 1, z := 1\}^2 \setminus Id$   
 $U = \{y := 1, z := 1\}$

# Preview

---

We first define a notion of “**equivalent**” runs.

We then consider some **conditions** guaranteeing that every equivalence class is preserved in the reduced system.

Finally, we consider some practical **implementation** issues are solved in Spin.

# Stuttering equivalence

---

Definition: Let  $\sigma, \rho$  be infinite sequences over  $2^{AP}$ . We call  $\sigma$  and  $\rho$  **stuttering equivalent** iff there are integer sequences

$$0 = i_0 < i_1 < i_2 < \dots \text{ and } 0 = k_0 < k_1 < k_2 < \dots,$$

such that for all  $\ell \geq 0$ :

$$\begin{aligned} \sigma(i_\ell) &= \sigma(i_\ell + 1) = \dots = \sigma(i_{\ell+1} - 1) = \\ \rho(k_\ell) &= \rho(k_\ell + 1) = \dots = \rho(k_{\ell+1} - 1) \end{aligned}$$

(I.e.,  $\sigma$  and  $\rho$  can be partitioned into “blocks” of possibly differing sizes, but with the same valuations.)

In our example: all infinite sequences of the Kripke structure are stuttering equivalent.

---

We extend this notion to Kripke structures:

Let  $\mathcal{K}, \mathcal{K}'$  be two Kripke structures with the same set of atomic propositions  $AP$ .

$\mathcal{K}$  and  $\mathcal{K}'$  are called **stuttering equivalent** iff for every sequence in  $[[\mathcal{K}]]$  there exists a stuttering equivalent sequence in  $[[\mathcal{K}']]$ , and vice versa.

I.e.,  $[[\mathcal{K}]]$  and  $[[\mathcal{K}']]$  contain the same equivalence classes of runs.

In our example: The Kripke structure containing only “the rightmost path” is stuttering equivalent to the original one.

# Invariance under stuttering

---

Let  $\phi$  be an LTL formula. We call  $\phi$  **invariant under stuttering** iff for all stuttering-equivalent pairs of sequences  $\sigma$  and  $\rho$ :

$$\sigma \in \llbracket \phi \rrbracket \quad \text{iff} \quad \rho \in \llbracket \phi \rrbracket.$$

Put differently:  $\phi$  cannot distinguish stuttering-equivalent sequences.  
(And neither stuttering-equivalent Kripke structures.)

**Theorem:** Any LTL formula that does not contain an **X** operator is invariant under stuttering.

**Proof:** Exercise.

# Strategy

---

We replace  $\mathcal{K}$  by a stuttering-equivalent, smaller structure  $\mathcal{K}'$ .

Then we check whether  $\mathcal{K}' \models \phi$ , which is equivalent to  $\mathcal{K} \models \phi$  (if  $\phi$  does not contain any  $\mathbf{X}$ ).

We generate  $\mathcal{K}'$  by performing a DFS on  $\mathcal{K}$ , and in each step eliminating certain successor states, based on the knowledge about properties of actions that is imparted by  $I$  and  $U$ .

The method presented here is called the **ample set** method.



# Ample sets

---

For every state  $s$  we compute a set  $red(s) \subseteq en(s)$ ;  $red(s)$  contains the actions whose corresponding successor states will be explored.

(partially conflicting) goals:

$red(s)$  must be chosen in such a way that stuttering equivalence is guaranteed.

The choice of  $red(s)$  should reduce  $\mathcal{K}$  strongly.

The computation of  $red(s)$  should be efficient.

# Conditions for Ample Sets

---

**C0:**  $red(s) = \emptyset$  iff  $en(s) = \emptyset$

**C1:** Every path of  $\mathcal{K}$  starting at a state  $s$  satisfies the following: no action that depends on some action in  $red(s)$  occurs before an action from  $red(s)$ .

**C2:** If  $red(s) \neq en(s)$  then all actions in  $red(s)$  are invisible.

**C3:** For all cycles in  $\mathcal{K}'$  the following holds: if  $a \in en(s)$  for some state  $s$  in the cycle, then  $a \in red(s')$  for some (possibly other) state  $s'$  in the cycle.

# Idea

---

**C0** ensures that no additional deadlocks are introduced.

**C1** and **C2** ensure that every stuttering-equivalence class of runs is preserved.

**C3** ensures that enabled actions cannot be omitted forever.

# Example

---

Pseudocode program with two concurrent processes:

```
int x,y init 0;  
cobegin { P || Q } coend
```

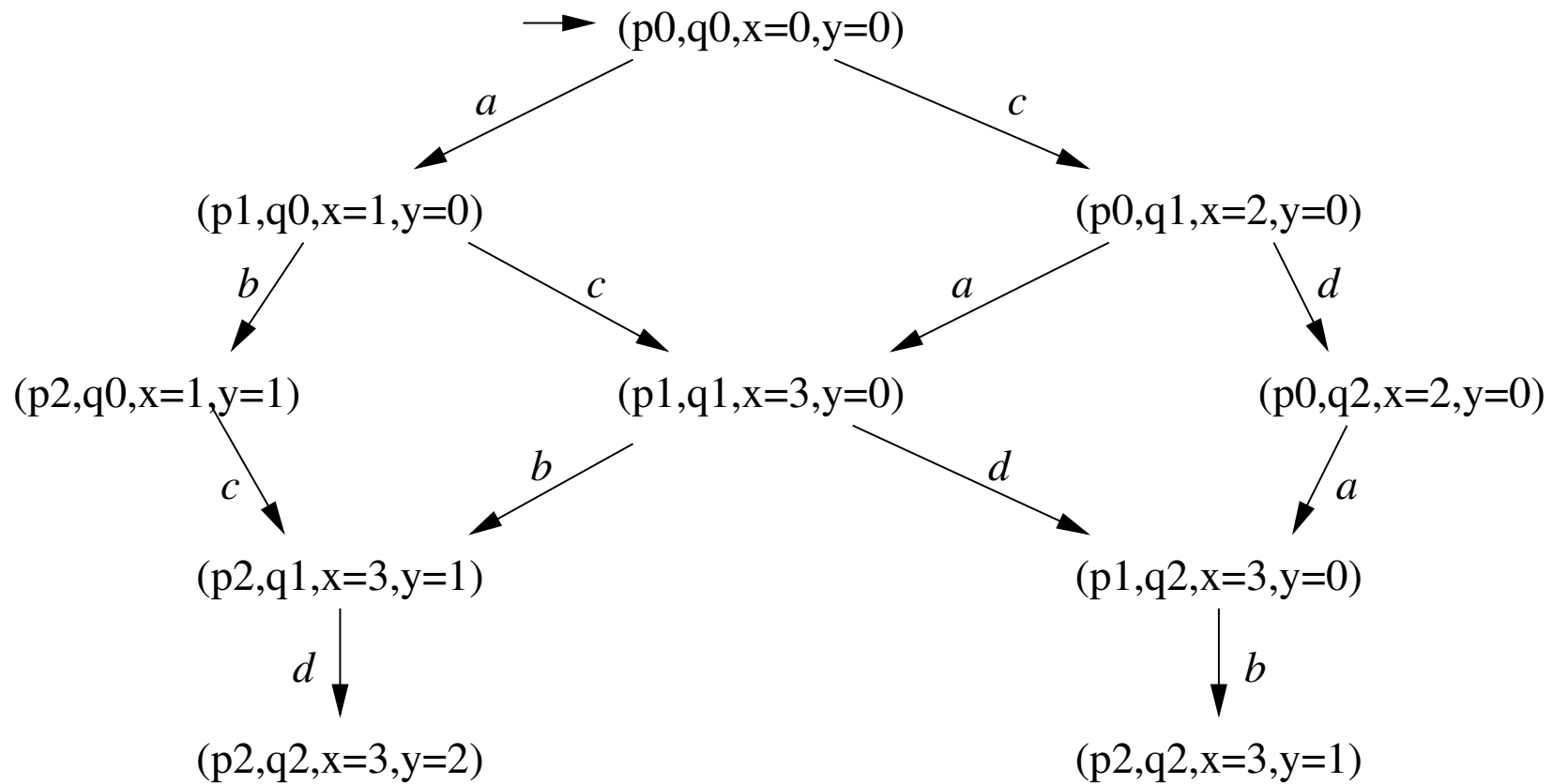
```
P =  
  p0:  $x := x + 1$ ; (action a)  
  p1:  $y := y + 1$ ; (action b)  
  p2: end
```

```
Q =  
  q0:  $x := x + 2$ ; (action c)  
  q1:  $y := y * 2$ ; (action d)  
  q2: end
```

Independent actions: all pairs but (*b,d*) and (*d,b*).

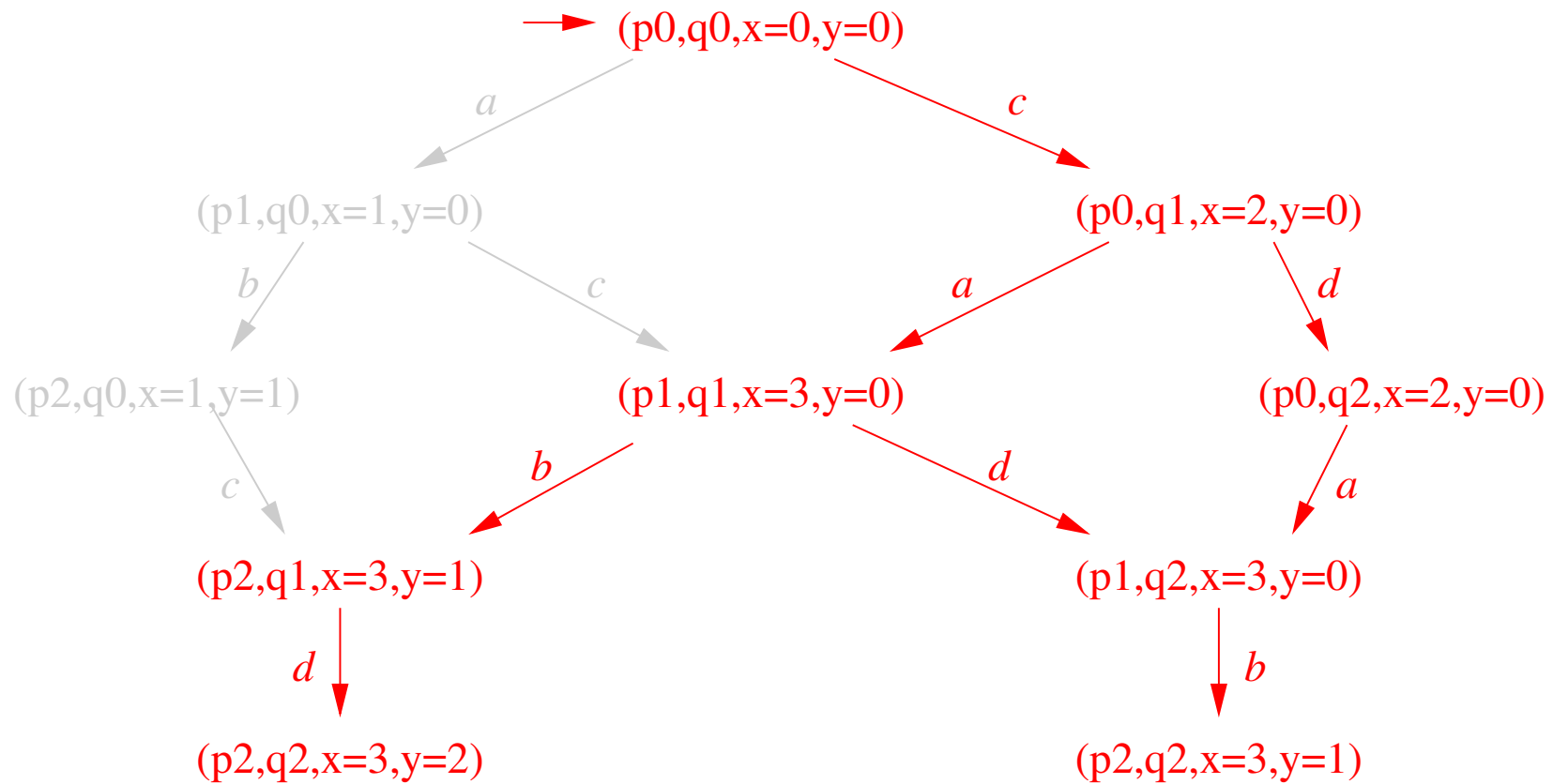
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Transition system of the example:



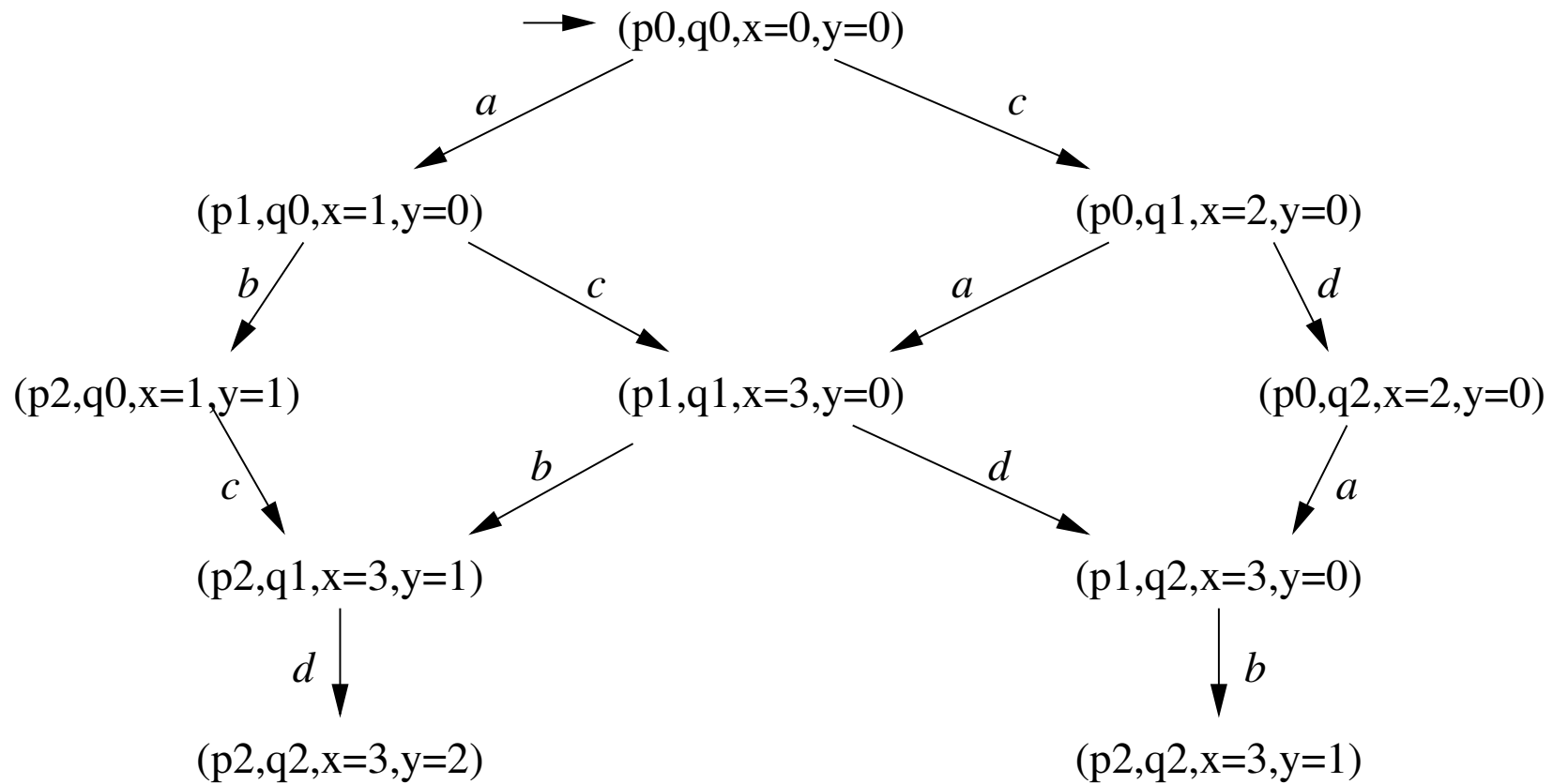
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Possible reduced structure if  $b, d$  are visible:



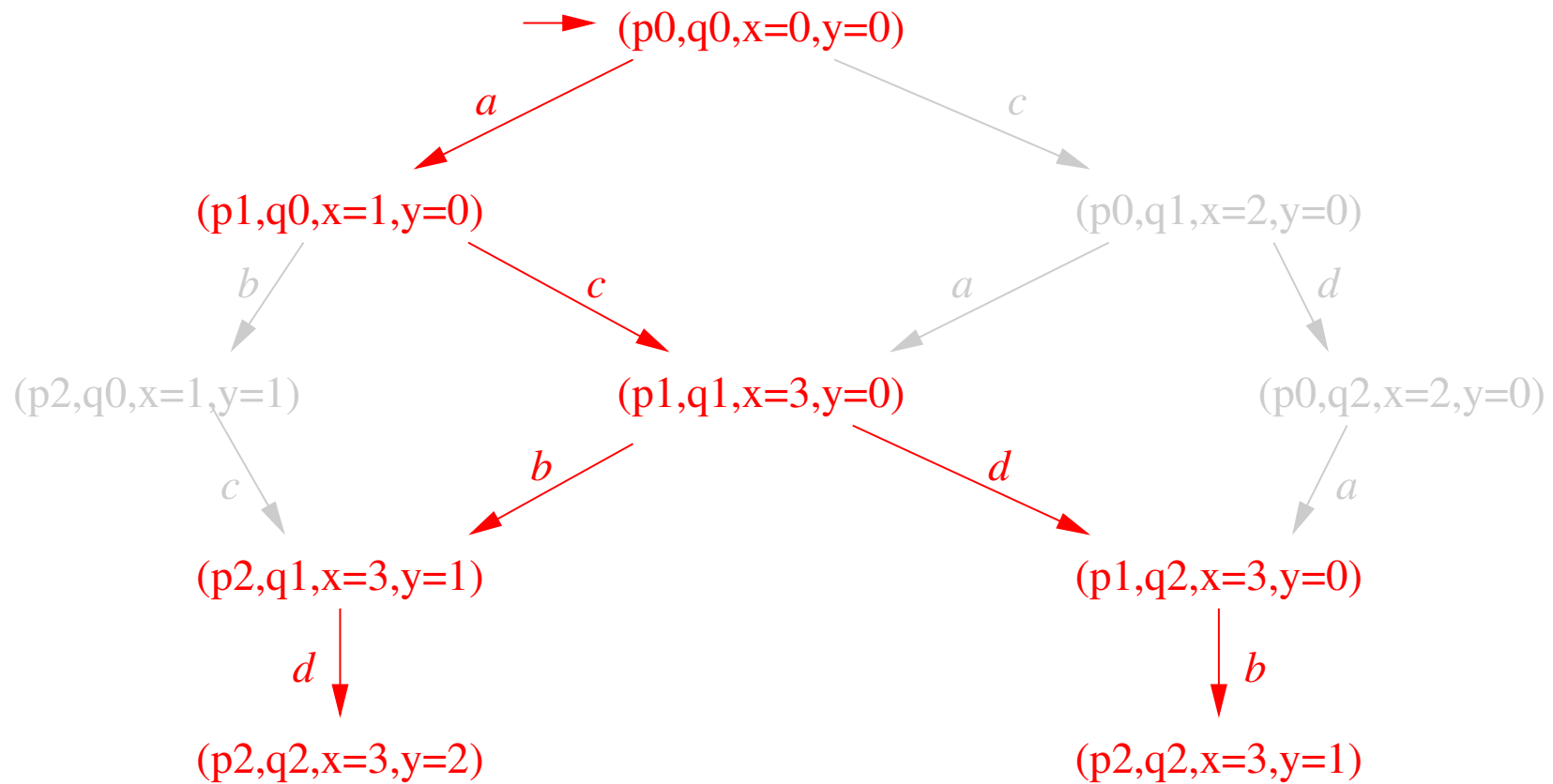
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No reduction is possible if  $a, c$  are visible:



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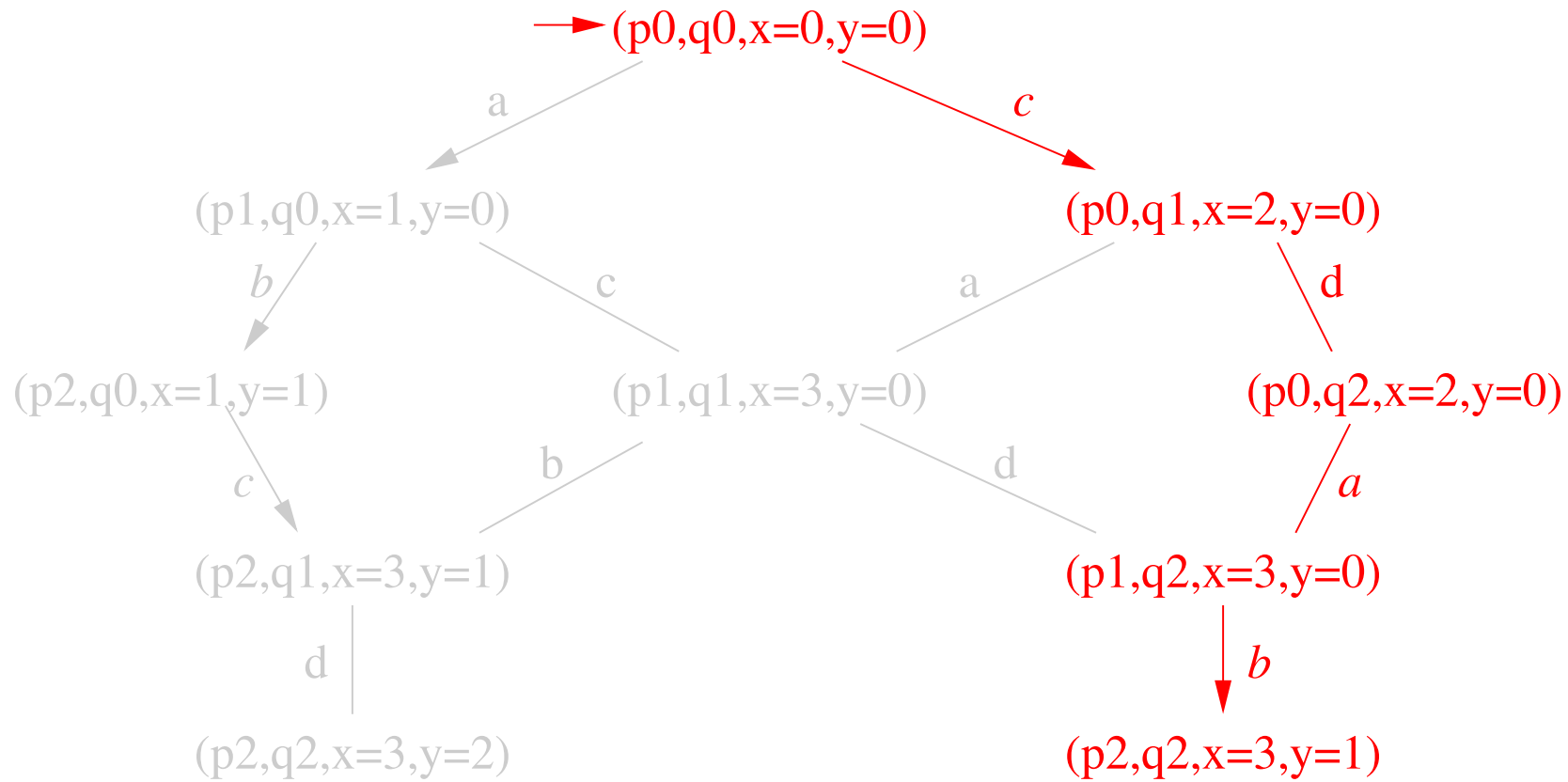
Possible reduced structure if only  $b$  (or even no action) is visible:





---

Can we do even better if only  $b$  (or even no action) is visible?



# (Non-)Optimality

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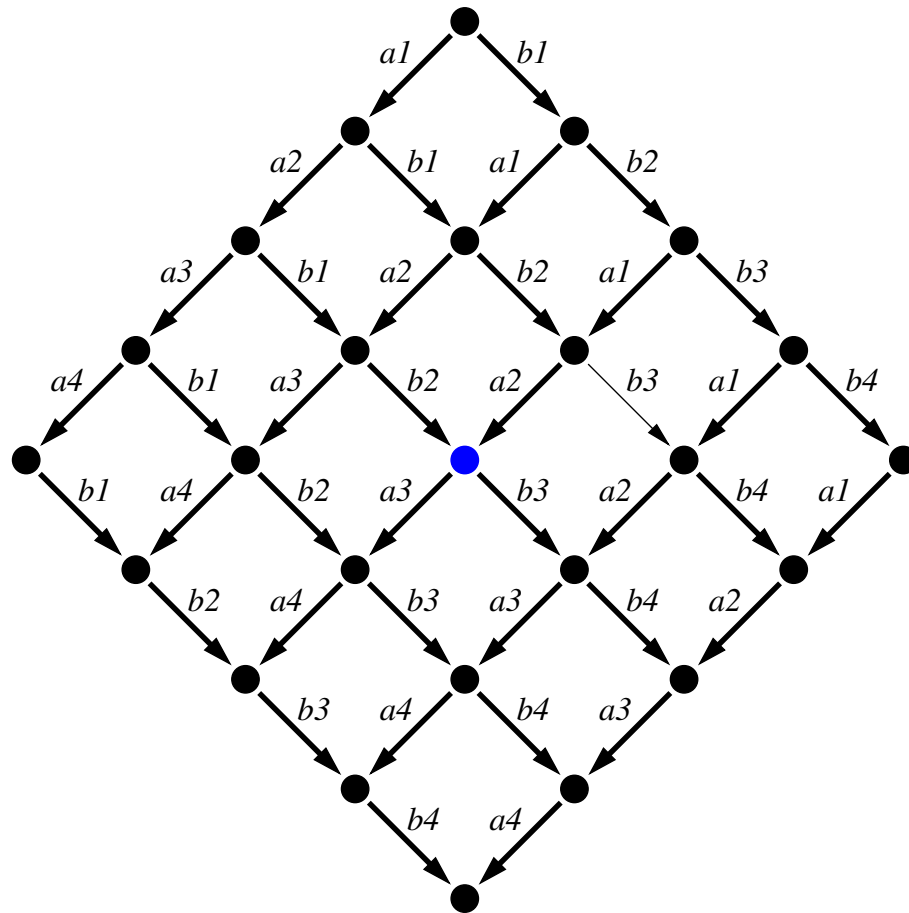
An ideal reduction would retain only one execution from each stuttering equivalence class.

C0–C3 do *not* ensure such an ideal reduction, i.e. the resulting reduced structure is not minimal in general.

Example (see next slide): two parallel processes with four actions each ( $a_1, \dots, a_4$  or  $b_1, \dots, b_4$ , resp.), all independent.

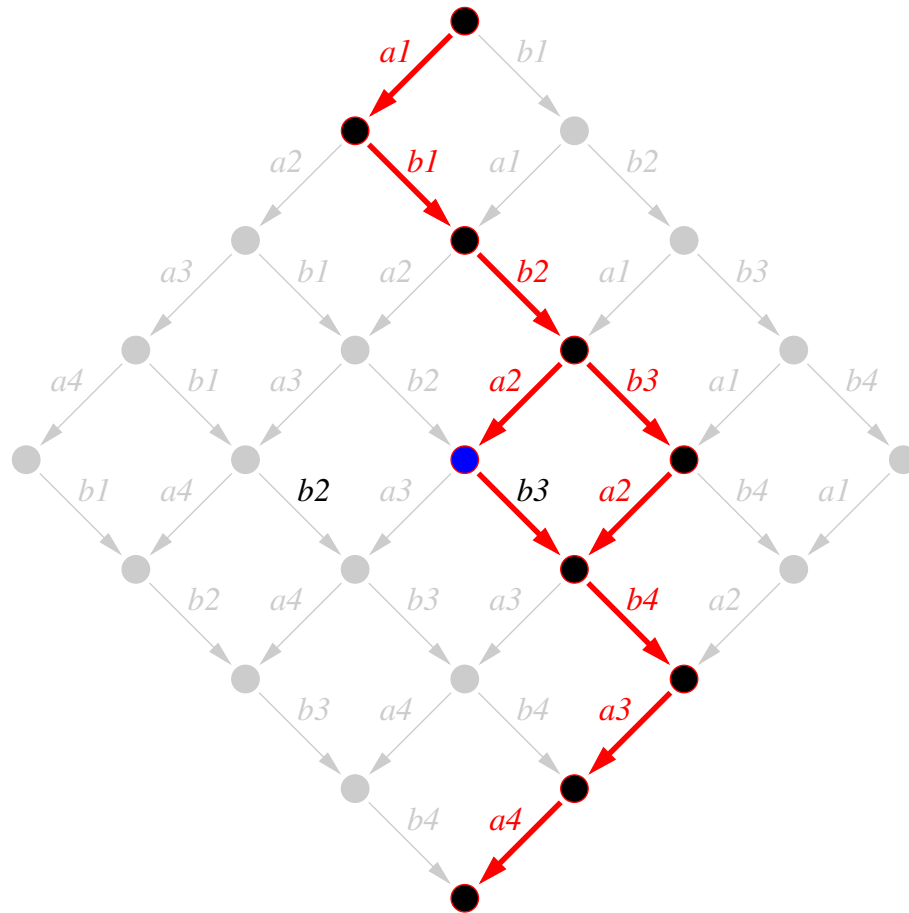
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The valuation of the blue state differs from the others:



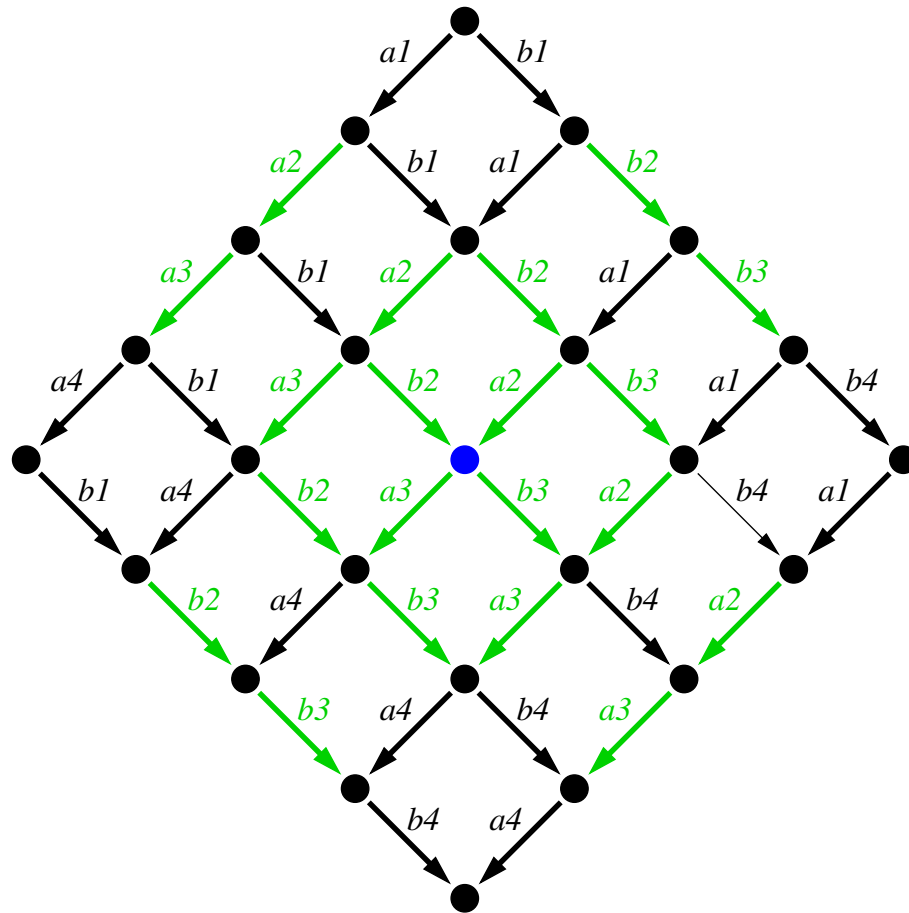
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Minimal stuttering-equivalent structure:



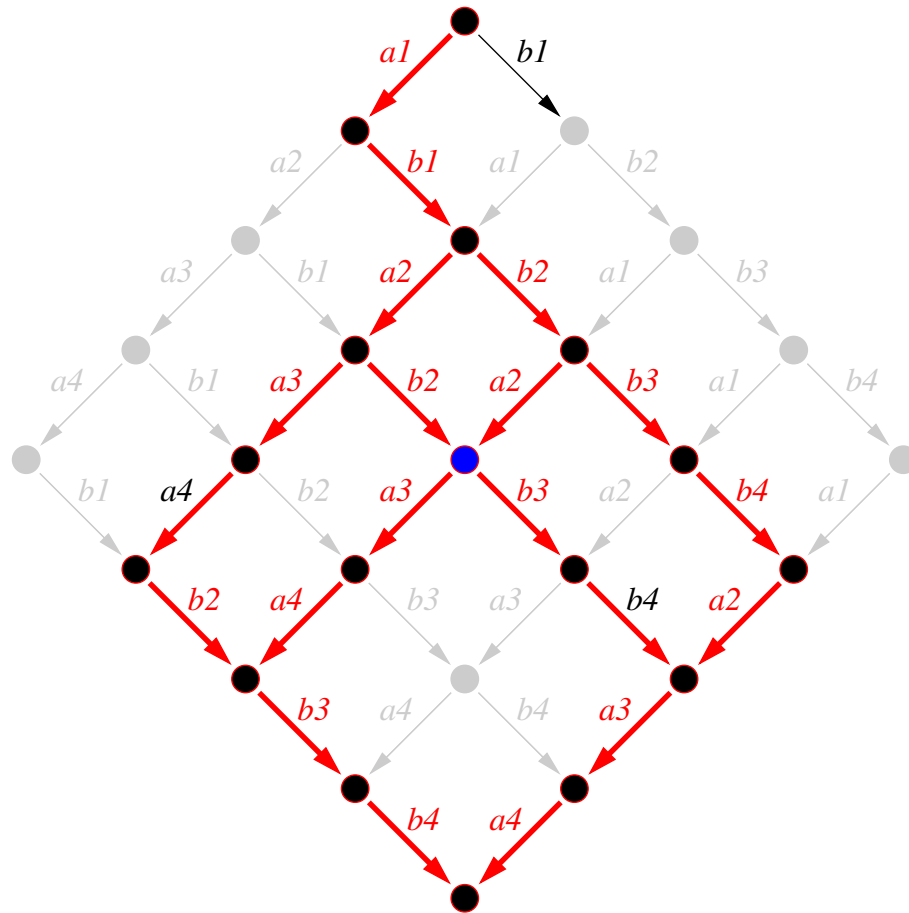
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Visible actions:  $a_2, a_3, b_2, b_3$  (in green):



---

Smallest structure satisfying C0–C3:



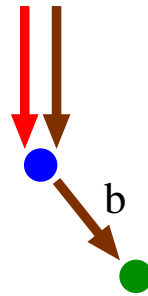
# Correctness

---

**Claim:** If *red* satisfies conditions C0 through C3, then  $\mathcal{K}'$  is stuttering-equivalent to  $\mathcal{K}$ .

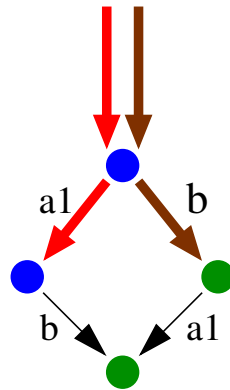
**Proof (idea):** Let  $\sigma$  be an infinite path in  $\mathcal{K}$ . We show that in  $\mathcal{K}'$  there exists an infinite path  $\tau$  such that  $\nu(\sigma)$  and  $\nu(\tau)$  are stuttering-equivalent.

In the following,  $\sigma$  is shown in **brown** and  $\tau$  in **red**. States known to fulfil the same atomic propositions are drawn in the same colours.

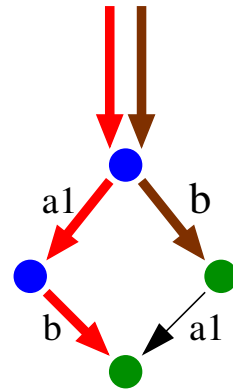


Suppose that the transition labelled with  $b$  is the first in  $\sigma$  that is not present in  $\mathcal{K}'$ .

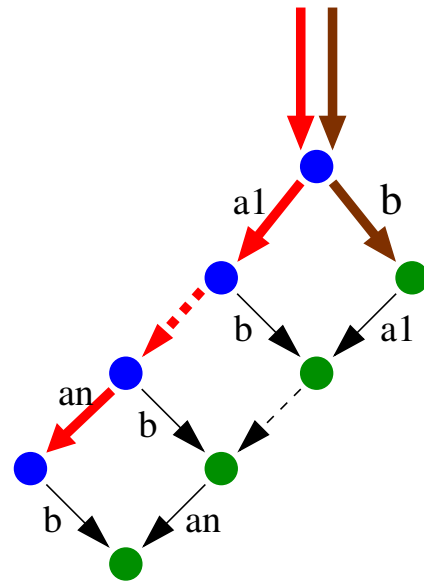




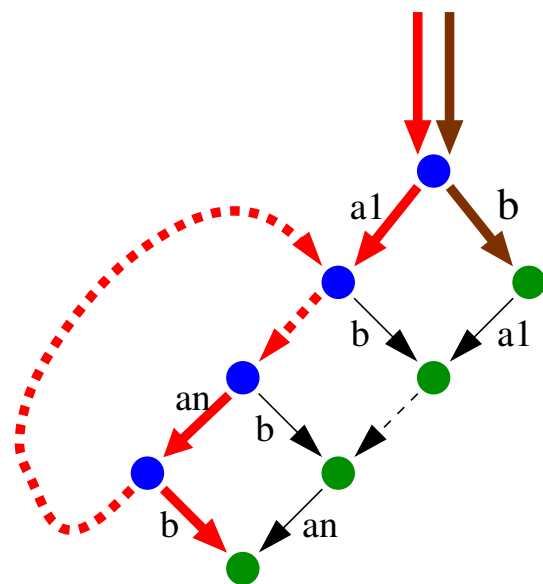
Because of C0 the blue state must have another enabled action, let us call it  $a_1$ .  
 $a_1$  is independent of  $b$  (C1) and invisible (C2).



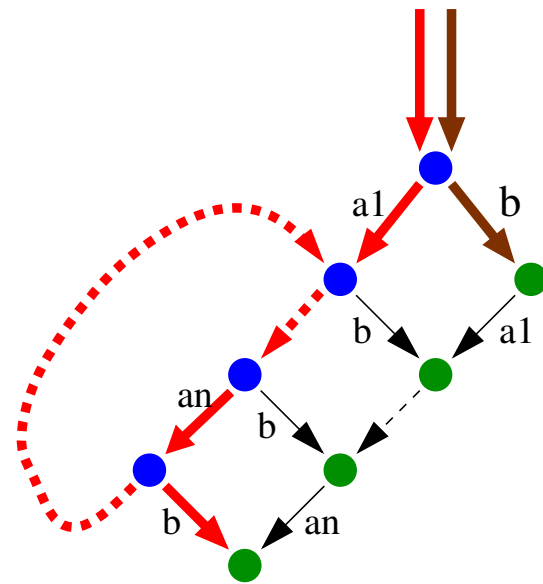
Either the second  $b$ -transition is in  $\mathcal{K}'$ , then we take  $\tau$  to be the sequence of red edges...



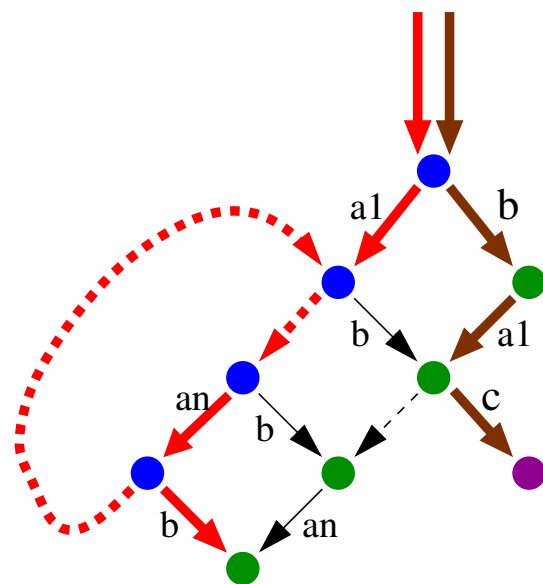
... or  $b$  will be “deferred” in favour of  $a_2, \dots, a_n$ , all of which are also invisible and independent of  $b$ .



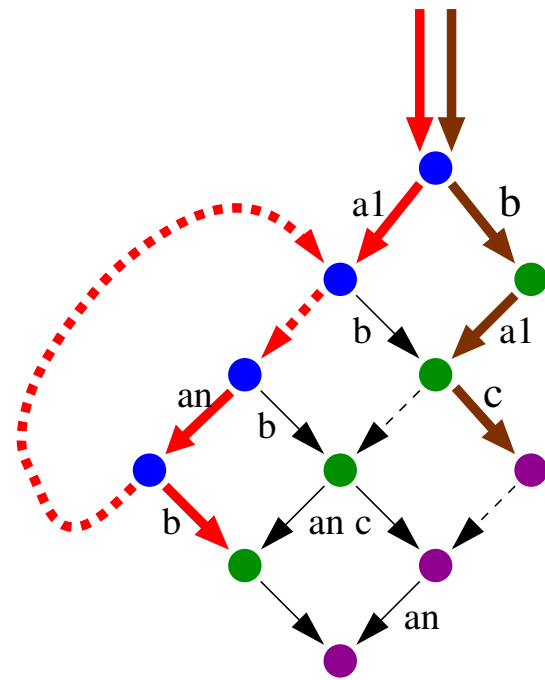
Since  $\mathcal{K}$  is finite, this process must either end or create a cycle (in  $\mathcal{K}'$ ). Because of C3,  $b$  must be activated in some state along the cycle.



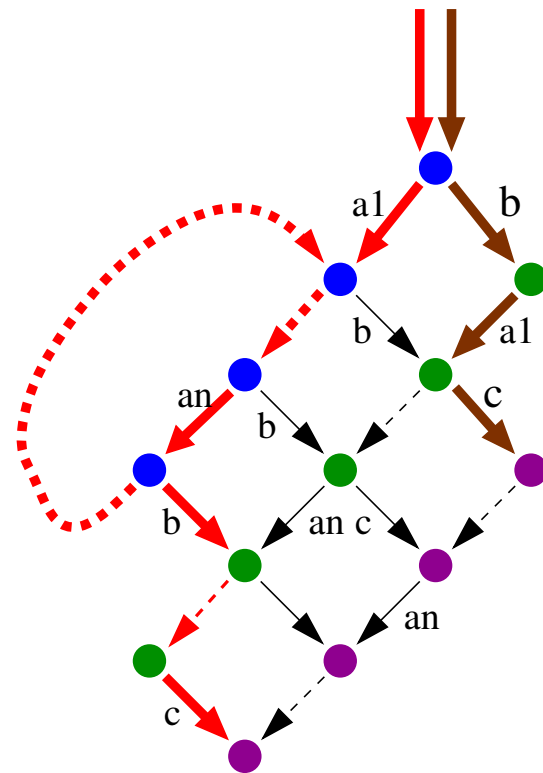
Both  $\sigma$  and  $\tau$  contain blue states followed by green ones.



$\sigma$  either continues with  $a_1, \dots, a_n$  until the paths “converge”, or it “diverges” again with an action  $c$ .

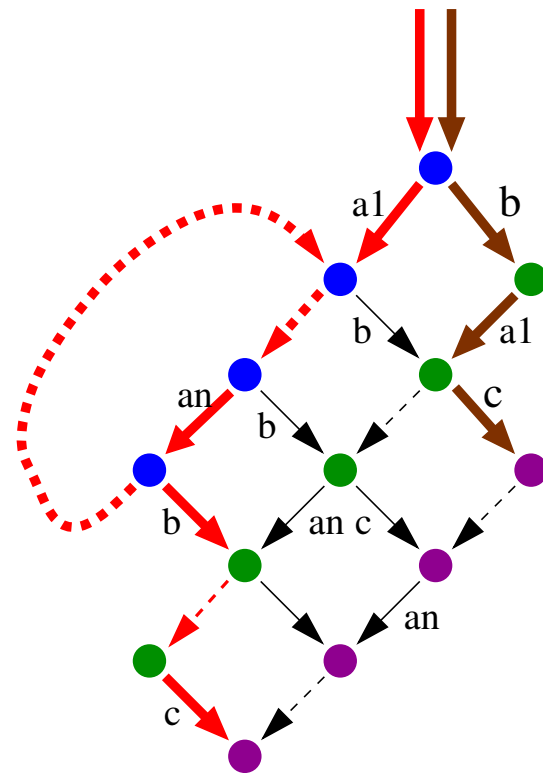


Then, however,  $c$  must be independent from  $a_2, \dots, a_n$ .



Repeating the previous arguments we can conclude that  $\mathcal{K}'$  also has a  $c$ -labelled transition along the red path.





Both the red and the brown path again contains blue, green, and purple states, in that order. The previous arguments can be repeated ad infinitum.

# Implementation issues

---

C0 and C2: obvious

C1 and C3 depend on the complete state graph

We shall find conditions that are stronger than C1 and C3. These will exclude certain reductions but can be efficiently implemented during DFS.

Replace C3 by C3':

If  $red(s) \neq en(s)$  for some state  $s$ , then no action of  $red(s)$  may lead to a state that is currently on the search stack of the DFS.

# Heuristic for C1

---

Depends on the underlying description of the system; here, we will discuss what Spin is doing for Promela models.

In general, a Promela model will contain multiple concurrent processes  $P_1, \dots, P_n$  communicating via global variables and message channels.

# Heuristic for C1 in Spin: Dependent actions

---

$dep(a) := \{ b \mid (a, b) \notin I \}$  contains the actions dependent on  $a$ .

In Spin, if  $a$  is an action of  $P_i$ , then  $dep(a)$  will be overapproximated by:

all other actions in  $P_i$ ;

actions in other processes that write to a variable from which  $a$  reads, or vice versa;

if  $a$  reads from a message channel, then all actions in other processes that read from the same channel.

if  $a$  writes to a message channel, then all actions in other processes that write to the same channel.

# Heuristic for C1 in Spin

---

Let  $E_i(s)$  denotes the actions of process  $P_i$  activated in  $s$ .

Spin tests the sets  $E_i(s)$ , for  $i = 1, \dots, n$ , as candidates for  $red(s)$ . If all of them fail, it “gives up” and takes  $red(s) = en(s)$ .

Recall:  $E_i(s)$  satisfies  $C_1$  if no action  $a$  that depends on  $E_i(s)$  may be executed before an action from  $E_i(s)$  itself.

A **sufficient** condition for  $E_i(s)$  to satisfy  $C_1$  is the conjunction of

- (i) No action of other processes depends on  $E_i(s)$   
(so  $a$  cannot be an action from other process)
- (ii) No action of  $P_i$  outside  $E_i(s)$  can become activated by an action of another process.  
(so  $a$  cannot be an action of  $P_i$  either)

# The approach in Spin

---

Let  $A_j$  denote the possible actions in process  $P_j$ .

Let  $Pc_j(s)$  denote the actions possible in  $P_j$  at label  $pc_j(s)$ .

Observe that some actions of  $Pc_j(s)$  may not be activated in  $s$  itself!

- Their guards may not be enabled at  $s$ , but may become enabled due to an action from **another** process.

# The approach in Spin

---

Let  $Pre(a)$  be the set of actions that might **activate**  $a$ , i.e. (some overapproximation of) the set

$$\{ b \mid \exists s: a \notin en(s), b \in en(s), a \in en(b(s)) \}$$

In Spin: if  $a$  is an action of  $P_i$ ; then we can choose  $Pre(a)$  as the set containing

all actions of  $P_i$  leading to a control-flow label in which  $a$  can be executed;

if the guard of  $a$  uses global variables, all actions in other processes that modify these variables;

if  $a$  reads from a message channel or writes into it, then all actions in other processes that do the opposite (write/read).

# Test for C1

---

```
function check_C1(s, Pi)
  for all Pj ≠ Pi do
    if  $\text{dep}(E_i(s)) \cap A_j \neq \emptyset$ 
      \* some action of Pj      *\
      \* is dependent of Ei(s) *\
    or  $\text{Pre}(Pc_i(s) \setminus E_i(s)) \cap A_j \neq \emptyset$ 
      \* some action of Pi outside Ei(s) can *\
      \* become activated by an action of Pj  *\
    then return False;
  return True;
end function
```



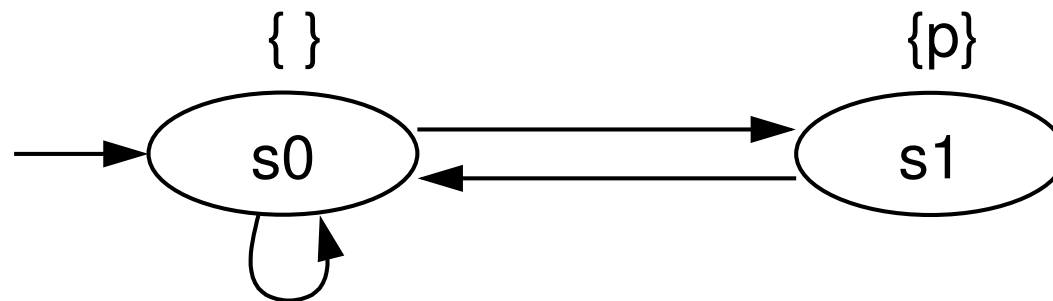
# Part 8: Branching-time logics

# Motivation

---

Linear-time logic describes properties of runs. However, it cannot describe the *options* that are possible in a system.

Example: The behaviour of the structure below cannot be adequately described using LTL. At any time, the system *may* change to a state satisfying  $p$ , however, it may never do so.



---

**Branching-time logics** allow to speak about the branching behaviour, i.e. about multiple possible futures. Hence, branching-time logics are evaluated over *trees* of valuations.

We shall first extend LTL into a branching-time logic called CTL\*. CTL\* is quite powerful, however, the associated model-checking problem is correspondingly hard.

We then introduce a restriction of CTL\*, called CTL. CTL is probably the most popular branching-time logic in verification because

- it is still expressive enough for most applications;

- it has useful algorithmic properties;

- correspondingly, it is often used in automated verification.

# Valuation trees

---

Let  $\mathcal{T} = (V, \rightarrow, r, AP, \nu)$  be a Kripke structure (where  $V$  may be an infinite set). We call  $\mathcal{T}$  a **valuation tree** iff

$(V, \rightarrow)$  is a directed tree with root  $r$  (i.e. for every node  $v \in V$  there is exactly one path from  $r$  to  $v$ ).

$\lceil v \rceil$  denotes the subtree whose root is  $v \in V$ .

# Computation tree

---

Let  $\mathcal{K} = (S, \rightarrow, s_0, AP, \nu)$  be a Kripke structure and  $s \in S$ .

By  $\mathcal{T}_{\mathcal{K}}(s)$  we denote the (unique) valuation tree with root  $r$  and the following properties:

$$\nu(r) = \nu(s)$$

$s \rightarrow s'$  holds in  $\mathcal{K}$  iff in  $\mathcal{T}_{\mathcal{K}}(s)$  there is a transition  $r \rightarrow r'$  such that  $\lceil r' \rceil$  is isomorphic to  $\mathcal{T}_{\mathcal{K}}(s')$ .

We call  $\mathcal{T}_{\mathcal{K}}(s)$  the **computation tree** of  $s$ .

(Note:  $\mathcal{T}_{\mathcal{K}}(s_0)$  is also sometimes called the **(acyclic) unfolding** of  $\mathcal{K}$ .)

# CTL\*: Syntax

---

Let  $AP$  be a set of atomic propositions. The set of CTL\* formulae over  $AP$  is inductively defined as follows:

if  $p \in AP$ , then  $p$  is a formula;

if  $\phi_1, \phi_2$  are formulae, then  $\neg\phi_1$  and  $\phi_1 \vee \phi_2$  are formulae.

Moreover, let  $\phi$  be an ‘extended’ LTL formula where atomic propositions are replaced by CTL\* formulae. Then  $\mathbf{E}\phi$  is a CTL\* formula.

Note: We use  $\mathbf{A}\phi$  as an abbreviation for  $\neg\mathbf{E}\neg\phi$ .

# CTL\*: Semantics

---

Let  $\mathcal{T}$  be a valuation tree with root  $r$  and  $\phi$  a CTL\* formula. We write  $\mathcal{T} \models \phi$  for “ $\mathcal{T}$  satisfies  $\phi$ .”

$\mathcal{T} \models p$                     iff  $p \in AP$  and  $p \in \nu(r)$

$\mathcal{T} \models \neg\phi$                     iff  $\mathcal{T} \not\models \phi$

$\mathcal{T} \models \phi_1 \vee \phi_2$             iff  $\mathcal{T} \models \phi_1$  or  $\mathcal{T} \models \phi_2$

$\mathcal{T} \models \mathbf{E}\phi$                     iff  $\mathcal{T}$  contains some infinite path  $\sigma$  starting at  $r$   
such that  $\nu(\sigma) \models \phi$ .

(Note: When  $\phi$  contains  $\mathbf{E}\phi'$  as an “atomic proposition”  
then we deem  $\nu(\sigma)^i \models \mathbf{E}\phi'$  iff  $[\sigma(i)] \models \phi'$ .)

We say  $\mathcal{K} \models \phi$  iff  $\mathcal{T}_{\mathcal{K}}(s_0) \models \phi$ , where  $\mathcal{K}$  is a Kripke structure with initial state  $s_0$ .

# A model-checking algorithm for CTL\*

---

Let  $\mathcal{K}$  be a Kripke structure. By  $\mathcal{K}[s]$ , where  $s$  is a state of  $\mathcal{K}$  we denote the same structure as  $\mathcal{K}$ , but with  $s$  as initial state.

Given an LTL formula  $\phi$  we require an algorithm that solves the **global model-checking problem** for LTL: Find the set  $\llbracket \phi \rrbracket_{\mathcal{K}}$  of all states in  $\mathcal{K}$  such that  $s \in \llbracket \phi \rrbracket_{\mathcal{K}}$  iff  $\mathcal{K}[s] \models \phi$ .

Now, let  $\phi$  be a CTL\* formula. We follow these steps:

Either  $\phi$  does not contain any **E**-subformula. Then the procedure is obvious.

Otherwise, let  $\phi' = \mathbf{E}\psi$  be a subformula of  $\phi$  such that  $\psi$  does not contain another **E**-subformula. We compute  $M := S \setminus \llbracket \neg\psi \rrbracket_{\mathcal{K}}$ . We then replace  $\phi'$  by a “fresh” atomic proposition  $p$  and modify  $\nu$  such that for all states  $s$ , we have  $p \in \nu(s)$  iff  $s \in M$ . We then repeat the procedure until all **E**s are eliminated.



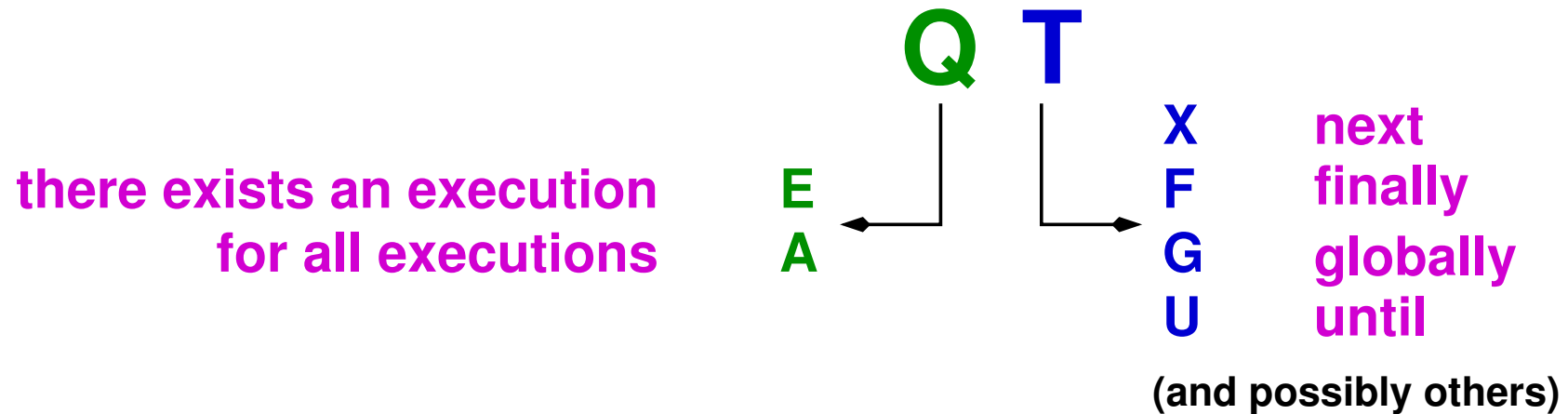
# Part 9: CTL

# CTL: Overview

---

We now define CTL (**Computation-Tree Logic**) as a syntactic restriction of CTL\*.

Operators are restricted to the following form:



# CTL: Syntax

---

We define a minimal syntax first. Later we define additional operators with the help of the minimal syntax.

Let  $AP$  be a set of atomic propositions: The set of **CTL formulas** over  $AP$  is as follows:

if  $a \in AP$ , then  $a$  is a CTL formula;

if  $\phi_1, \phi_2$  are CTL formulas, then so are

$\neg\phi_1$ ,  $\phi_1 \vee \phi_2$ , **EX**  $\phi_1$ , **EG**  $\phi_1$ ,  $\phi_1$  **EU**  $\phi_2$

---

It is easy to see that every CTL formula is also a CTL\* formula.

Previously, we defined the satisfaction relationship between valuation trees and CTL\* formulae. Since each state of a Kripke structure has a clearly defined computation tree, we may just as well say that a *state* satisfies a CTL/CTL\* formula, meaning that its computation tree does.

Let  $\mathcal{K}$  be a Kripke structure, let  $s$  one of its states, and let  $\phi$  be a CTL formula. On the following slide, we define a set  $[[\phi]]_{\mathcal{K}}$  in such a way that  $s \in [[\phi]]_{\mathcal{K}}$  iff  $\mathcal{T}_{\mathcal{K}}(s) \models \phi$ .

# CTL: Semantics

---

Let  $\mathcal{K} = (\mathcal{S}, \rightarrow, r, AP, \nu)$  be a Kripke structure.

We define the semantic of every CTL formula  $\phi$  over  $AP$  w.r.t.  $\mathcal{K}$  as a set of states  $\llbracket \phi \rrbracket_{\mathcal{K}}$ , as follows:

$$\llbracket a \rrbracket_{\mathcal{K}} = \{ s \mid a \in \nu(s) \} \quad \text{for } a \in AP$$

$$\llbracket \neg \phi_1 \rrbracket_{\mathcal{K}} = \mathcal{S} \setminus \llbracket \phi_1 \rrbracket_{\mathcal{K}}$$

$$\llbracket \phi_1 \vee \phi_2 \rrbracket_{\mathcal{K}} = \llbracket \phi_1 \rrbracket_{\mathcal{K}} \cup \llbracket \phi_2 \rrbracket_{\mathcal{K}}$$

$$\llbracket \mathbf{EX} \phi_1 \rrbracket_{\mathcal{K}} = \{ s \mid \text{there is a } t \text{ s.t. } s \rightarrow t \text{ and } t \in \llbracket \phi_1 \rrbracket_{\mathcal{K}} \}$$

$$\llbracket \mathbf{EG} \phi_1 \rrbracket_{\mathcal{K}} = \{ s \mid \text{there is a run } \rho \text{ with } \rho(0) = s$$

$$\text{and } \rho(i) \in \llbracket \phi_1 \rrbracket_{\mathcal{K}} \text{ for all } i \geq 0 \}$$

$$\llbracket \phi_1 \mathbf{EU} \phi_2 \rrbracket_{\mathcal{K}} = \{ s \mid \text{there is a run } \rho \text{ with } \rho(0) = s \text{ and } k \geq 0 \text{ s.t.} \\ \rho(i) \in \llbracket \phi_1 \rrbracket_{\mathcal{K}} \text{ for all } i < k \text{ and } \rho(k) \in \llbracket \phi_2 \rrbracket_{\mathcal{K}} \}$$

---

We say that  $\mathcal{K}$  satisfies  $\phi$  (denoted  $\mathcal{K} \models \phi$ ) iff  $r \in \llbracket \phi \rrbracket_{\mathcal{K}}$ .

The **local model-checking problem** is to check whether  $\mathcal{K} \models \phi$ .

The **global model-checking problem** is to compute  $\llbracket \phi \rrbracket_{\mathcal{K}}$ .

We declare two formulas equivalent (written  $\phi_1 \equiv \phi_2$ ) iff for every Kripke structure  $\mathcal{K}$  we have  $\llbracket \phi_1 \rrbracket_{\mathcal{K}} = \llbracket \phi_2 \rrbracket_{\mathcal{K}}$ .

In the following, we omit the index  $\mathcal{K}$  from  $\llbracket \cdot \rrbracket_{\mathcal{K}}$  if  $\mathcal{K}$  is understood.

# CTL: Extended syntax

---

$$\phi_1 \wedge \phi_2 \equiv \neg(\neg\phi_1 \vee \neg\phi_2)$$

$$\text{true} \equiv a \vee \neg a$$

$$\text{false} \equiv \neg\text{true}$$

$$\phi_1 \text{ EW } \phi_2 \equiv \text{EG } \phi_1 \vee (\phi_1 \text{ EU } \phi_2)$$

$$\text{EF } \phi \equiv \text{true EU } \phi$$

$$\text{AX } \phi \equiv \neg \text{EX } \neg\phi$$

$$\text{AG } \phi \equiv \neg \text{EF } \neg\phi$$

$$\text{AF } \phi \equiv \neg \text{EG } \neg\phi$$

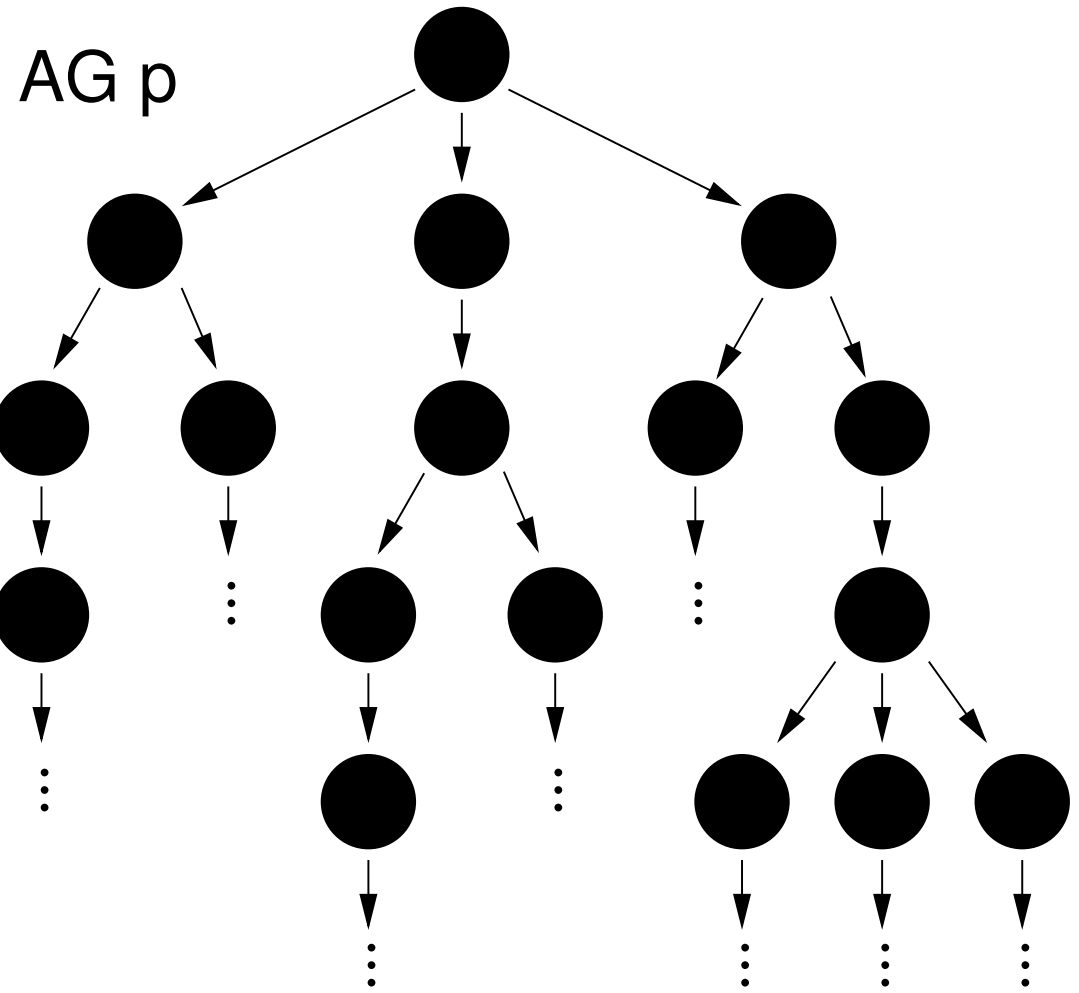
$$\phi_1 \text{ AW } \phi_2 \equiv \neg(\neg\phi_2 \text{ EU } \neg(\phi_1 \vee \phi_2))$$

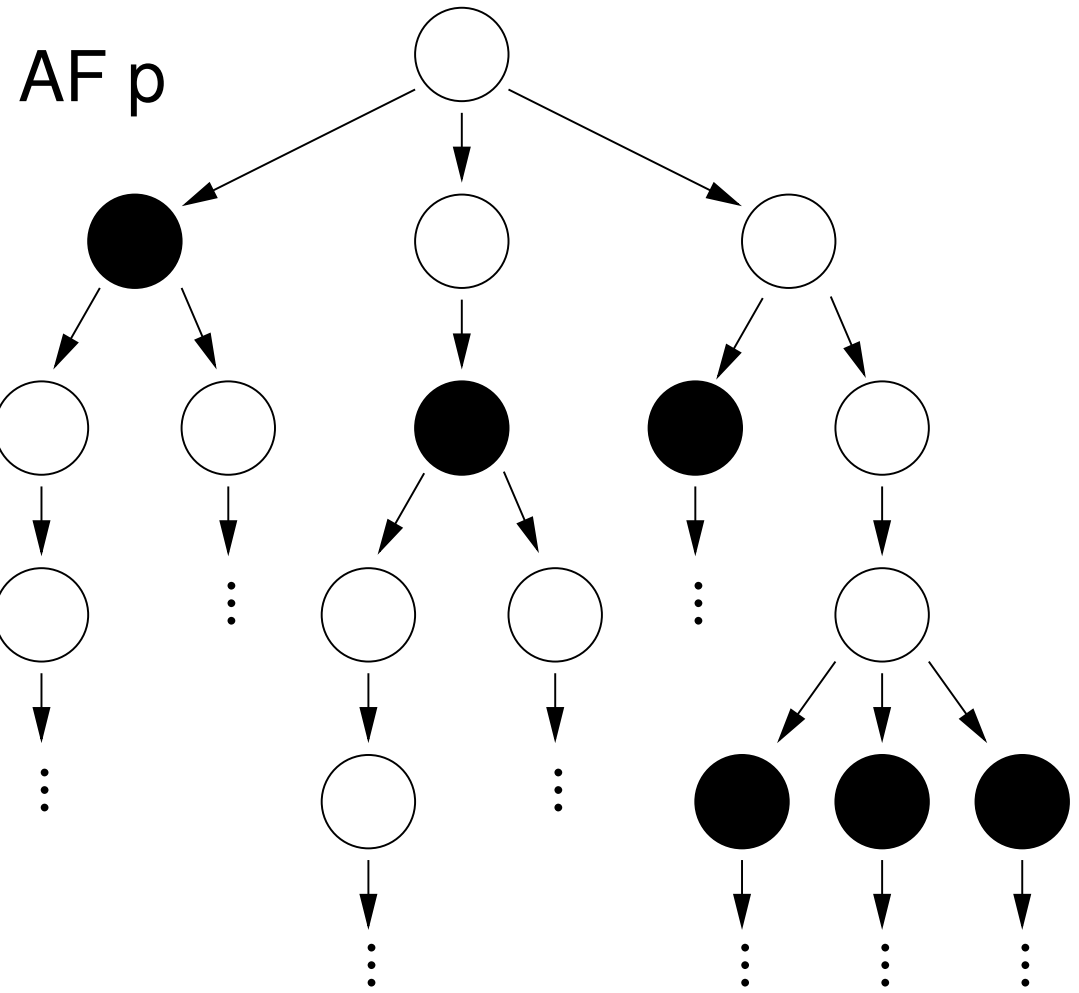
$$\phi_1 \text{ AU } \phi_2 \equiv \text{AF } \phi_2 \wedge (\phi_1 \text{ AW } \phi_2)$$

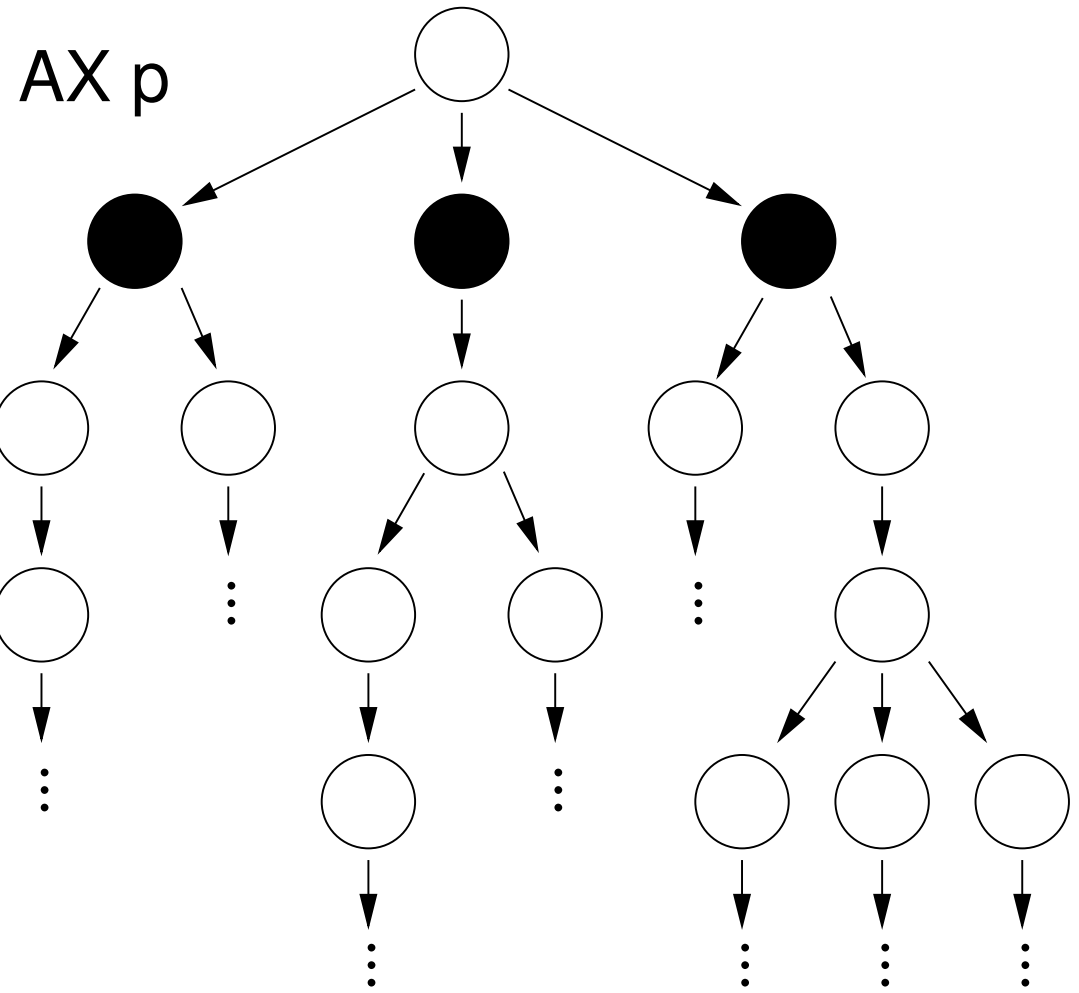
Other logical and temporal operators (e.g.  $\rightarrow$ , **ER**, **AR**), ... may also be defined.

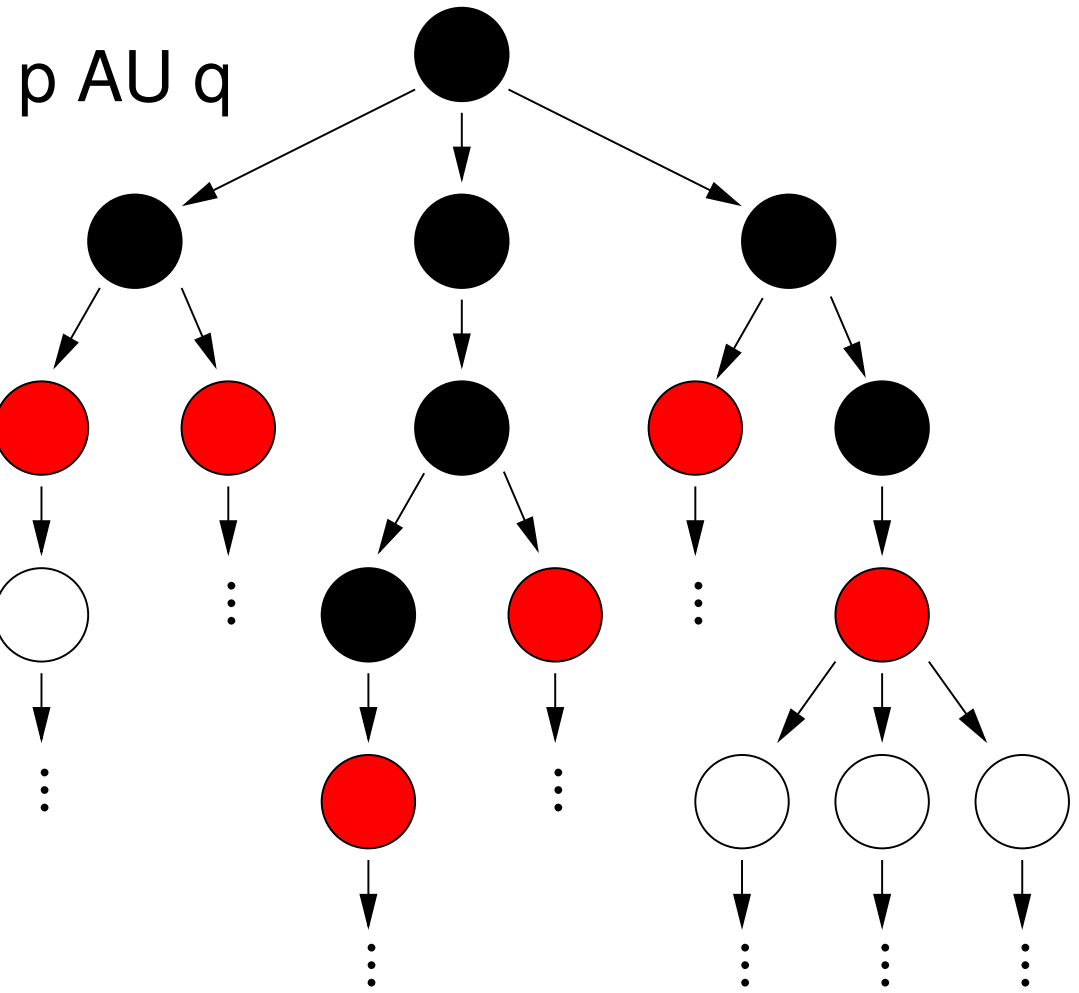


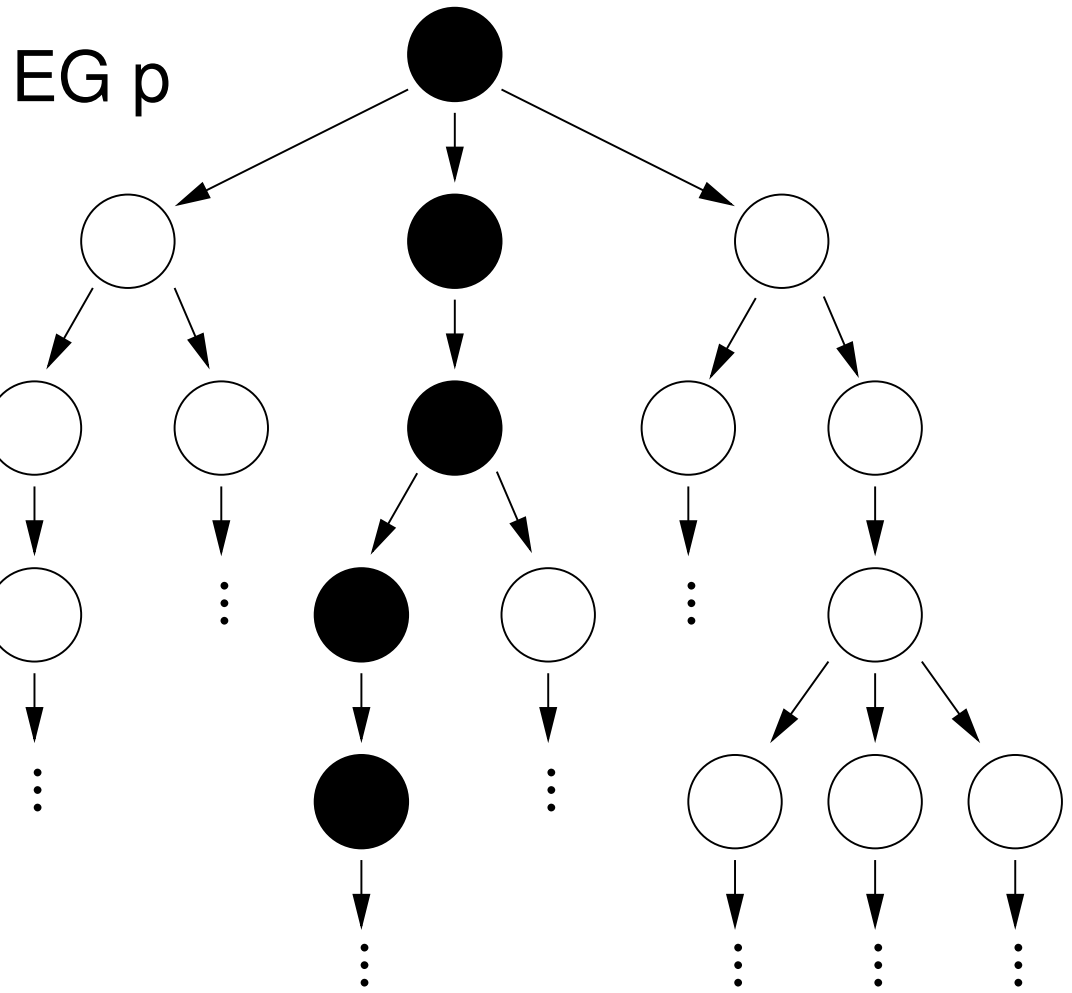






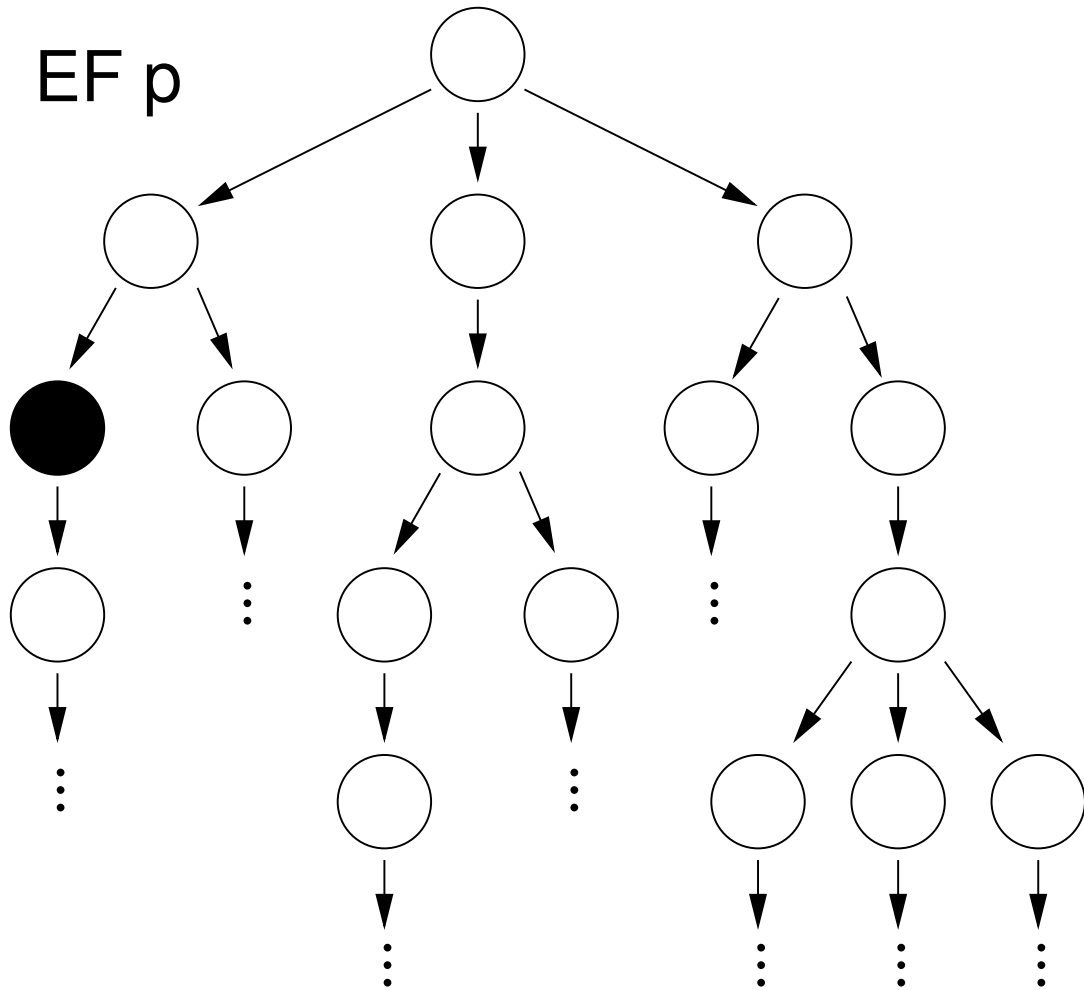






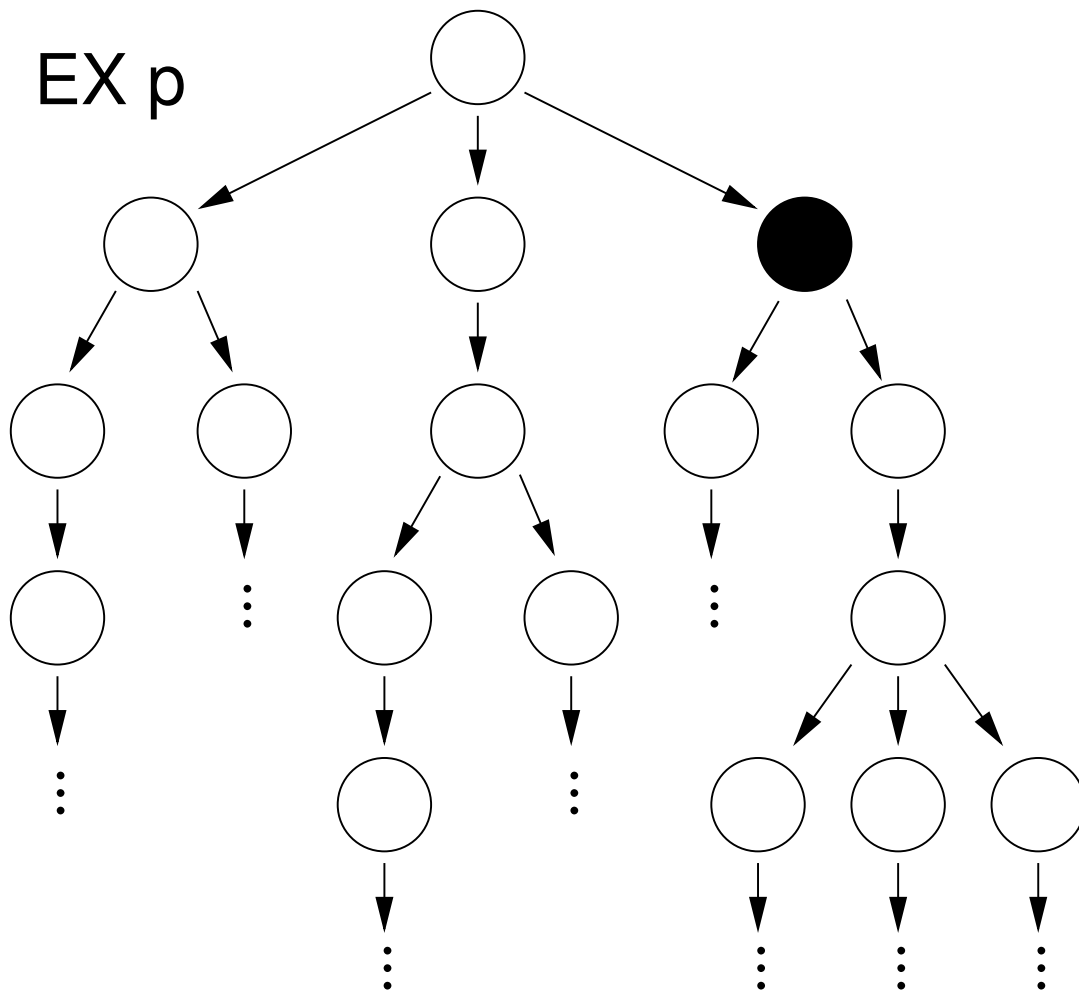
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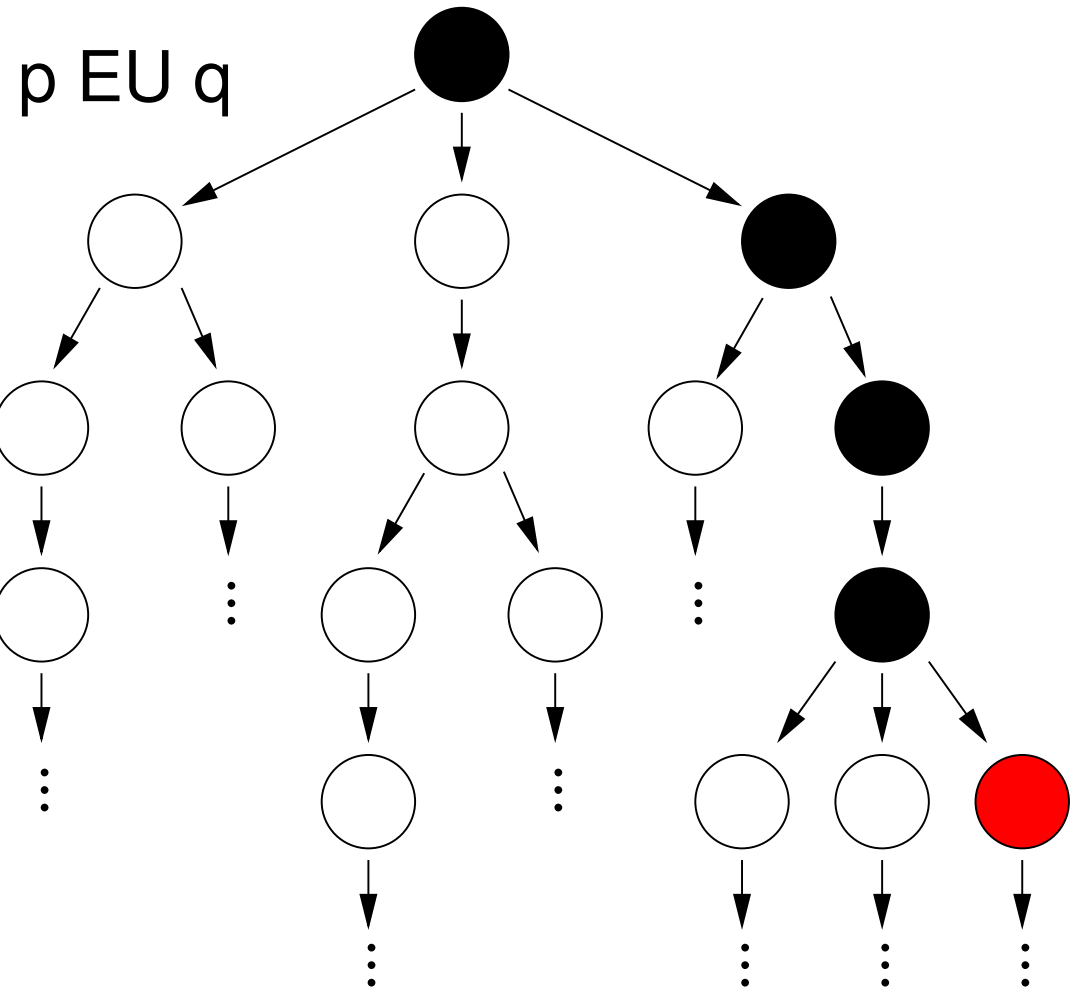
EF p



---

EX p

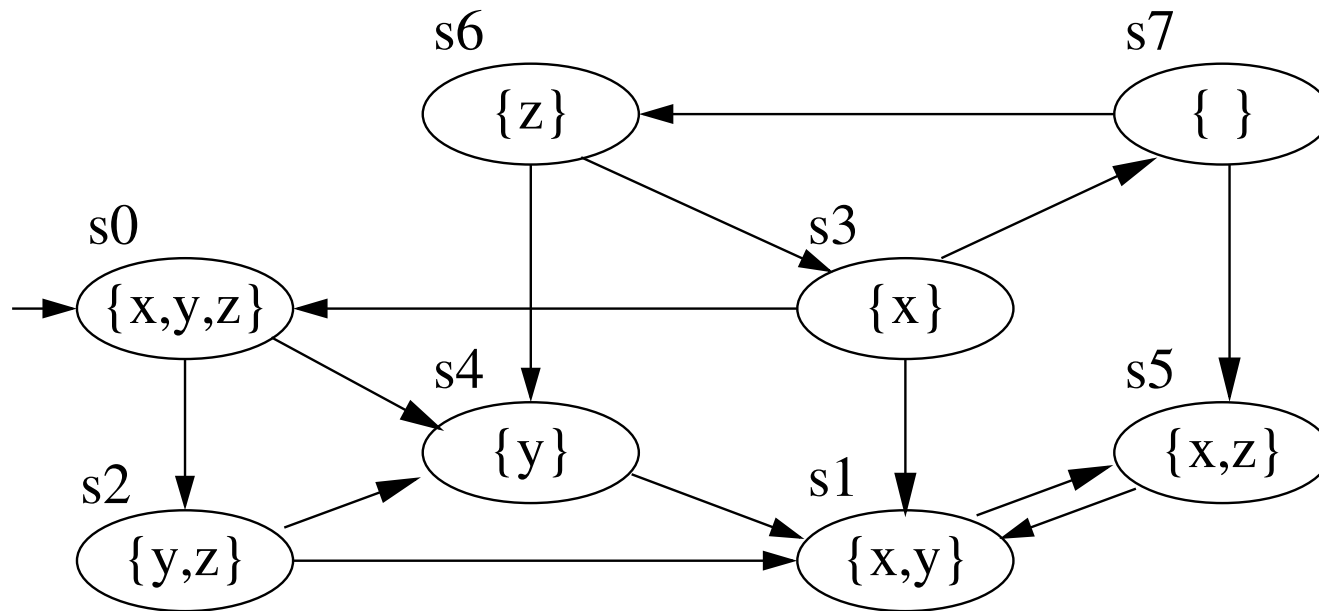






# Solving nested formulas: Is $s_0 \in \llbracket \text{AF AG } x \rrbracket$ ?

---

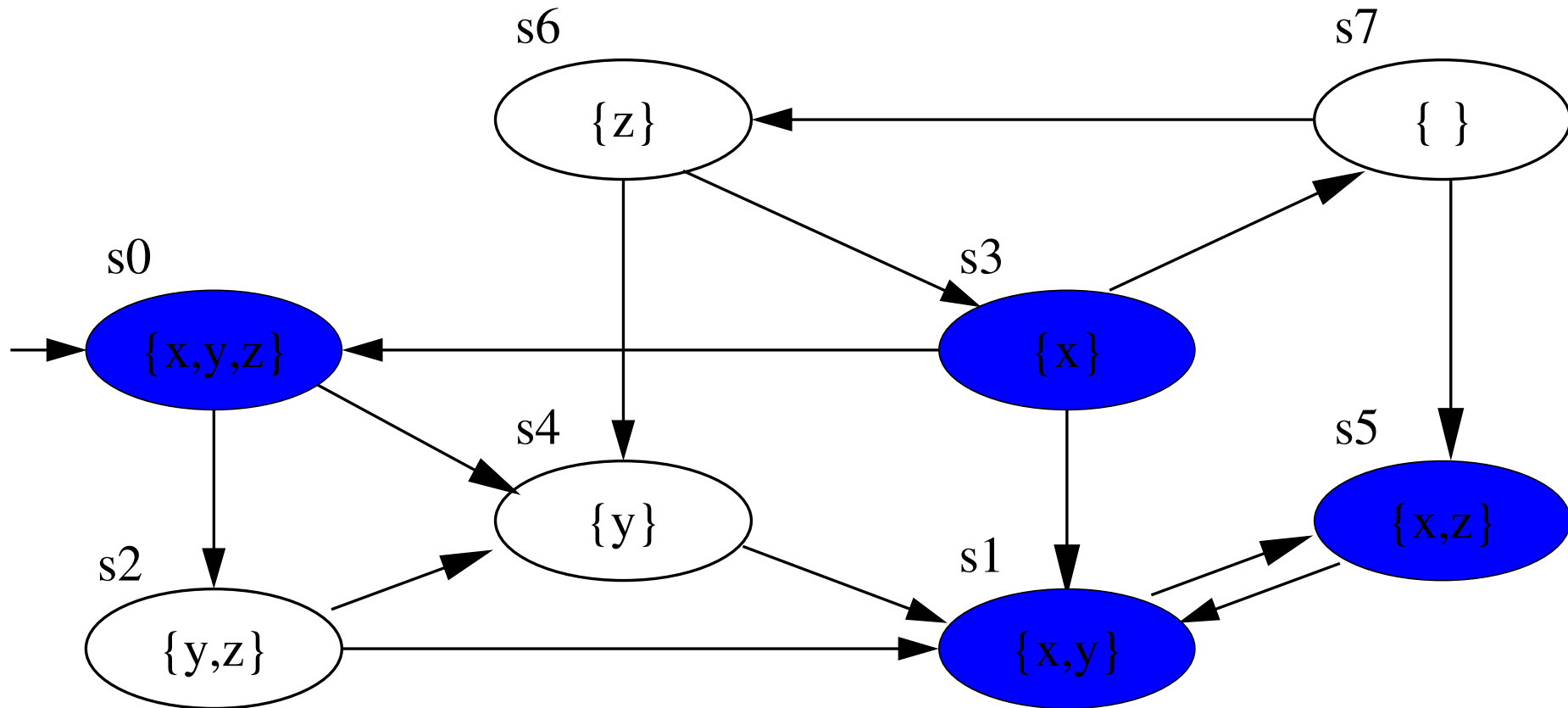


To compute the semantics of formulas with nested operators, we first compute the states satisfying the innermost formulas; then we use those results to solve progressively more complex formulas.

In this example, we compute  $\llbracket x \rrbracket$ ,  $\llbracket \text{AG } x \rrbracket$ , and  $\llbracket \text{AF AG } x \rrbracket$ , in that order.

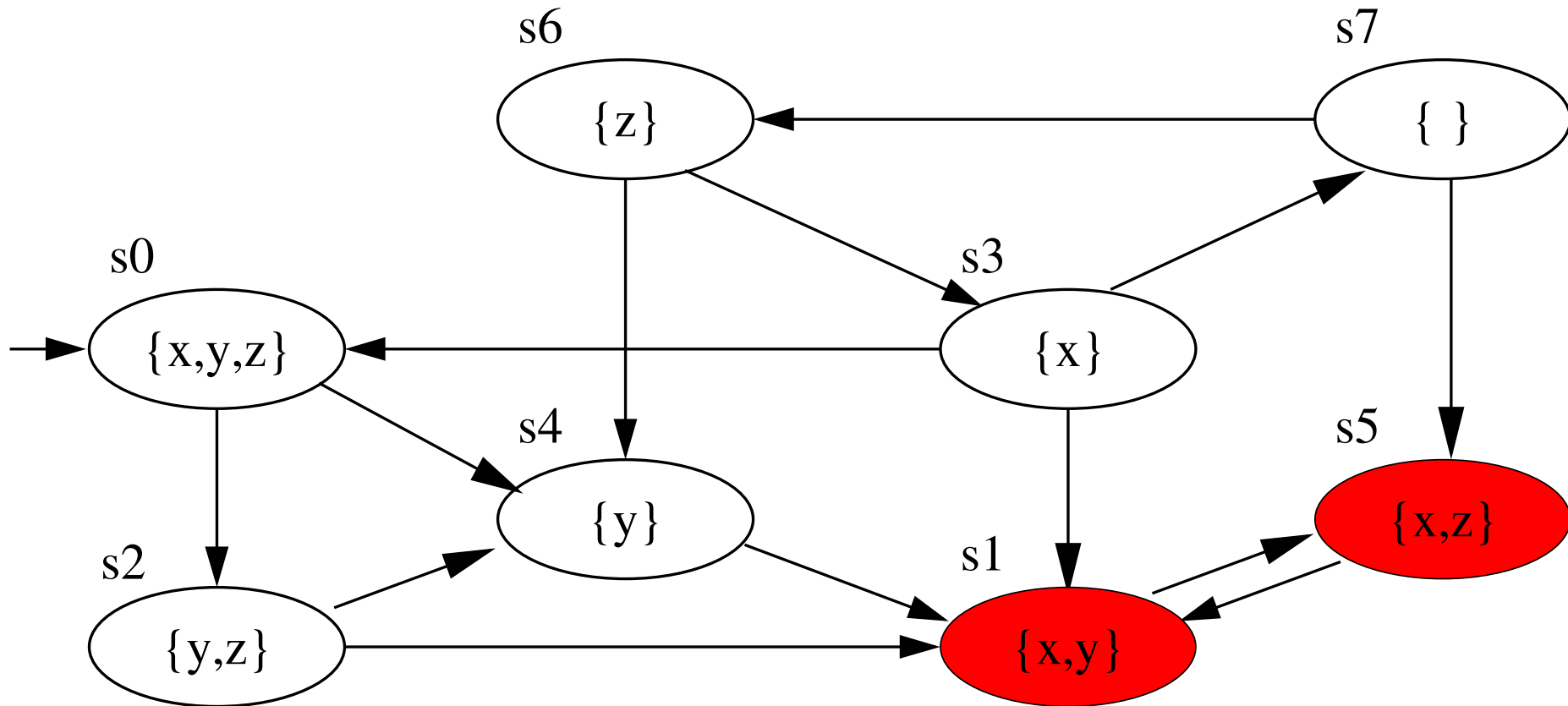
# Bottom-up method (1): Compute $[[x]]$

---



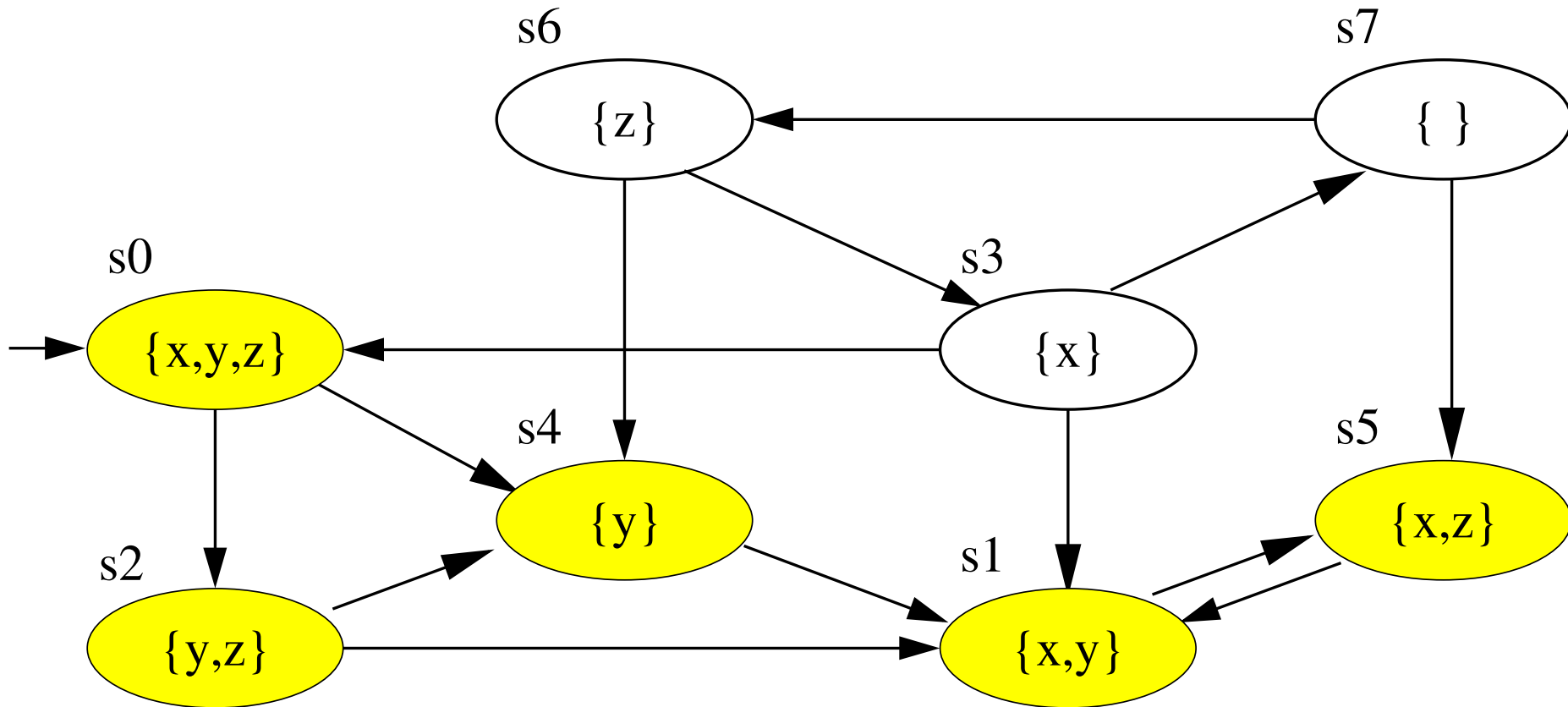
## Bottom-up method (2): Compute $[[AG\ x]]$

---



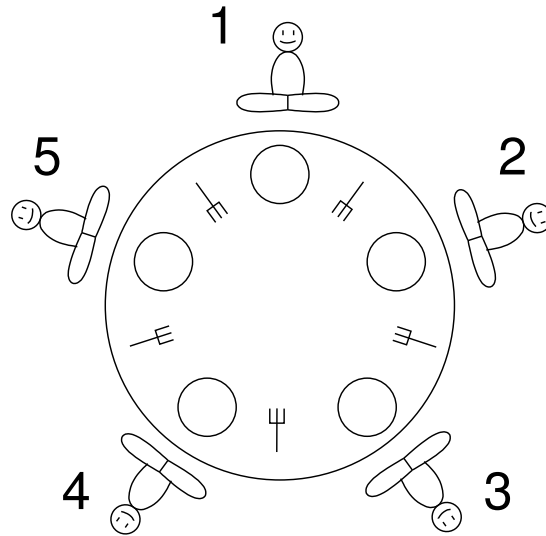
## Bottom-up method (3): Compute $[[AF AG x]]$

---



# Example: Dining Philosophers

---



Five philosophers are sitting around a table, taking turns at thinking and eating.

We shall express a couple of properties in CTL. Let us assume the following atomic propositions:

$e_i \equiv$  philosopher  $i$  is currently eating

# Properties of the Dining Philosophers

---

“Philosophers 1 and 4 will never eat at the same time.”

# Properties of the Dining Philosophers

---

“Philosophers 1 and 4 will never eat at the same time.”

$$AG \neg(e_1 \wedge e_4)$$

“It is possible that Philosopher 3 never eats.”

# Properties of the Dining Philosophers

---

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“It is possible that Philosopher 3 never eats.”

$$EG \neg e_3$$

“From every situation on the table it is possible to reach a state where only philosopher 2 is eating.”



# Properties of the Dining Philosophers

---

“Philosophers 1 and 4 will never eat at the same time.”

$$\mathbf{AG} \neg(e_1 \wedge e_4)$$

“It is possible that Philosopher 3 never eats.”

$$\mathbf{EG} \neg e_3$$

“From every situation on the table it is possible to reach a state where only philosopher 2 is eating.”

$$\mathbf{AG} \mathbf{EF}(\neg e_1 \wedge e_2 \wedge \neg e_3 \wedge \neg e_4)$$

# Part 10: Algorithms for CTL

# CTL-Model-Checking

---

In the following, let  $\mathcal{K} = (S, \rightarrow, r, AP, \nu)$  be a Kripke structure (where  $S$  is finite) and  $\phi$  a CTL formula over  $AP$ .

We shall solve the *global* model-checking problem for CTL, i.e. to compute  $\llbracket \phi \rrbracket_{\mathcal{K}}$  (all states of  $\mathcal{K}$  whose computation tree satisfies  $\phi$ ).

Our solution works “bottom-up”, i.e. it considers simple subformulae first, and then successively more complex ones.

The solution shown here considers only the minimal syntax. For additional efficiency one could extend it by treating some cases of the extended syntax more directly.

# The ‘bottom-up’ algorithm for CTL

---

The algorithm reduces  $\phi$  step by step to a single atomic proposition. Reminder:  $\llbracket p \rrbracket_{\mathcal{K}} = \{s \mid p \in \nu(s)\}$  for  $p \in AP$ . In the following, we abbreviate this set as  $\mu(p)$ .

1. Check whether  $\phi = p$ , where  $p \in AP$ . If yes, output  $\mu(p)$  and stop.
2. Otherwise,  $\phi$  contains some subformula  $\psi$  of the form  $\neg p$ ,  $p \vee q$ , **EX**  $p$ , **EG**  $p$ , or  $p$  **EU**  $q$ , where  $p, q \in AP$ . Compute  $\llbracket \psi \rrbracket_{\mathcal{K}}$  using the algorithms on the following slides.
3. Let  $p' \notin AP$  be a “fresh” atomic proposition. Add  $p'$  to  $AP$  and set  $\mu(p') := \llbracket \psi \rrbracket_{\mathcal{K}}$ . Replace all occurrences of  $\psi$  in  $\phi$  by  $p'$  and continue at step 1.

# Computation of $\llbracket \psi \rrbracket_{\mathcal{K}}$ : simple cases

---

Case 1:  $\psi \equiv \neg p$ ,  $p \in AP$

By definition,  $\llbracket \psi \rrbracket_{\mathcal{K}} = S \setminus \mu(p)$ .

Case 2:  $\psi \equiv p \vee q$ ,  $p, q \in AP$

Then  $\llbracket \psi \rrbracket_{\mathcal{K}} = \mu(p) \cup \mu(q)$ .

Case 3:  $\psi \equiv \mathbf{EX} p$ ,  $p \in AP$

In the following, let  $pre(X)$ , for  $X \subseteq S$ , denote the set

$$pre(X) := \{s \mid \exists t \in X: s \rightarrow t\}.$$

Then by definition  $\llbracket \psi \rrbracket_{\mathcal{K}} = pre(\mu(p))$ .

# Computation of $[[\psi]]_{\mathcal{K}}$ : EU and EG

---

We shall first define **EU** and **EG** in terms of fixed points.

**EU** is characterized by a **smallest** fixed point: We first assume that no state satisfies the EU formula and then, one by one, identify those that do satisfy it after all.

By contrast, **EG** can be characterized by a **largest** fixed point: We first assume that all states satisfy a given EG formula and then, one by one, eliminate those that do not.

Based on this, we then derive algorithms for EG and EU.

# Computation of $\llbracket \psi \rrbracket_{\mathcal{K}}$ : EG

---

Case 4:  $\psi \equiv \mathbf{EG} \rho$ ,  $\rho \in AP$

Lemma 1:  $\llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}}$  is the largest solution (w.r.t.  $\subseteq$ ) of the equation

$$X = \mu(\rho) \cap pre(X).$$

**Proof:** We proceed in two steps:

1. We show that  $\llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}}$  is indeed a solution of the equation, i.e.

$$\llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}} = \mu(\rho) \cap pre(\llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}}).$$

Reminder:  $\llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}} = \{s \mid \exists \rho: \rho(0) = s \wedge \forall i \geq 0: \rho(i) \in \mu(\rho)\}$ .

“ $\Rightarrow$ ” Let  $s \in \llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}}$  and  $\rho$  a “witness” path. Then obviously  $s \in \mu(\rho)$ .  
Moreover,  $\rho(1) \in \llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}}$  (because of  $\rho^1$ ), hence  $s \in pre(\llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}})$ .

---

Continuation of the proof of Lemma 1:

1. “ $\Leftarrow$ ” Let  $s \in \mu(p) \cap pre(\llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}})$ . Then  $s$  has a direct successor  $t$ , where a path  $\rho$  starts proving that  $t \in \llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}}$ . Thus,  $s\rho$  is a path witnessing that  $s \in \llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}}$ .

2. We show that  $\llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}}$  is indeed the *largest* solution, i.e., if  $M$  is a solution of the equation, then  $M \subseteq \llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}}$ .

Let  $M \subseteq S$  be a solution of the equation, i.e.  $M = \mu(p) \cap pre(M)$ , and let  $s \in M$ . We shall show  $s \in \llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}}$ .

- Since  $s \in M$ , we have  $s \in \mu(p)$  and  $s \in pre(M)$ .
- Since  $s \in pre(M)$ , there exist  $s_1 \in M$  with  $s \rightarrow s_1$ .
- Repeating this argument, we can construct an infinite path  $\rho = ss_1 \cdots$  in which all states are contained in  $\mu(p)$ . Therefore,  $s \in \llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}}$ .



---

**Lemma 2:** Consider the sequence  $S, \pi(S), \pi(\pi(S)), \dots$ , i.e.  $(\pi^i(S))_{i \geq 0}$ ,

where  $\pi(X) := \mu(p) \cap pre(X)$ .

For all  $i \geq 0$  we have  $\pi^i(S) \supseteq \llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}}$ .

We state the following two facts:

(1)  $\pi$  is *monotone*: if  $X \supseteq X'$ , then  $\pi(X) \supseteq \pi(X')$ .

(2) The sequence is *descending*:  $S \supseteq \pi(S) \supseteq \pi(\pi(S)) \dots$  (follows from (1)).

**Proof of Lemma 2:** (induction over  $i$ )

Base:  $i = 0$ : obvious.

Step:  $i \rightarrow i + 1$ :

$$\begin{aligned} \pi^{i+1}(S) &= \mu(p) \cap pre(\pi^i(S)) \\ &\supseteq \mu(p) \cap pre(\llbracket \mathbf{EG} \varphi \rrbracket_{\mathcal{K}}) \quad (\text{i.h. and monotonicity}) \\ &= \llbracket \mathbf{EG} p \rrbracket_{\mathcal{K}} \end{aligned}$$

---

**Lemma 3:** There exists an index  $i$  such that  $\pi^i(S) = \pi^{i+1}(S)$ , and  $\llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}} = \pi^i(S)$ .

**Proof:** Since  $S$  is finite, the descending sequence must reach a fixed point, say after  $i$  steps. Then we have  $\pi^i(S) = \pi(\pi^i(S)) = \mu(\rho) \cap pre(\pi^i(S))$ . Therefore,  $\pi^i(S)$  is a solution of the equation from Lemma (1).

Because of Lemma 1, we have  $\pi^i(S) \subseteq \llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}}$ .

Because of Lemma 2, we have  $\pi^i(S) \supseteq \llbracket \mathbf{EG} \rho \rrbracket_{\mathcal{K}}$ .

# An algorithm for EG

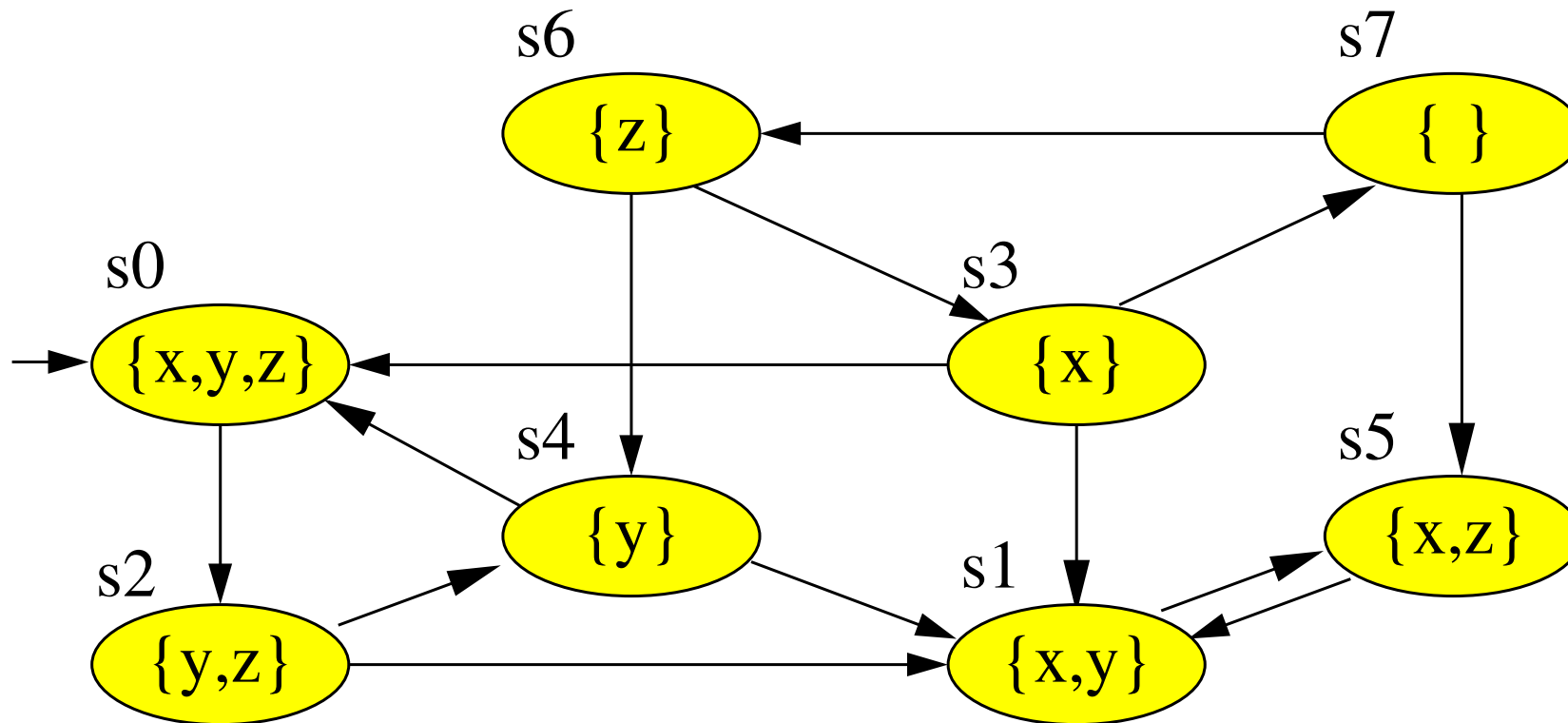
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Lemma 3 gives us a strategy for computing  $\llbracket \text{EG } p \rrbracket_{\mathcal{K}}$ : compute the sequence  $S, \pi(S), \dots$  until a fixed point is reached.

For practicality, one would start immediately with  $X := \mu(p)$ . Then, in each round, one eliminates those states having no successors in  $X$ .

This can be efficiently implemented in  $\mathcal{O}(|\mathcal{K}|)$  time (“reference counting”).

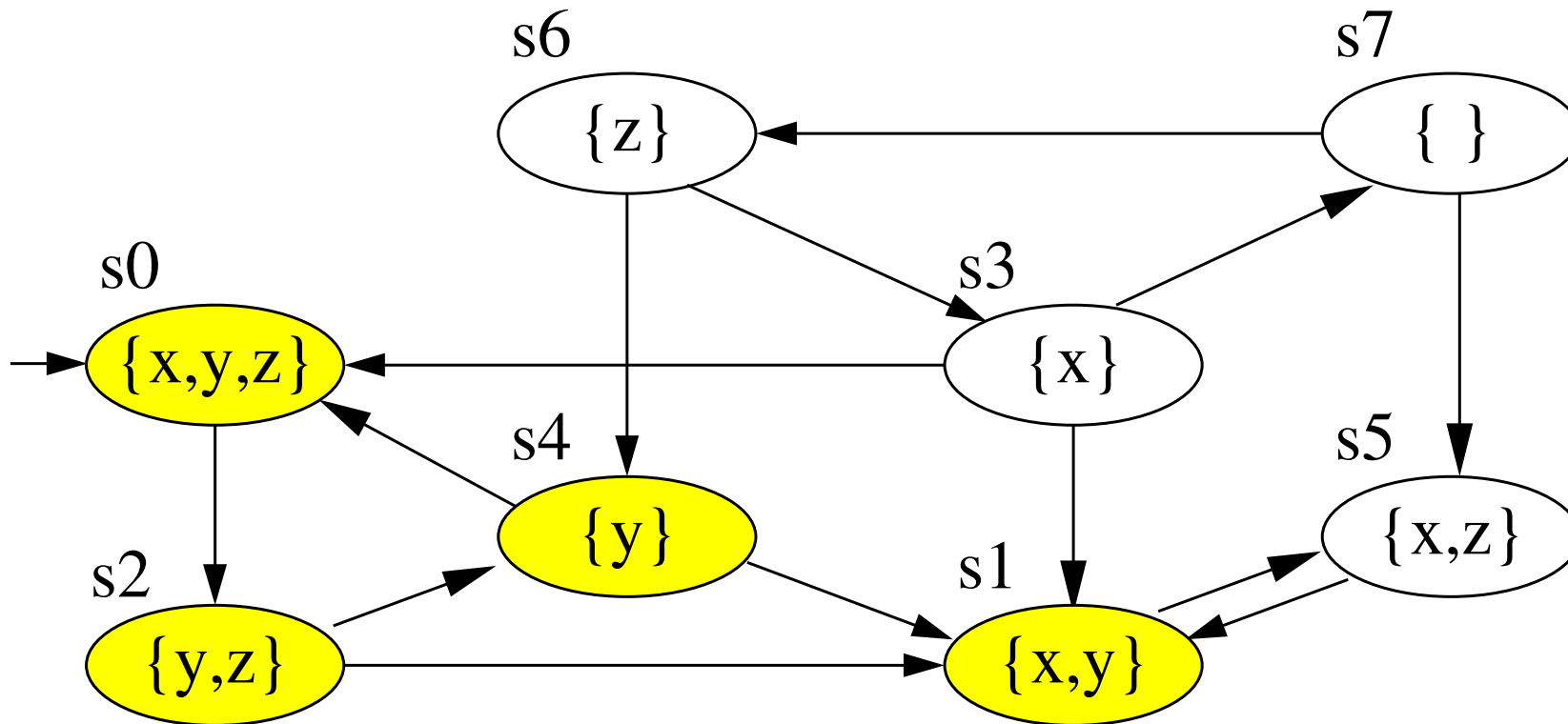
# Example: Computation of $[[EG\ y]]_{\mathcal{K}}$ (1/4)



$$\pi^0(S) = S$$

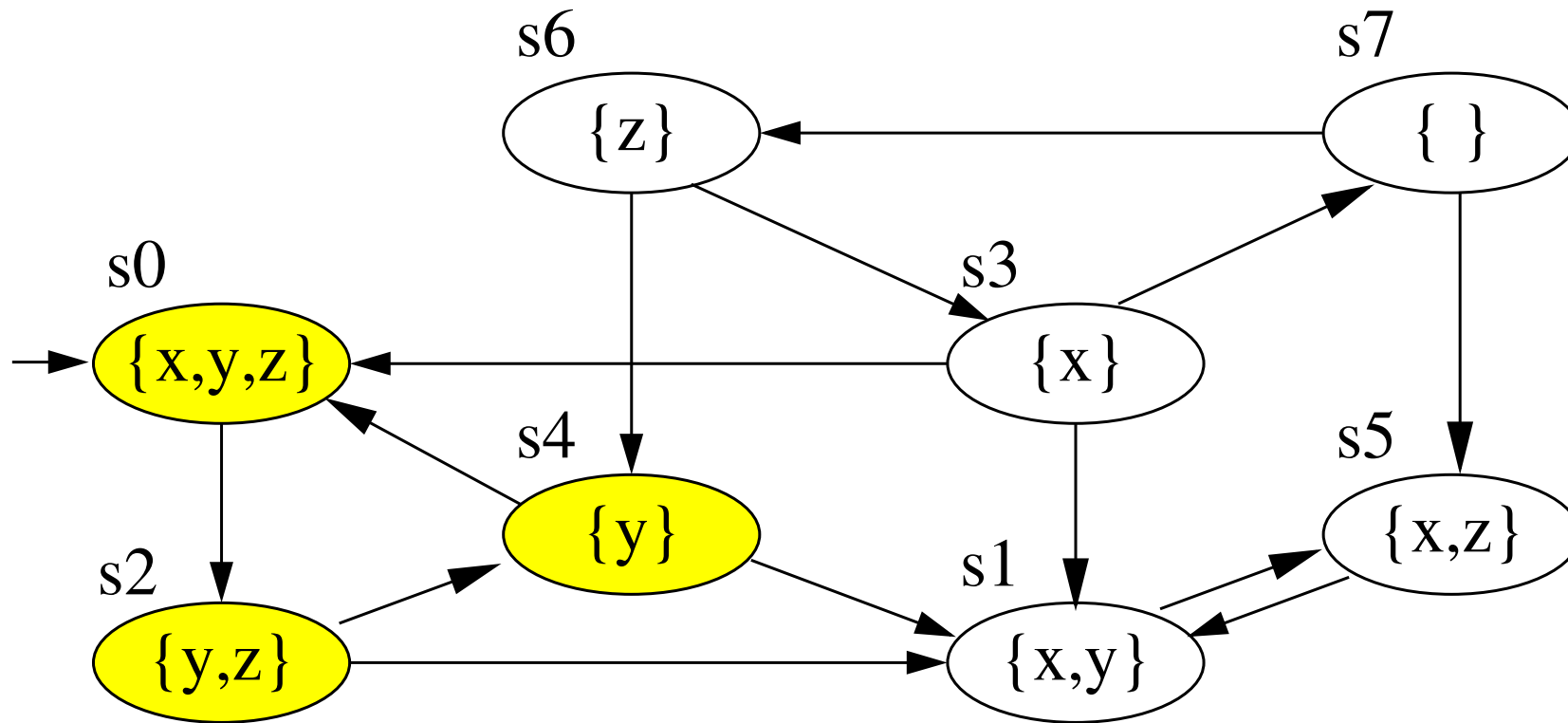
## Example: Computation of $[[EG\ y]]_{\mathcal{K}}$ (2/4)

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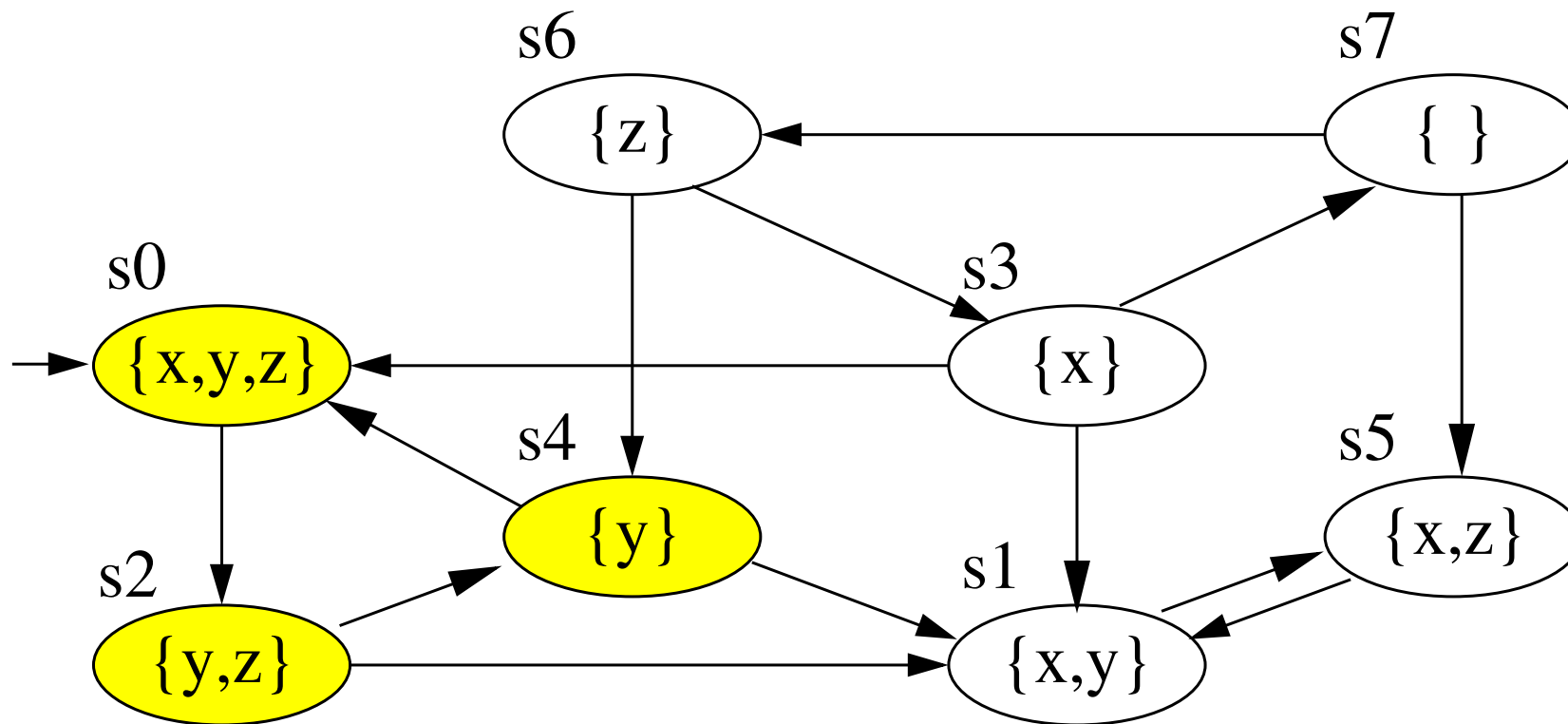
$$\pi^1(S) = \mu(y) \cap pre(S)$$

# Example: Computation of $[[EG\ y]]_{\mathcal{K}}$ (3/4)



$$\pi^2(S) = \mu(y) \cap pre(\pi^1(S))$$

## Example: Computation of $\llbracket \mathbf{EG} y \rrbracket_{\mathcal{K}}$ (4/4)



$$\pi^3(\mathcal{S}) = \mu(y) \cap pre(\pi^2(\mathcal{S})) = \pi^2(\mathcal{S}): \llbracket \mathbf{EG} y \rrbracket_{\mathcal{K}} = \{s_0, s_2, s_4\}$$

# Computation of EU

---

Case 5:  $\psi \equiv p \text{ EU } q$ ,  $p, q \in AP$

Analogous to EG (proofs omitted):

Lemma 4:  $\llbracket p \text{ EU } q \rrbracket_{\mathcal{K}}$  is the smallest solution (w.r.t.  $\subseteq$ ) of the equation

$$X = \mu(q) \cup (\mu(p) \cap \text{pre}(X)).$$

Lemma 5:  $\llbracket p \text{ EU } q \rrbracket_{\mathcal{K}}$  is the fixed point of the sequence

$$\emptyset, \xi(\emptyset), \xi(\xi(\emptyset)), \dots \text{ where } \xi(X) := \mu(q) \cup (\mu(p) \cap \text{pre}(X))$$



# An algorithm for EU

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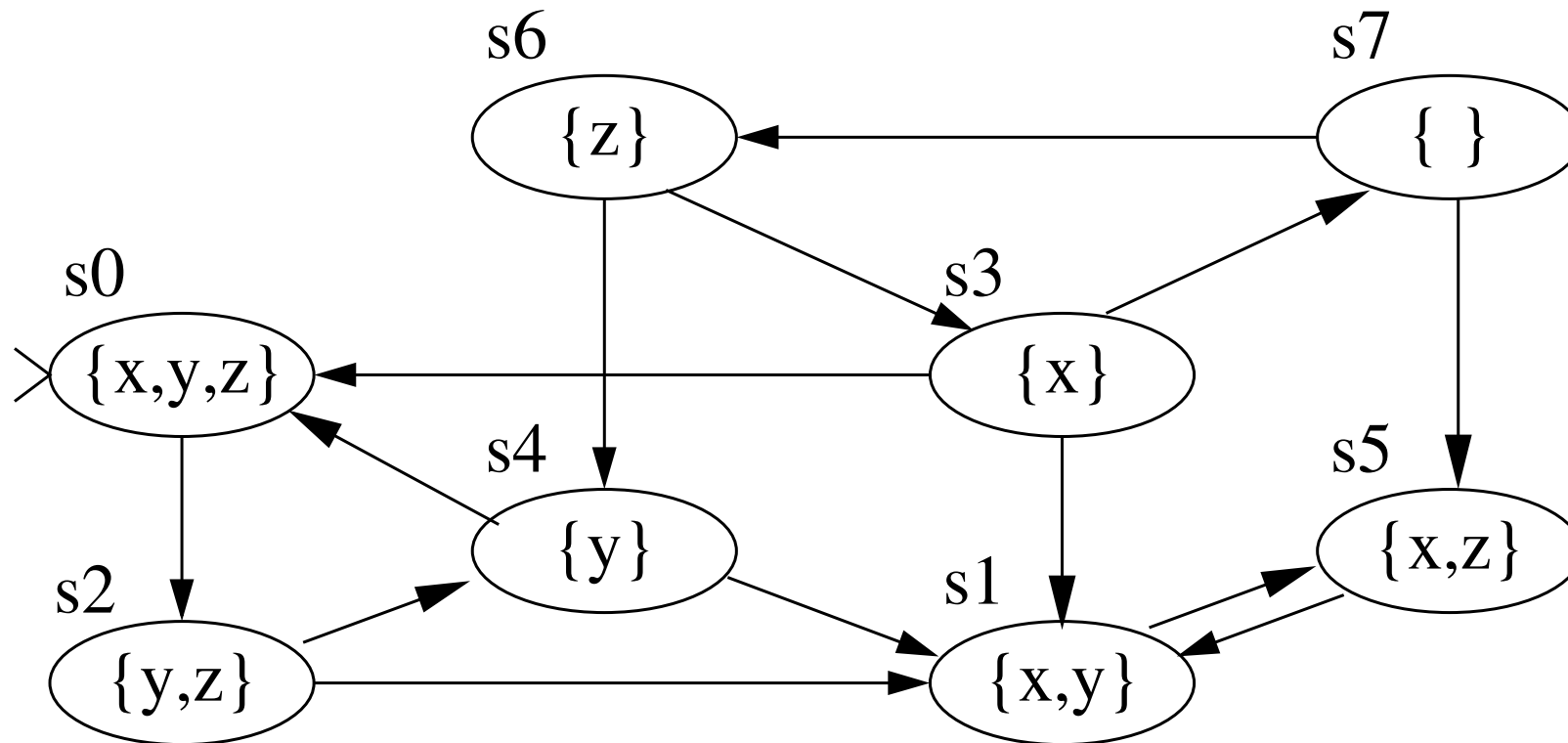
Lemma 5 proposes a strategy: Compute the sequence  $\emptyset, \xi(\emptyset), \dots$  until a fixed point is reached.

In practice one would start with  $X := \mu(q)$ . Then, in each step, one can add those direct successors that are in  $\mu(p)$ .

Can be done efficiently in  $\mathcal{O}(|\mathcal{K}|)$  time (multiple backwards DFS).

# Example: Computation of $\llbracket z \text{ EU } y \rrbracket_{\mathcal{K}}$ (1/4)

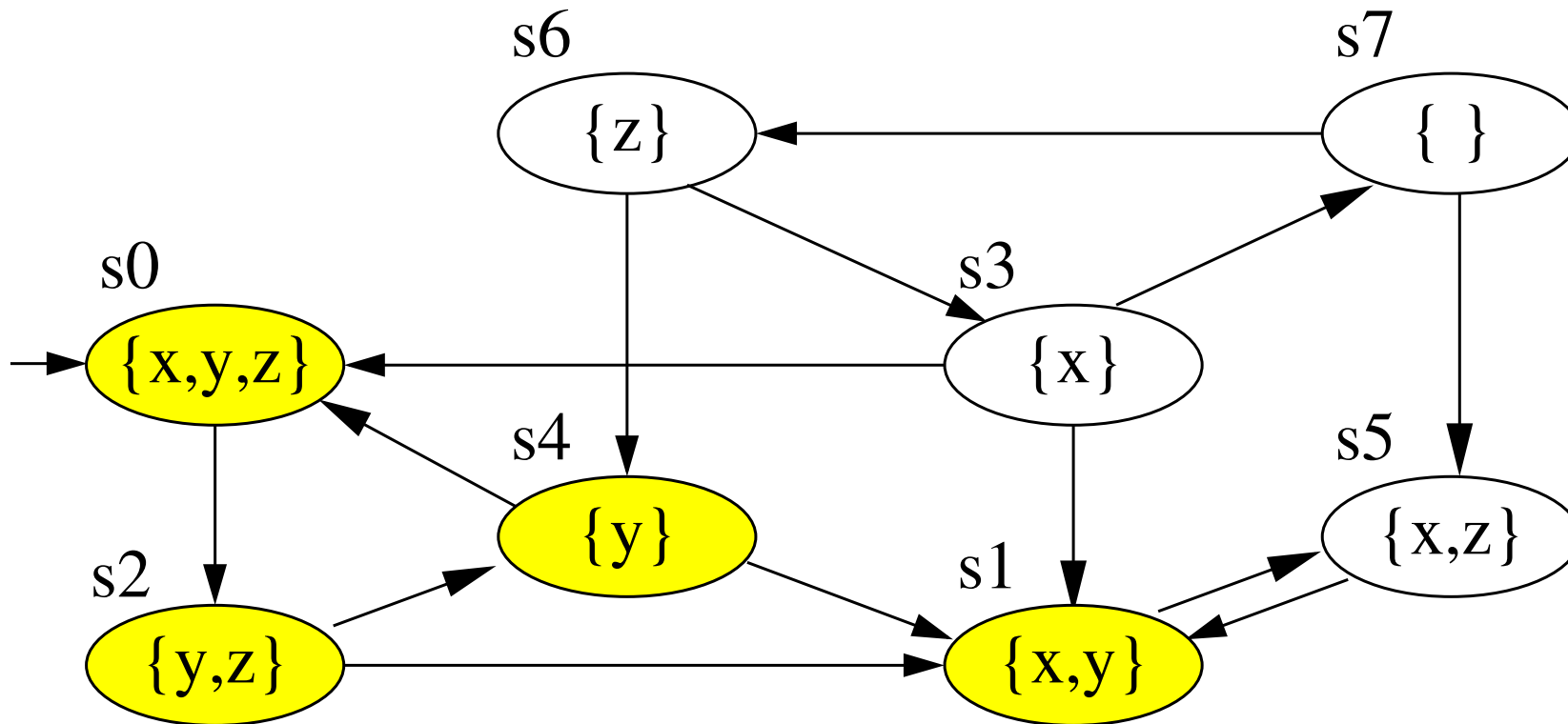
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$$\xi^0(\emptyset) = \emptyset$$

## Example: Computation of $\llbracket z \text{ EU } y \rrbracket_{\mathcal{K}}$ (2/4)

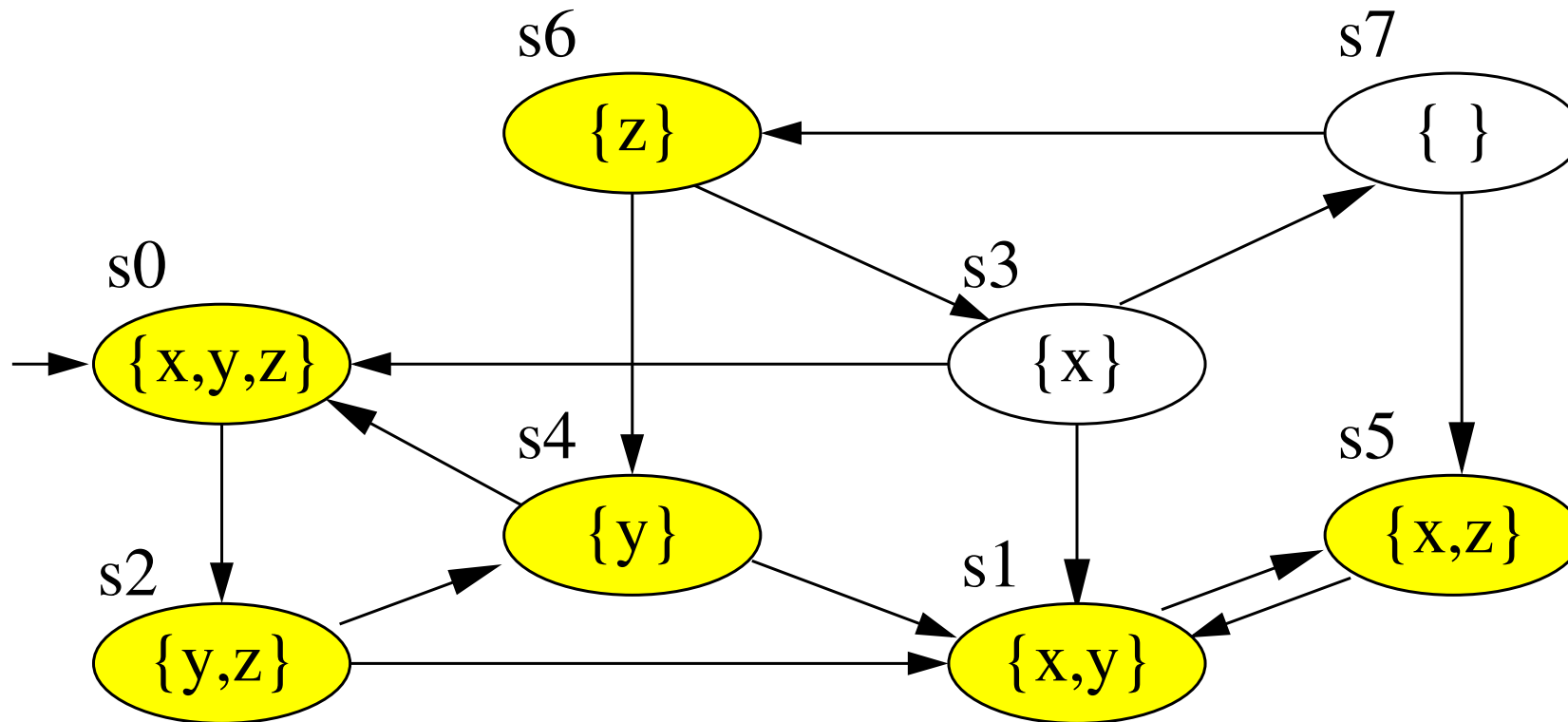
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$$\xi^1(\emptyset) = \mu(y) \cup (\mu(z) \cap \text{pre}(\xi^0(\emptyset)))$$

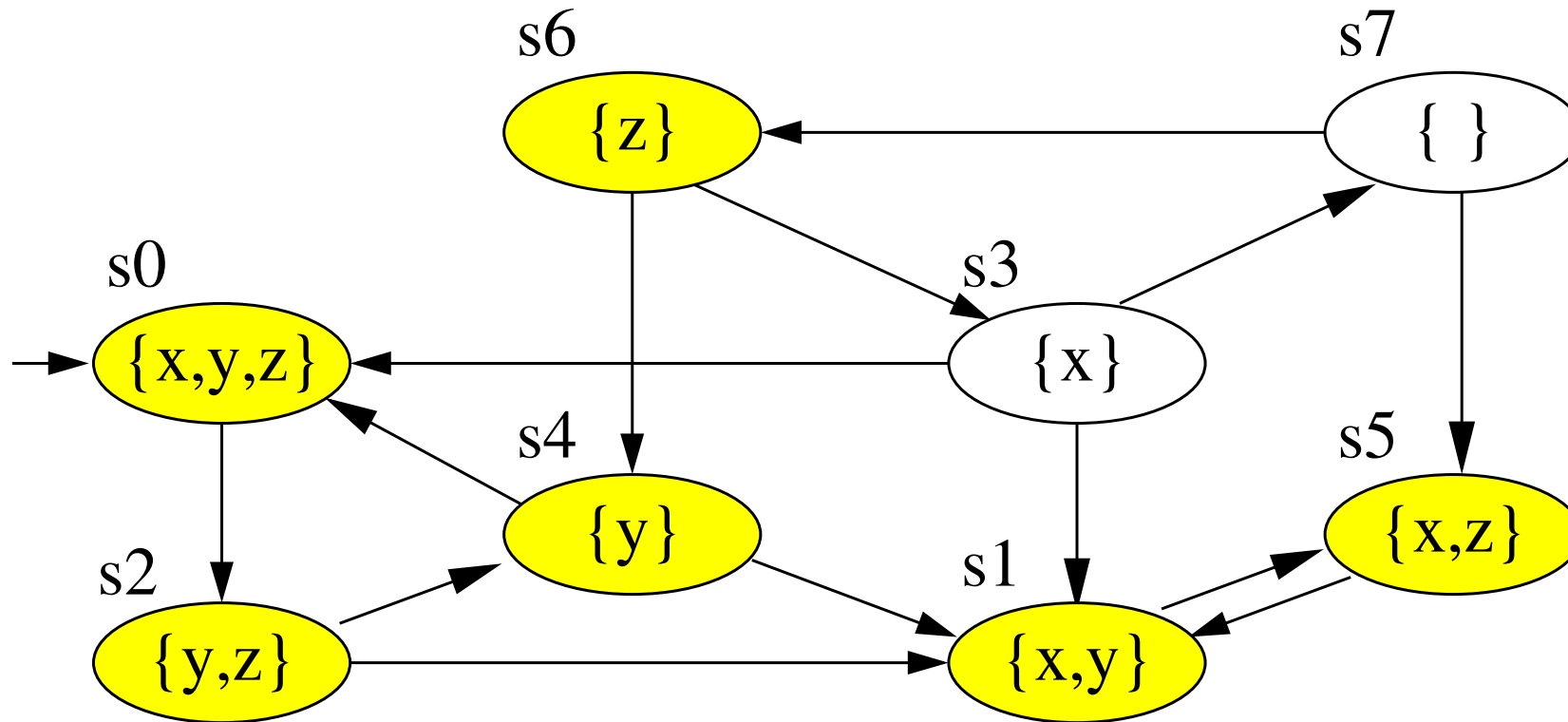
## Example: Computation of $\llbracket z \text{ EU } y \rrbracket_{\mathcal{K}}$ (3/4)

---



$$\xi^2(\emptyset) = \mu(y) \cup (\mu(z) \cap \text{pre}(\xi^1(\emptyset)))$$

## Example: Computation of $\llbracket z \text{ EU } y \rrbracket_{\mathcal{K}}$ (4/4)



$$\xi^3(\emptyset) = \mu(y) \cup (\mu(z) \cap \text{pre}(\xi^2(\emptyset))) = \xi^2(\emptyset)$$

$$\llbracket z \text{ EU } y \rrbracket_{\mathcal{K}} = \{s_0, s_1, s_2, s_4, s_5, s_6\}$$

## Example: Dekker's mutex algorithm

---

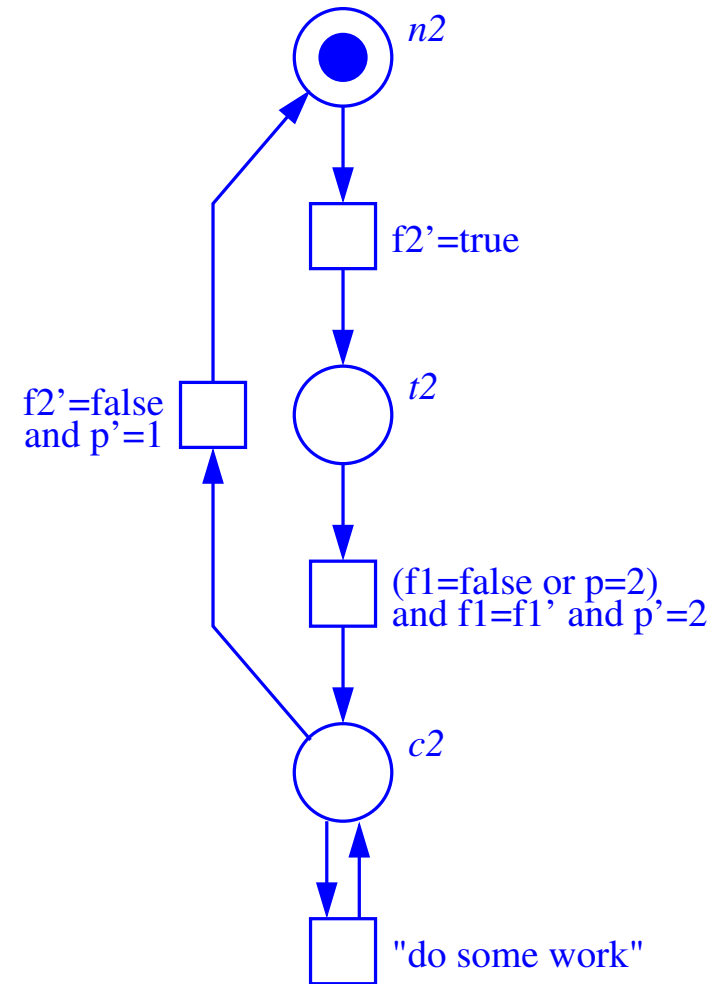
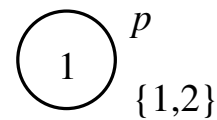
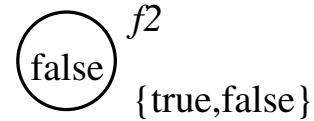
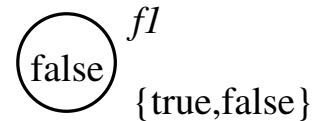
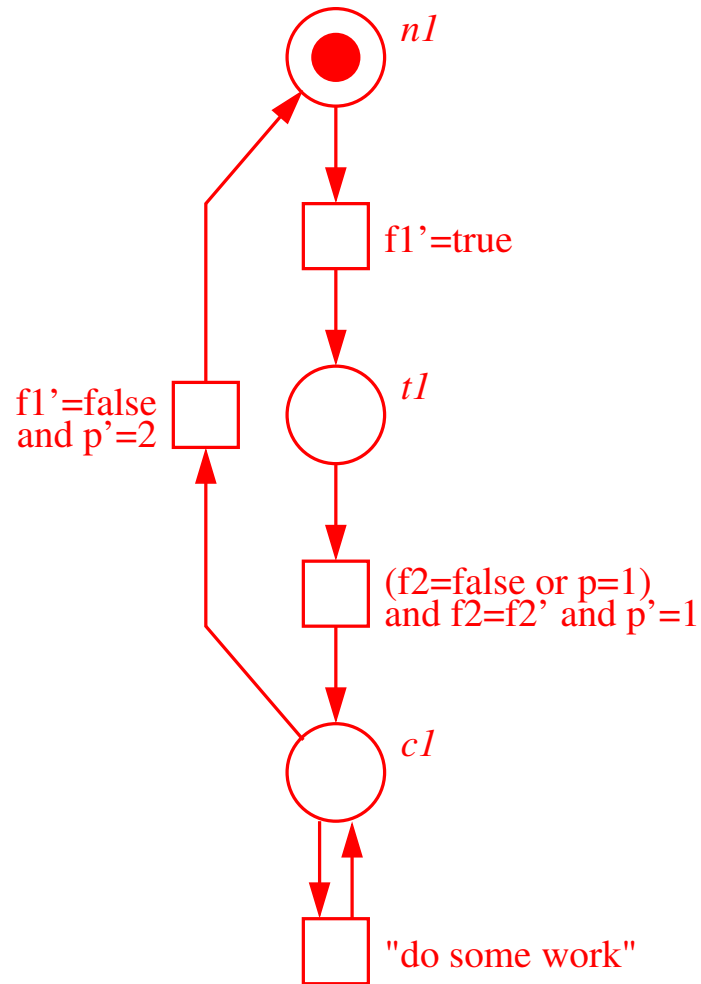
The next slides show a Petri net implementing a (fair) mutual exclusion protocol for two processes (red and blue), and the reachability graph of the net.

The places  $n1$ ,  $n2$  denote the non-critical sections,  $c1$ ,  $c2$  the critical sections, and  $t1$ ,  $t2$  indicate that a process is *trying* to enter its critical section.

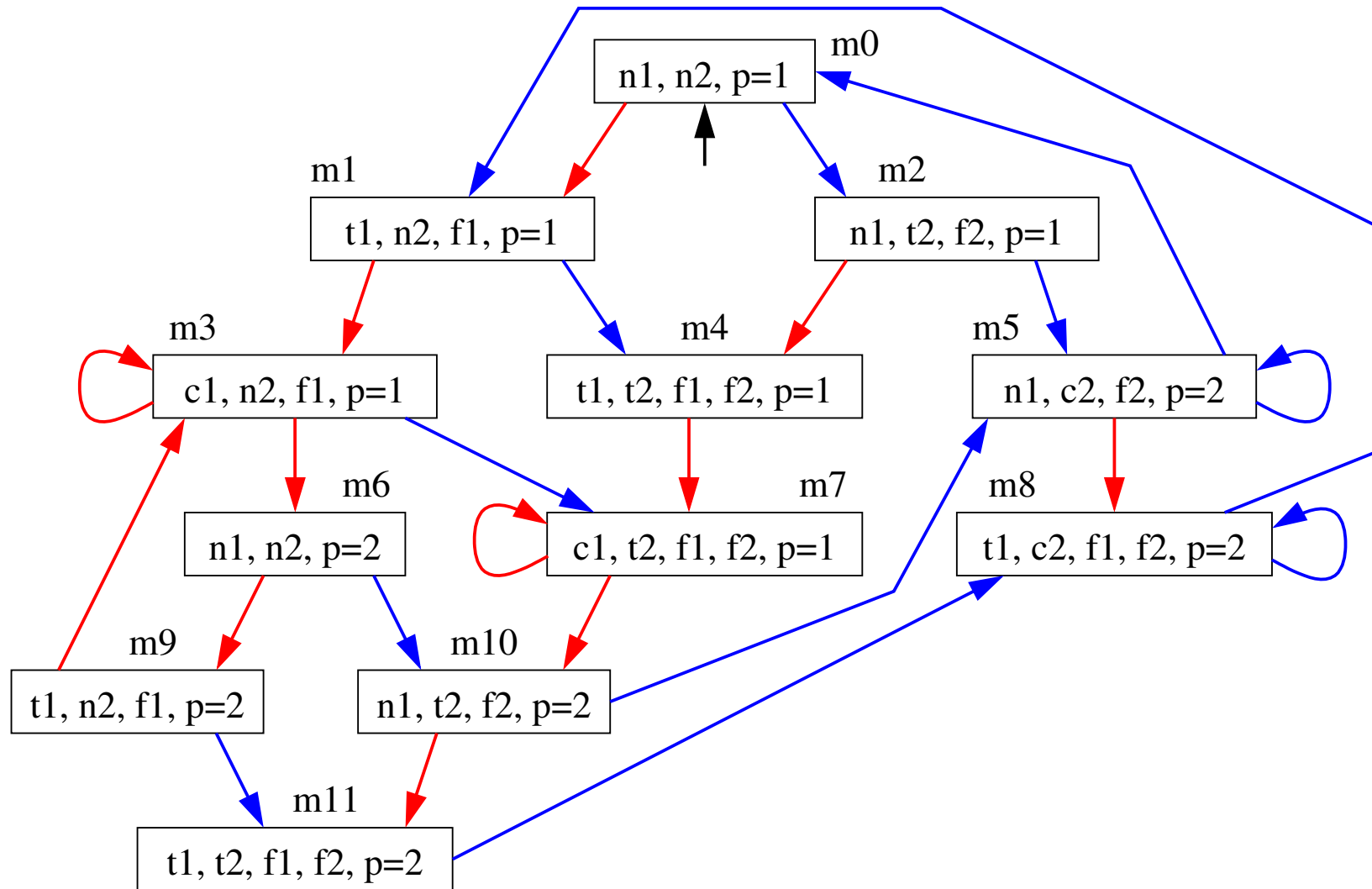
For space reason, the arcs to places in the middle ( $f1$ ,  $f2$ ,  $p$ ) were omitted.

In the reachability graph,  $f1$  and  $f2$  are only mentioned when true.

# Example: Petri net



# Example: Reachability graph





# Specification

---

The satisfactoin mutual exclusion property is directly observable in the reachability graph. Moreover, we might be interested in the following property:

“Whenever a process wants to enter a critical section, it will eventually succeed in doing so.”

A plausible formulation in CTL is as follows:

$$\mathbf{AG}(t1 \rightarrow \mathbf{AF} c1) \quad \text{and} \quad \mathbf{AG}(t2 \rightarrow \mathbf{AF} c2)$$

For this, we extend the reachability graph into a Kripke structure with four atomic propositions:

$$c1 \text{ with } \mu(c1) := \{m_3, m_7\}; \quad t1 \text{ with } \mu(t1) := \{m_1, m_4, m_8, m_9, m_{11}\}$$

$$c2 \text{ with } \mu(c2) := \{m_5, m_8\}; \quad t2 \text{ with } \mu(t2) := \{m_2, m_4, m_7, m_{10}, m_{11}\}$$

# Checking the formula

---

We rewrite the first formula into the minimal syntax:

$$\neg(\text{true EU } (t1 \wedge \text{EG } \neg c1))$$

(The second formula is analogous, so we consider just the first.)

When checking this formula, we shall observe that it does not hold – the system may remain in states  $m_5$  and  $m_8$  forever.

This happens, when one process remains in its critical section forever.

# Fairness

---

Thus, the property is not satisfied because the blue process may exhibit an “unfair” behaviour, i.e. does not leave the critical section.

Can we – in analogy to LTL – consider only those runs that satisfy some fairness constraint (here: processes eventually leave their critical section)?

Answer 1: No. This is not expressible in CTL (e.g.,  $(\mathbf{AG AF fair}) \rightarrow \phi$  won't do).

Answer 2: Yes. We can extend CTL accordingly.

# CTL with fairness

---

Let  $\mathcal{K}$  and  $\phi$  be the same as before. Additionally, let  $F_1, \dots, F_n \subset S$  be **fairness constraints**.

In the following, we shall call a path **fair** iff it visits each fairness constraint infinitely often.

In our example, the fairness constraints would be as follows:

$$F_1 = S \setminus \mu(c1)$$

$$F_2 = S \setminus \mu(c2)$$

---

Our problem is to compute  $\llbracket \phi \rrbracket_{\mathcal{K}}$  for the case where **EG** and **EU** quantify only over “fair” paths. We introduce modified operators **EG<sub>f</sub>** and **EU<sub>f</sub>** with the following meaning:

$\mathcal{T} \models \mathbf{EG}_f \phi$       iff  $\mathcal{T}$  has a **fair** infinite path  $r = v_0 \rightarrow v_1 \rightarrow \dots$ ,  
 where for all  $i \geq 0$  we have:  $\llbracket v_i \rrbracket \models \phi$

$\mathcal{T} \models \phi_1 \mathbf{EU}_f \phi_2$       iff  $\mathcal{T}$  has a **fair** infinite path  $r = v_0 \rightarrow v_1 \rightarrow \dots$ ,  
 s.t.  $\exists i: \llbracket v_i \rrbracket \models \phi_2 \wedge \forall k < i: \llbracket v_k \rrbracket \models \phi_1$

---

We make the following observations:

- (1)  $\rho$  is fair iff  $\rho^i$  is fair for all  $i \geq 0$ .
- (2)  $\rho$  is fair iff there is a fair suffix  $\rho^i$  for some  $i \geq 0$ .

Thus, we can rewrite  $\mathbf{EU}_f$  as follows:

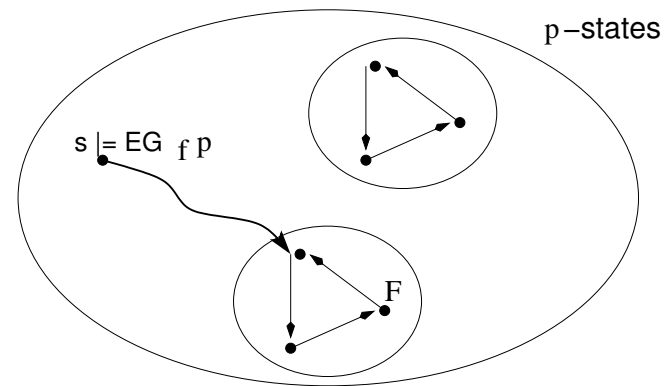
$$\phi_1 \mathbf{EU}_f \phi_2 \equiv \phi_1 \mathbf{EU} (\phi_2 \wedge \mathbf{EG}_f \text{true})$$

It therefore suffices to find a modified algorithm for  $\mathbf{EG}_f$ .

# An algorithm for $[[EG_f p]]_{\mathcal{K}}$

---

1. Let  $\mathcal{K}_p$  be the restriction of  $\mathcal{K}$  to the states  $\mu(p)$ .
2. Compute the SCCs of  $\mathcal{K}_p$ .
3. Find the non-trivial SCCs intersecting all fairness constraints.
4.  $[[EG_f p]]_{\mathcal{K}}$  contains exactly those states in  $\mathcal{K}_p$  from which such an SCC is reachable.



**Complexity:** still linear in  $|\mathcal{K}|$ .

# Comparison of CTL and LTL



---

Many properties can be expressed equivalently in both CTL and LTL, e.g.

**Invariants** (e.g., “ $p$  never holds.”)

$$AG \neg p \quad \text{or} \quad G \neg p$$

**Reactivity** (“Whenever  $p$  happens, eventually  $q$  will happen.”)

$$AG(p \rightarrow AF q) \quad \text{or} \quad G(p \rightarrow F q)$$

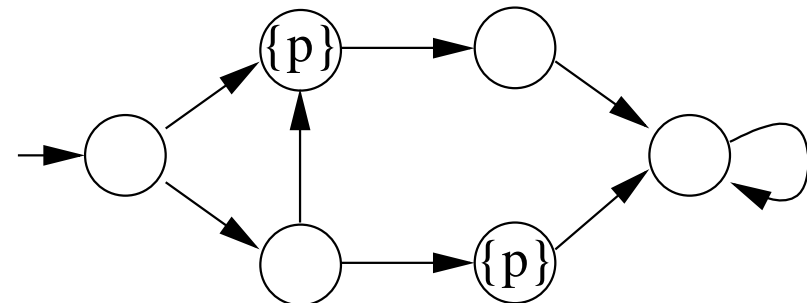
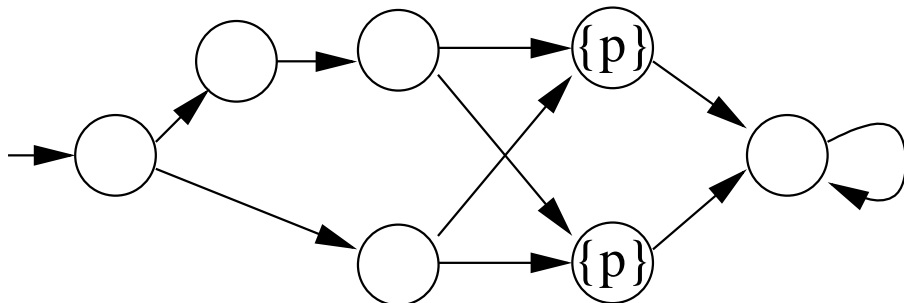
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CTL considers the whole computation tree, LTL the set of runs. Hence, CTL can reason about the branching behaviour (the *possibilities*), which LTL cannot.

Examples:

The CTL formula  $AG\ EF\ p$  (“reset”) is not expressible in LTL.

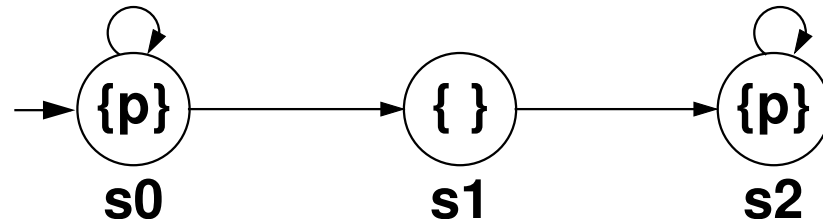
The CTL formula  $AF\ AX\ p$  distinguishes the following structures, but the LTL formula  $FX\ p$  does not:



---

However, the syntactic restriction in CTL (paired quantors and path operators) means that, in turn, some LTL properties are inexpressible in CTL. Hence, the two logics are *incomparable* in their expressive power.

The LTL property  $\mathbf{F G p}$  is inexpressible in CTL:



$$\mathcal{K} \models \mathbf{F G p} \quad \text{but} \quad \mathcal{K} \not\models \mathbf{A F A G p}$$

(Emerson, Halpern 1986)

# Computational complexity

---

CTL model checking:  $\mathcal{O}(|\mathcal{K}| \cdot |\phi|)$

LTL model checking:  $\mathcal{O}(|\mathcal{K}| \cdot 2^{|\phi|})$

Does this mean that CTL model checking is more efficient?  
(for properties expressible in both logics)

Answer: **not necessarily**

LTL enables on-the-fly model checking and partial-order reduction

CTL: whole structure must be constructed and traversed multiple times

CTL: instead of p.o.reduction  $\rightarrow$  efficient data structures for sets (to be done)

# Literature

---

More about this topic:

M. Vardi, *Branching vs. Linear Time: Final Showdown*, 2001

Specification Patterns Database:

<http://patterns.projects.cis.ksu.edu/>

# Tool demonstration: SMV

# The SMV system

---

SMV was designed by [Ken McMillan](#) (CMU, 1992)

Model-checking tool for **CTL** (with fairness)

Useful for describing **finite** structures, especially synchronous or asynchronous circuits.

SMV was the first tool suitable for verifying large hardware systems (by employing **BDDs**).

# Information/downloading SMV

---

In the world-wide web:

`http://www-2.cs.cmu.edu/~modelcheck/smv.html`

(includes extensive manual)



# CTL model checking in SMV

---

The problem: Given  $\mathcal{K}$  (with multiple initial states) and  $\phi$ , do all initial states of  $\mathcal{K}$  satisfy  $\phi$ ?

In SMV, structures consist of “modules” and “processes”, that manipulate a number of variables.

Transitions are specified by declaring their ‘next’ value, depending on their current value.

Atomic propositions may talk about variables and are interpreted ‘naturally’ (as in Spin).

# Syntax example (1/3)

---

```
-- This is a comment.
```

```
MODULE main
```

```
VAR
```

```
    x : boolean;
```

```
    y : {q1, q2};
```

```
-- to be continued
```

Remarks:

- Data types: boolean, integer, enumerations
- all variable with *finite* range

## Syntax example (2/3)

---

ASSIGN

```
init(x) := 1;           -- initial value
next(x) := case        -- transition relation
    x: 0;
    !x: 1;
esac;

next(y) := case
    x & y=q1: q2;
    x & y=q2: {q1, q2}; -- non-determinism
    1      : y;
esac;
```

# Syntax example: Remarks

---

Initial states are given by the `init` predicates; uninitialized variables can take any initial value.

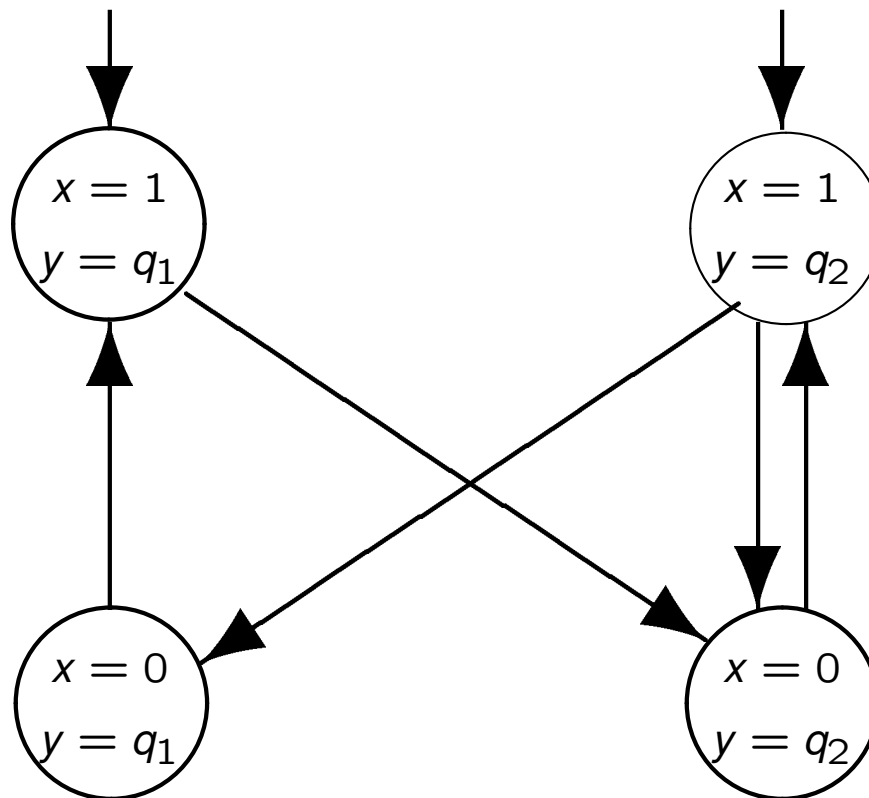
The transitions of the system the `synchronous` composition of the `next` predicates.

`case` expressions are evaluated top to bottom, the first case that fits is taken.

Non-determinism can be introduced by giving multiple successor values.

# The resulting structure

---



## Syntax example (3/3)

---

-- Is q1 always reachable from q2?

```
SPEC AG (y=q2 -> EF y=q1)
```

-- Is x true infinitely often in every execution?

```
SPEC AG AF x
```

Remarks:

- One can give multiple CTL formulae, SMV will check them all one by one.

**Demonstration:** xy.smv

# Modules (1/2)

---

Modules may be **parametrized**. Example:

```
MODULE counter_cell(carry_in)
VAR
    value : boolean;
ASSIGN
    init(value) := 0;
    next(value) := (value + carry_in) mod 2;
DEFINE
    carry_out := value & carry_in;
```

Remark:

- The parameter of this module is `carry_in`.
- `DEFINE` declares a 'macro'.

## Modules (2/2)

---

Modules may be **instantiated** like variables:

```
MODULE main
VAR
  bit0 : counter_cell(1);
  bit1 : counter_cell(bit0.carry_out);
  bit2 : counter_cell(bit1.carry_out);
```

This system behaves like a three-bit counter, i.e. “counts” from 0 to 7 and then resets.

**Remark:** There must be one module named `main`, and SMV will evaluate the specifications of this module.

**Demonstration:** counter.smv



# Asynchronous systems

---

All examples up to now were **synchronous**, i.e. all variables and modules take a transition *at the same time*.

When modules are instantiated with the keyword `process` (see next example), then in each step one process makes a step while the others do nothing (*asynchronous* composition, interleaving).

Alternatively, no process may make a step (“stuttering”).

# Example: Mutex

---

**var** turn : {0,1};

**while true do**

$q_0$  non-critical section

$q_1$  **await** (turn=0);

$q_2$  critical section

$q_3$  turn:=1;

**od**

**while true do**

$r_0$  non-critical section

$r_1$  **await** (turn=1);

$r_2$  critical section

$r_3$  turn:=0;

**od**

# Mutex in SMV (1/2)

---

```
MODULE main
```

```
VAR
```

```
  turn: boolean;
```

```
  p0: process p(0,turn);
```

```
  p1: process p(1,turn);
```

```
SPEC
```

```
  AG !(p0.state = critical & p1.state = critical)
```

## Mutex in SMV (2/2)

---

```
MODULE p (nr,turn)
VAR
  state: {non_critical, critical};
ASSIGN
  init(state) := non_critical;
  next(state) := case
    state = non_critical & turn != nr: non_critical;
    state = non_critical & turn = nr : critical;
    state = critical: {critical,non_critical};
  esac;
  next(turn) := case
    state = critical & next(state) = non_critical: !nr;
    1 : turn;
  esac;
```

# Fairness

---

In the mutex example, the following specification is evaluated to false.

SPEC

```
AG (p0.state = non_critical -> AF p0.state = critical)
```

This is because SMV allows the system to “stutter” forever (i.e. to do nothing).

One can exclude such behaviours using the keyword `FAIRNESS`, e.g. as follows:

FAIRNESS

```
p0.running & p1.running
```

The internal variable `running` becomes true whenever the corresponding process makes a step. With this addition, the verification succeeds.

# Part 12: Binary Decision Diagrams

# Literature

---

Some pointers:

H.R. Andersen, *An Introduction to Binary Decision Diagrams*, Lecture notes,  
Department of Information Technology, IT University of Copenhagen

URL: <http://www.itu.dk/people/hra/bdd97-abstract.html>

Tools:

DDcal (“BDD calculator”)

URL: <http://vlsi.colorado.edu/~fabio/>

# Set representations

---

As we have seen, the solution of the model-checking problem for CTL can be expressed by operations on sets:

states satisfying some atomic proposition:  $\mu(p)$  for  $p \in AP$

states satisfying (sub)formulae:  $\llbracket \psi \rrbracket_{\mathcal{K}}$

computation by set operations:  $pre, \cap, \cup, \dots$

How can such sets be represented:

**explicit list:**  $S = \{s_1, s_2, s_4, \dots\}$

**symbolic representation:** compact notation or data structure



# Symbolic representation

---

There are many ways of representing sets symbolically. Some are commonly used in mathematics:

**Intervals:**  $[1, 10]$  for  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

**Characterizations:** “odd numbers” for  $\{1, 3, 5, \dots\}$

Every symbolic representation is suitable for some sets and less so for others (for instance, intervals for odd numbers).

We are interested in a data structure suitable for representing sets of states in hardware systems, and where the necessary operations (*pre*,  $\cap$  etc) can be implemented efficiently.

---

In the following, we assume that states can be represented as Boolean vectors

$$S = \{0, 1\}^m \quad \text{for some } m \geq 1$$

Example:

1-safe Petri nets (every place marked with at most one token)

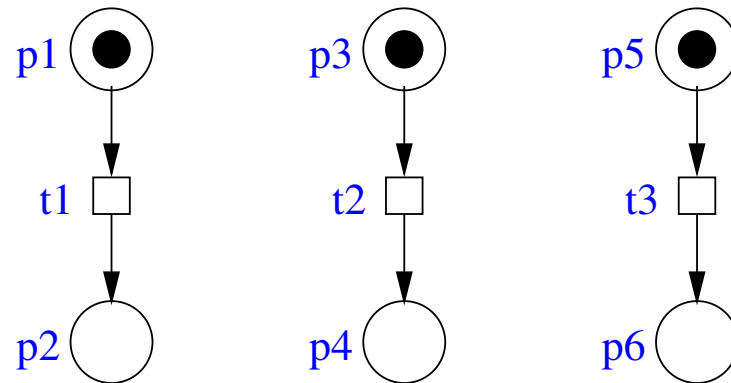
circuits (all inputs and outputs are 0 or 1)

Remark: In general, the elements of *any* finite set can be represented by Boolean vectors if  $m$  is chosen large enough. (However, this may not be adequate for all sets.)

# Example 1: Petri-net

---

Consider the following Petri net:



A state can be written as  $(p_1, p_2, \dots, p_6)$ , where  $p_i$ ,  $1 \leq i \leq 6$  indicates whether there is a token on  $P_i$ .

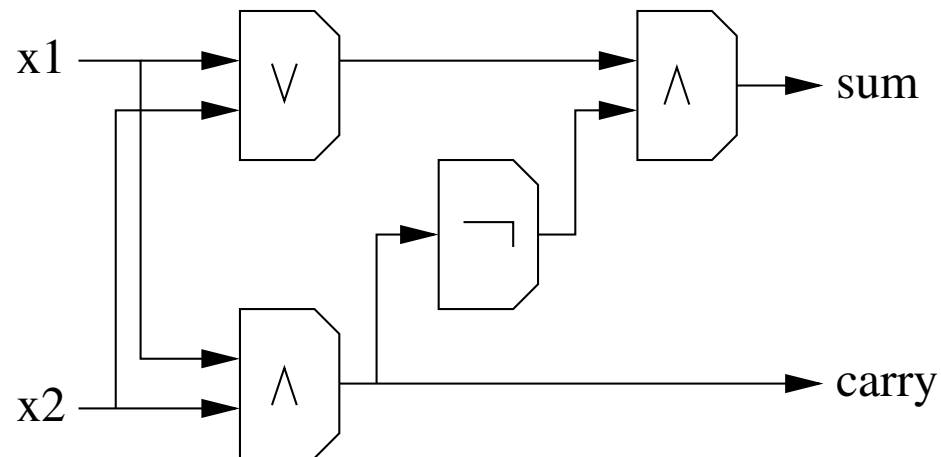
Initial state  $(1, 0, 1, 0, 1, 0)$ ;

other reachable states are, e.g.,  $(0, 1, 1, 0, 1, 0)$  or  $(1, 0, 0, 1, 0, 1)$ .

## Example 2: Circuit

---

Half-adder:



The circuit has got two inputs ( $x_1, x_2$ ) and two outputs ( $carry, sum$ ). Their admissible combinations can be denoted by Boolean 4-tuples, e.g.  $(1, 0, 0, 1)$  ( $x_1 = 1, x_2 = 0, carry = 0, sum = 1$ ) is a possible combination.

# Characteristic functions

---

Let  $C \subseteq S = \{0, 1\}^m$ .

(i.e., a set of Boolean vectors.)

The set  $C$  is uniquely defined by its **characteristic function**  $f_C: S \rightarrow \{0, 1\}$  given by

$$f_C(s) := \begin{cases} 1 & \text{if } s \in C \\ 0 & \text{if } s \notin C \end{cases}$$

Remark:  $f_C$  is a Boolean function with  $m$  inputs and therefore corresponds to a formula  $F$  of propositional logic with  $m$  atomic propositions.

---

The characteristic function of the admissible combinations in Example 2 corresponds to the following formula of propositional logic:

$$F \equiv \left( \textit{carry} \leftrightarrow (x_1 \wedge x_2) \right) \wedge \left( \textit{sum} \leftrightarrow (x_1 \vee x_2) \wedge \neg \textit{carry} \right)$$

In the following, we shall treat

sets of states (i.e. sets of Boolean vectors)

characteristic functions

Formulae of propositional logic

simply as different representations of the same objects.

# Representing formulae

---

Truth table:

$x_1$	$x_2$	$carry$	$sum$	$F$
0	0	0	0	1
0	0	0	1	0
		...		
0	1	0	1	1
		...		

A truth table is obviously *not* a compact representation.

However, we use it as a starting point for a graphical, more compact representation.

# Binary decision graphs

---

Let  $V$  be a set of variables (atomic propositions) and  $<$  a total order on  $V$ , e.g.

$$x_1 < x_2 < \textit{carry} < \textit{sum}$$

A **binary decision graph** (w.r.t.  $<$ ) is a directed, connected, acyclic graph with the following properties:

there is exactly one **root**, i.e. a node without incoming arcs;

there are at most two leaves, labelled by **0** or **1**;

all non-leaves are labelled with variables from  $V$ ;

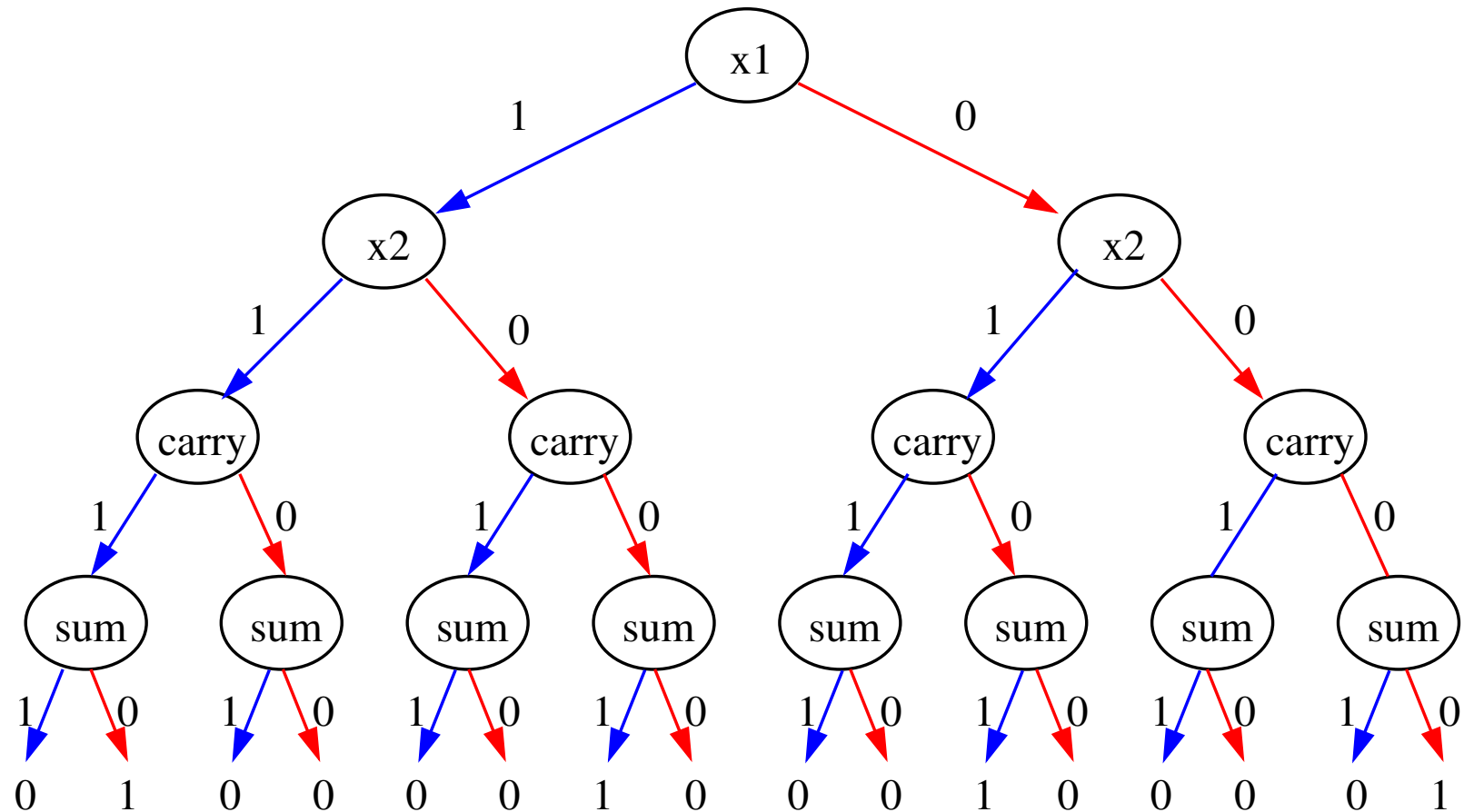
every non-leaf has two outgoing arcs labelled by **0** and **1**;

if there is an edge from an  $x$ -labelled node to a  $y$ -labelled node, then  $x < y$ .



## Example 2: Binary decision graph (here: a full tree)

---



Paths ending in **1** correspond to vectors whose entry in the truth table is 1.

# Binary decision diagrams

---

A **binary decision diagram** (BDD) is a binary decision graph with two additional properties:

no two subgraphs are isomorphic;

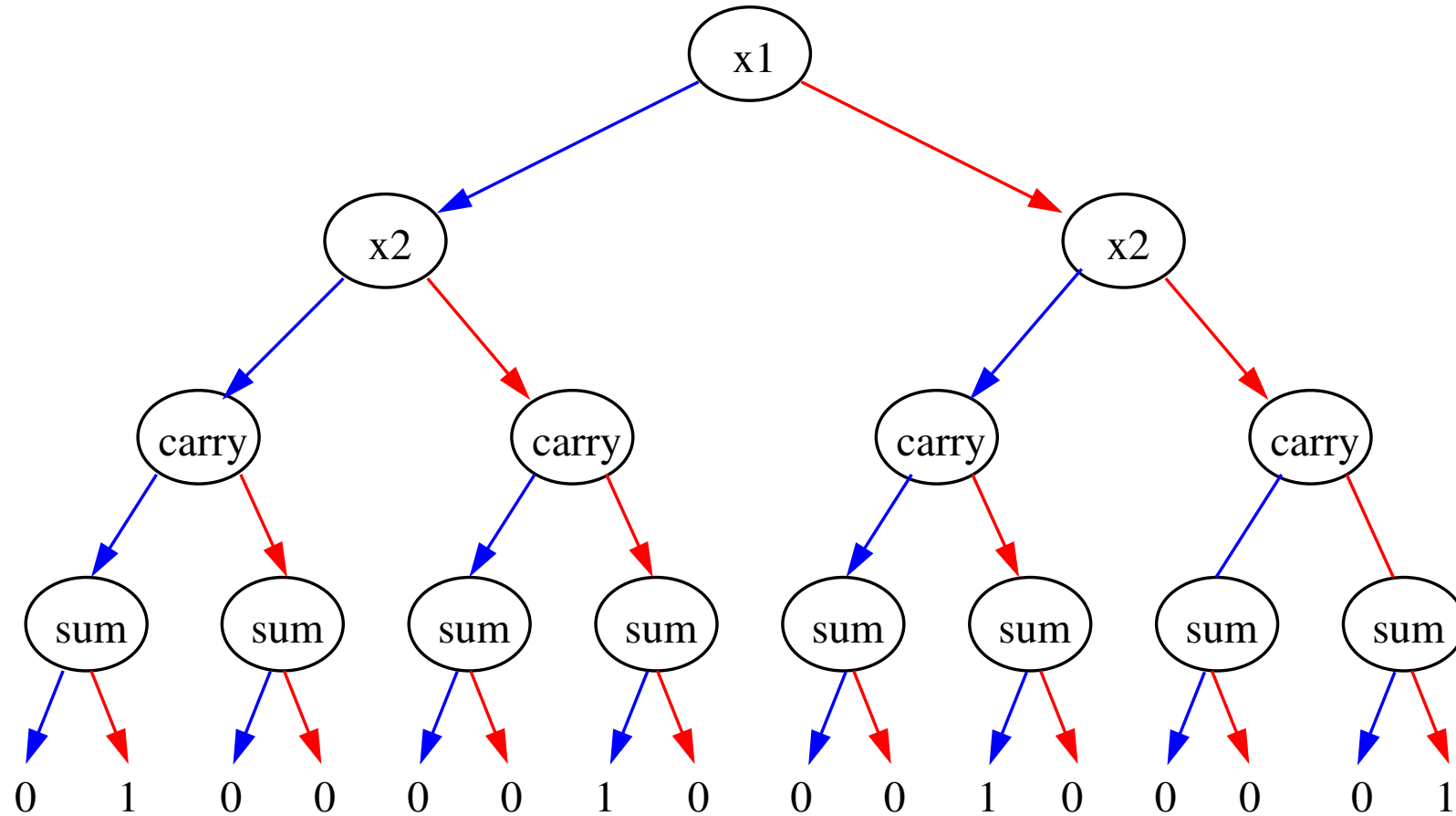
there are no *redundant* nodes, where both outgoing edges lead to the same target node.

We also allow to omit the **0**-node and the edges leading there.

Remarks: On the following slides, the **blue** edges are meant to be labelled by 1, the **red** edges by 0.

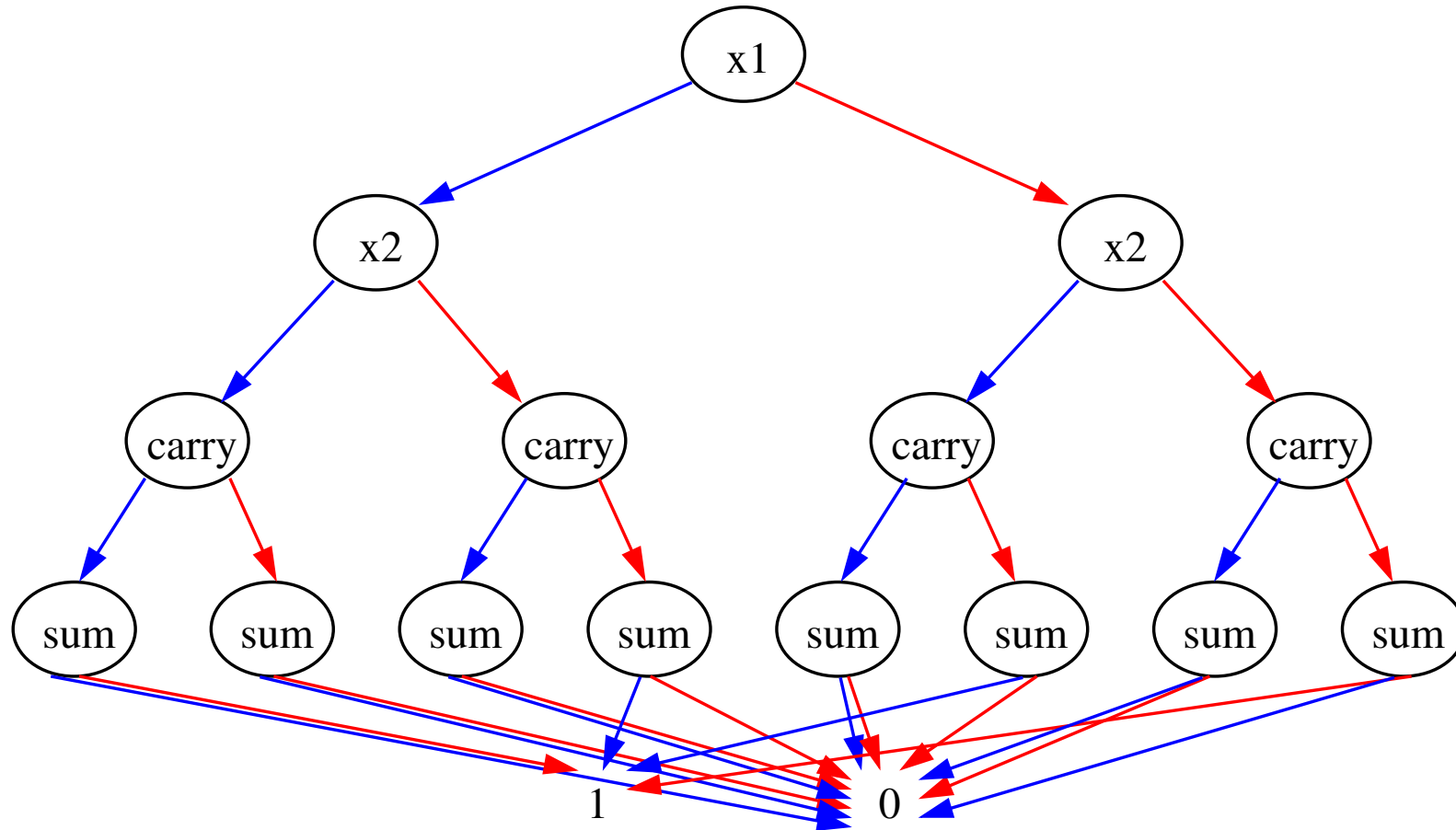
## Example 2: Eliminate isomorphic subgraphs (1/4)

---



## Example 2: Eliminate isomorphic subgraphs (2/4)

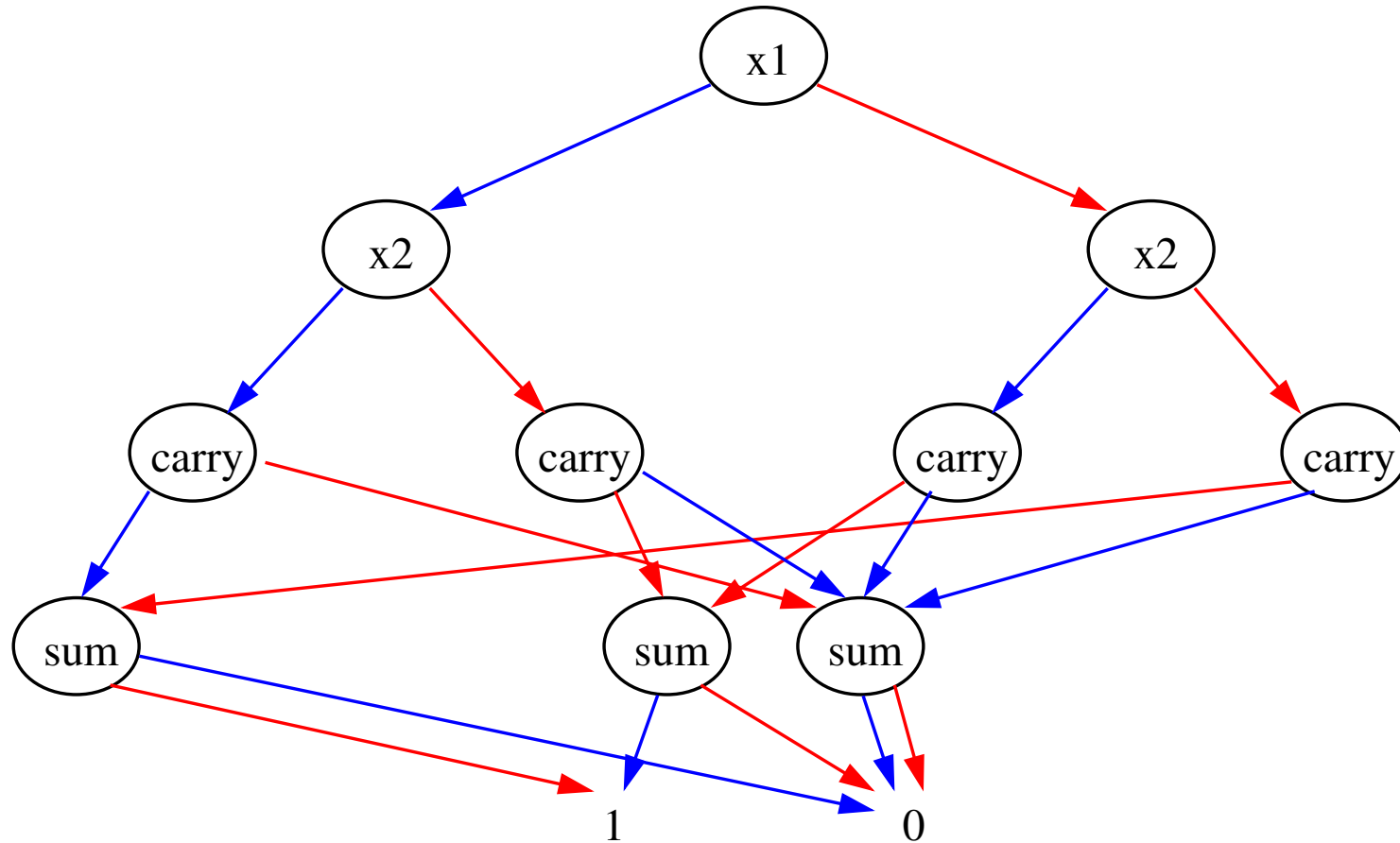
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0-nodes and 1-nodes merged, respectively.

## Example 2: Eliminate isomorphic subgraphs (3/4)

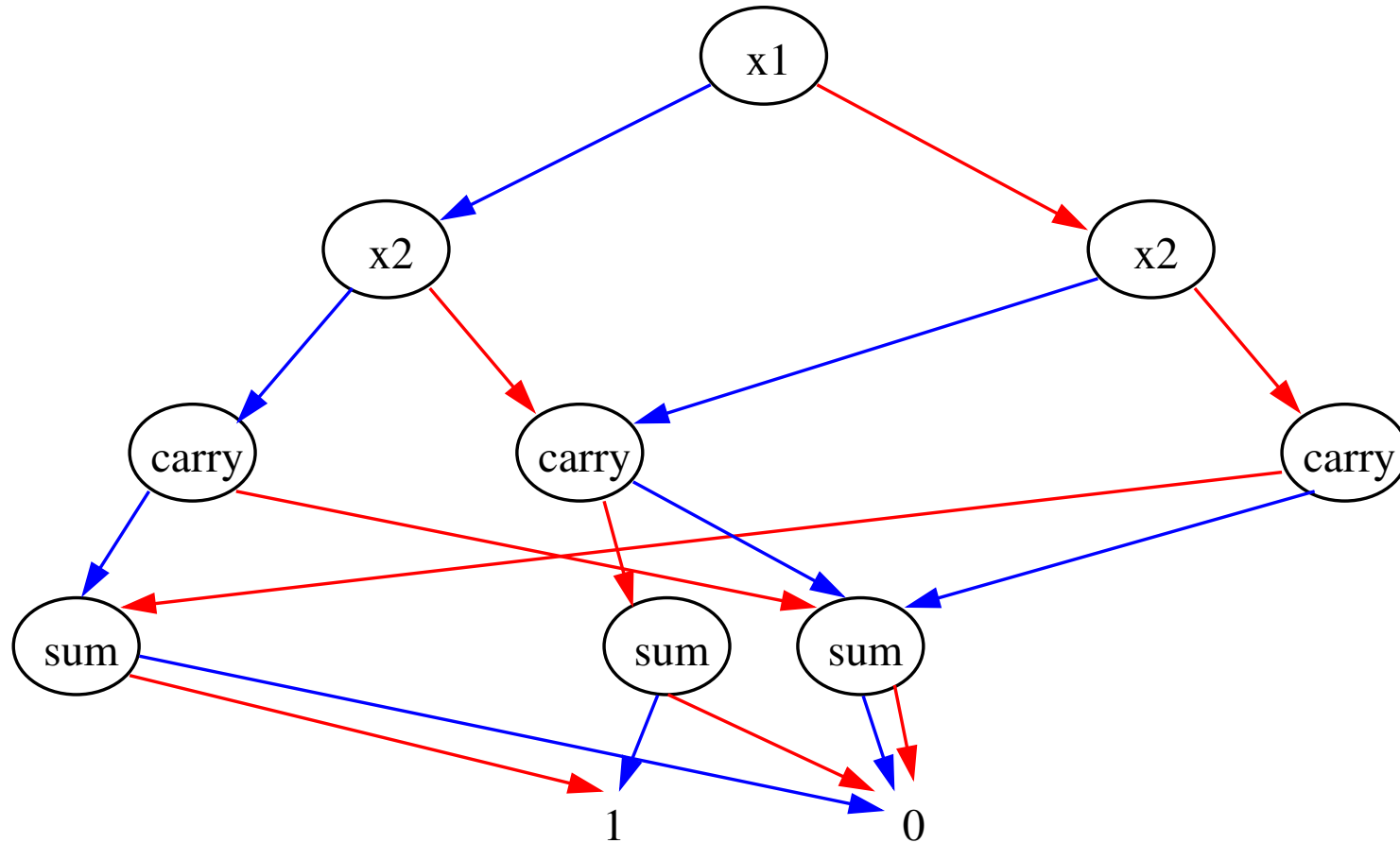
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Merged the isomorphic *sum*-nodes.

## Example 2: Eliminate isomorphic subgraphs (4/4)

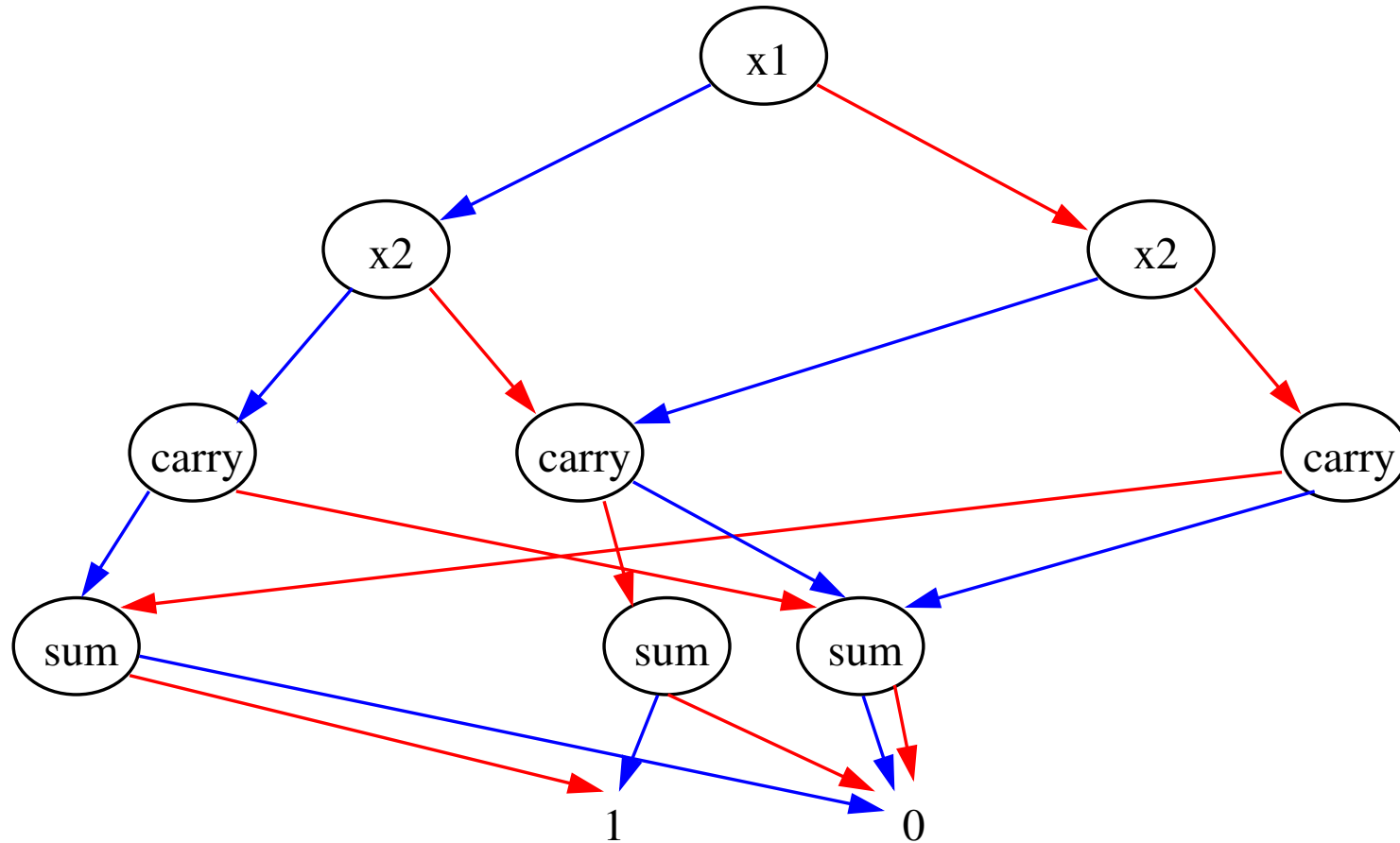
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No isomorphic subgraphs are left → we are done.

## Example 2: Remove redundant nodes (1/2)

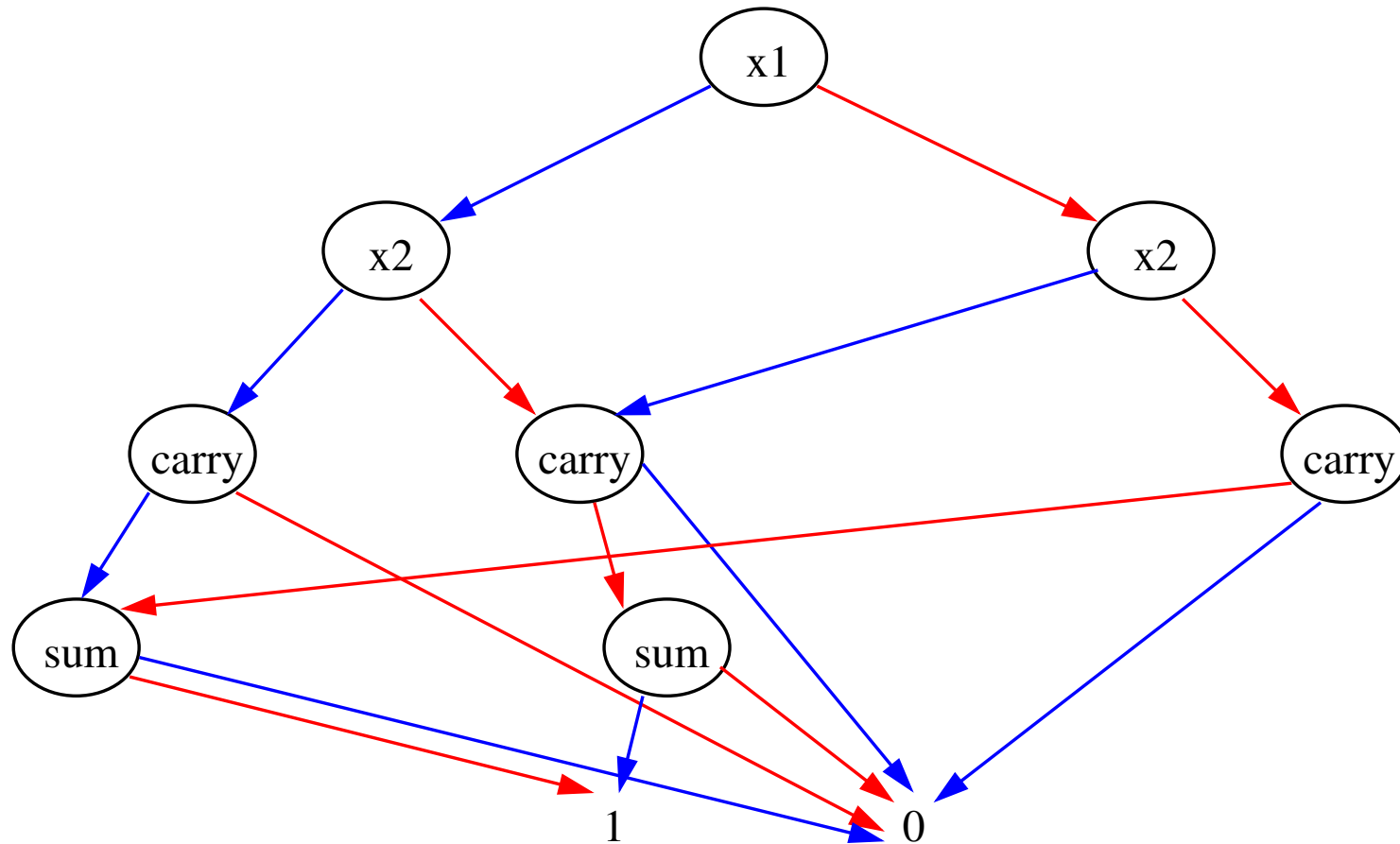
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Both edges of the right *sum*-node point to 0.

## Example 2: Remove redundant nodes (2/2)

---

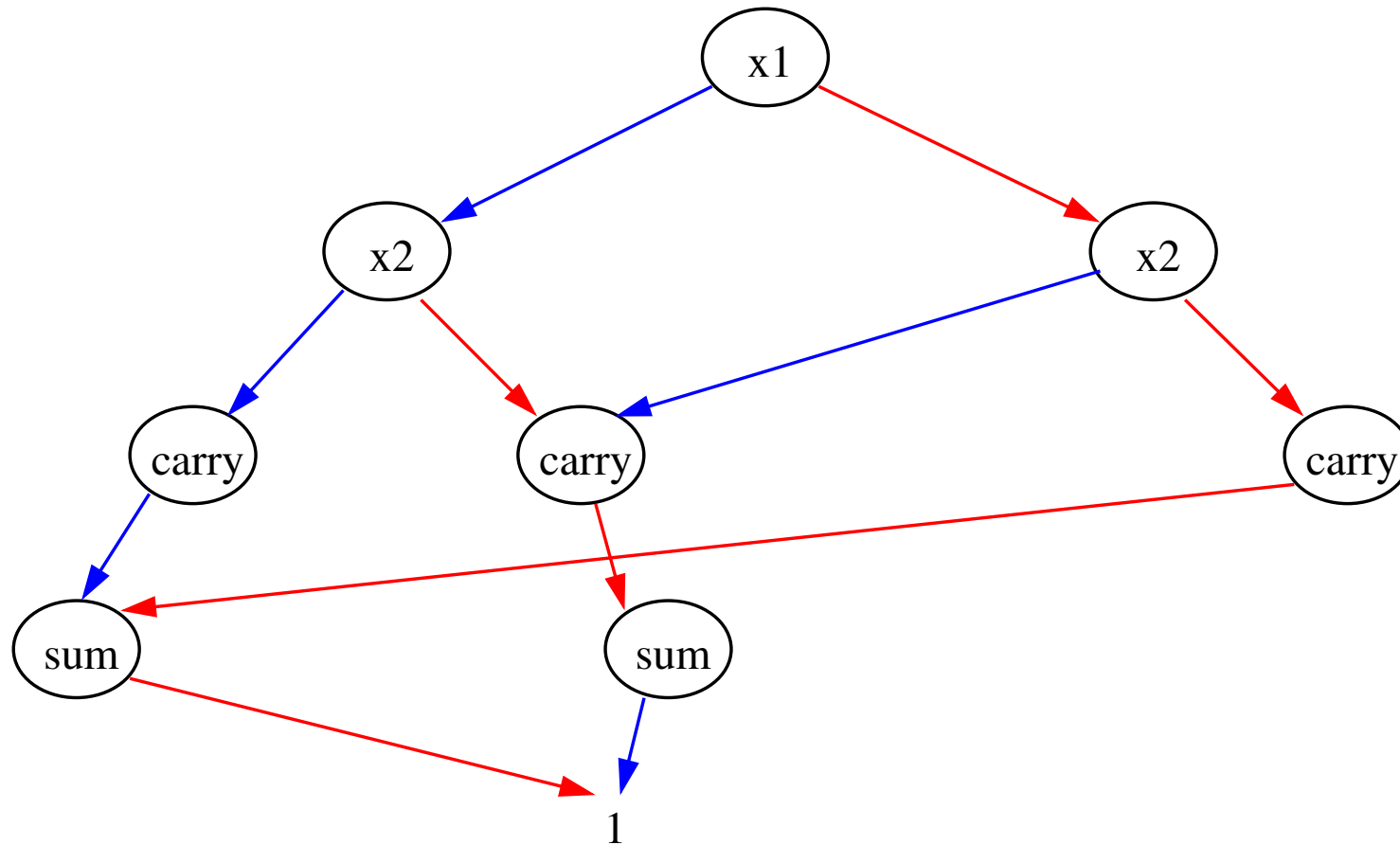


No more redundant nodes → we are done.



## Example 2: Omit 0-node

---



Optionally, we can remove the 0-node and edges leading to it, which makes the representation clearer.

# Semantics of a BDD

---

Let  $B$  be a BDD with order  $x_1 < \dots < x_n$ . Let  $P_B$  be the set of paths in  $B$ , leading from the root to the (unique)  $\mathbf{1}$ -node.

Let  $p \in P_B$  be such a path, e.g.  $x_{i_1} \xrightarrow{b_{i_1}} \dots \xrightarrow{b_{i_m}} \mathbf{1}$ . Then the “meaning” of  $p$  (denoted  $\llbracket p \rrbracket$ ) is the conjunction of all  $x_{i_j}$  with  $b_{i_j} = 1$  and all  $\neg x_{i_j}$  with  $b_{i_j} = 0$  (where  $j = 1, \dots, m$ ).

We then say that  $B$  represents the following PL formula:

$$F = \bigvee_{p \in P_B} \llbracket p \rrbracket$$

# Preview

---

In the following, we shall investigate operations on BDDs that are needed for CTL model checking.

Construction of a BDD (from a PL formula)

Equivalence check

Intersection, complement, union

Relations, computing predecessors

# Propositional logic with constants

---

In the following, we will consider formulae of propositional logic (PL), extended with the constants **0** and **1**, where:

**0** is an unsatisfiable formula;

**1** is a tautology.

# Substitution

---

Let  $F$  and  $G$  be formulae of PL (possibly with constants), and let  $x$  be an atomic proposition.

$F[x/G]$  denotes the PL formula obtained by replacing each occurrence of  $x$  in  $F$  by  $G$ .

In particular, we will consider formulae of the form  $F[x/0]$  and  $F[x/1]$ .

**Example:** Let  $F = x \wedge y$ . Then  $F[x/1] = 1 \wedge y \equiv y$  and  $F[x/0] = 0 \wedge y \equiv 0$ .

# If-then-else

---

Let us introduce a new, ternary PL operator. We shall call it *ite* (if-then-else).

Note: *ite* does not extend the expressiveness of PL, it is simply a convenient shorthand notation.

Let  $F, G, H$  be PL formulae. We define

$$ite(F, G, H) := (F \wedge G) \vee (\neg F \wedge H).$$

The set of **INF formulae** (if-then-else normal form) is inductively defined as follows:

**0** and **1** are INF formulae;

if  $x$  is an atomic proposition and  $G, H$  are INF formulae, then  $ite(x, G, H)$  is an INF formula.

# Shannon partitioning

---

Let  $F$  be a PL formula and  $x$  an atomic proposition. We have:

$$F \equiv \text{ite}(x, F[x/1], F[x/0])$$

**Proof:** In the following,  $G$  denotes the right-hand side of the equivalence above. Let  $\nu$  be a valuation s.t.  $\nu \models F$ . Either  $\nu(x) = 1$ , then  $\nu$  is also a model of  $F[x/1]$  and of  $x$  and therefore also of  $G$ . The case  $\nu(x) = 0$  is analogous. For the other direction, suppose  $\nu \models G$ . Then either  $\nu(x) = 1$  and the “rest” of  $\nu$  is a model of  $F[x/1]$ . Then, however,  $\nu$  will be a model for any formula in which some of the ones in  $F[x/1]$  are replaced by  $x$ , in particular also for  $F$ . The case  $\nu(x) = 0$  is again analogous.

**Remark:**  $G$  is called the **Shannon partitioning** of  $F$ .

**Corollary:** Every PL formula is equivalent to an INF formula.

(Proof: apply the equivalence above multiple times.)

# Construction of BDDs

---

We can now solve our first BDD-related problem: Given a PL formula  $F$  and some ordering of variables  $<$ , construct a BDD w.r.t.  $<$  that represents  $F$ .

If  $F$  does not contain any atomic propositions at all, then either  $F \equiv 0$  or  $F \equiv 1$ , and the corresponding BDD is simply the corresponding leaf node.

Otherwise, let  $x$  be the smallest variable (w.r.t.  $<$ ) occurring in  $F$ . Construct BDDs  $B_0$  and  $B_1$  for  $F[x/1]$  and  $F[x/0]$ , respectively (these formulae have one variable less than  $F$ ).

Because of the Shannon partitioning,  $F$  is representable by a binary decision *graph* whose root is labelled by  $x$  and whose subtrees are  $B_0$  and  $B_1$ . To obtain a BDD, we check whether  $B_0$  and  $B_1$  are isomorphic; if yes, then  $F$  is represented by  $B_0$ . Otherwise we merge all isomorphic subtrees in  $B_0$  and  $B_1$ .



# Example: BDD construction

---

Consider again the formula from Example 2:

$$F \equiv (\textit{carry} \leftrightarrow (x_1 \wedge x_2)) \wedge (\textit{sum} \leftrightarrow (x_1 \vee x_2) \wedge \neg \textit{carry})$$

We have, e.g.:

$$F[x_1/0] \equiv \neg \textit{carry} \wedge (\textit{sum} \leftrightarrow x_2)$$

$$F[x_1/1] \equiv (\textit{carry} \leftrightarrow x_2) \wedge (\textit{sum} \leftrightarrow \neg \textit{carry})$$

$$F[x_1/0][x_2/0] \equiv \neg \textit{carry} \wedge \neg \textit{sum}$$

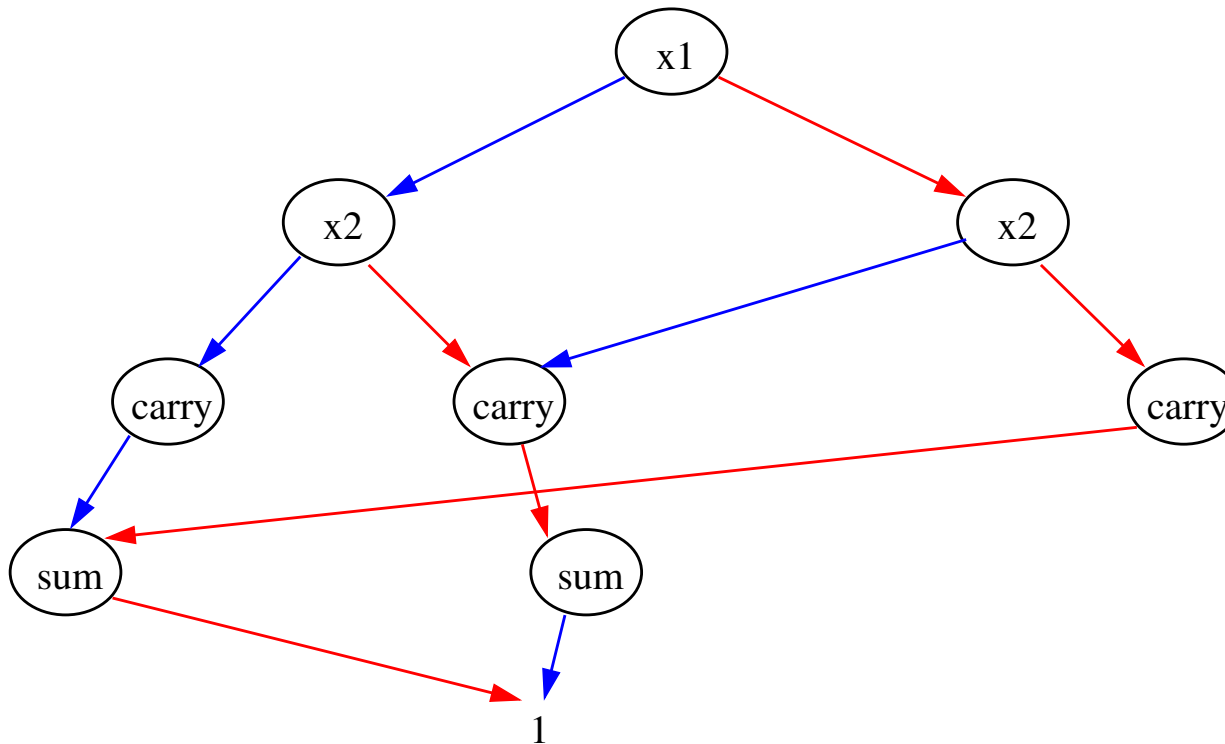
$$F[x_1/0][x_2/1] \equiv F[x_1/1][x_2/0] \equiv \neg \textit{carry} \wedge \textit{sum}$$

$$F[x_1/1][x_2/1] \equiv \textit{carry} \wedge \textit{sum}$$

# Example: BDD construction

---

By applying the construction, we obtain the same BDD as before:



**Remark:** Obviously, we can also obtain an INF formula from each BDD.

# BDDs are unique

---

**Remark:** The result of the previously given construction is *unique* (up to isomorphism).

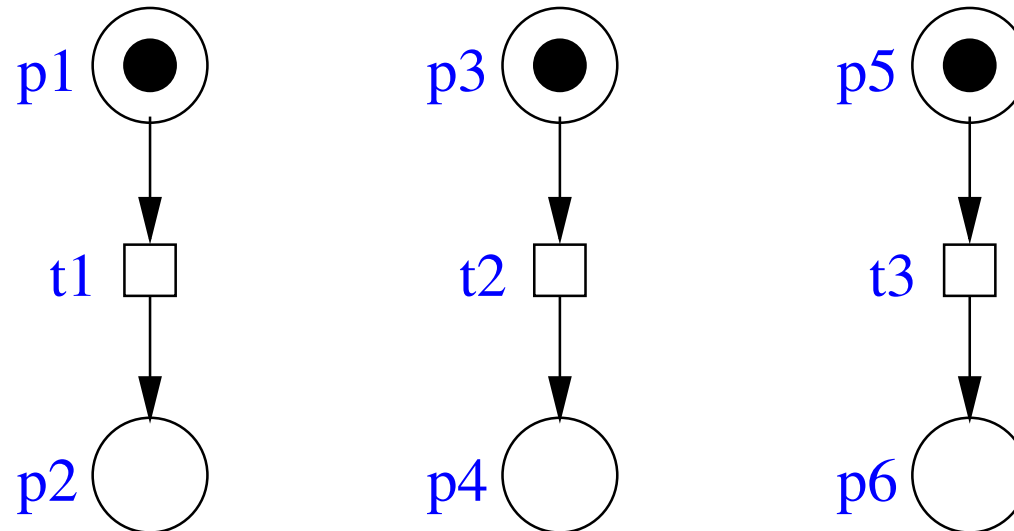
In other words, given  $F$  and  $<$ , there is (up to isomorphism) exactly one BDD that respects  $<$  and represents  $F$ .

Remark: Different orderings still lead to different BDDs.  
(possibly with vastly different sizes!)

# Example: Variable orderings

---

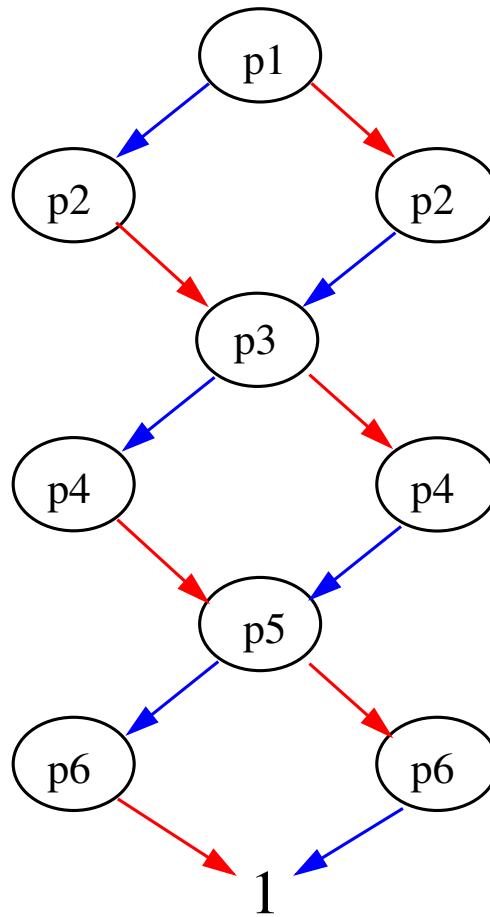
Recall Example 1 (the Petri net), and let us construct a BDD representing the reachable markings:



Remark:  $P_1$  is marked iff  $P_2$  is not, etc.

---

The corresponding BDD for the ordering  $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$ :



---

Remarks:

If we increase the number of components from 3 to  $n$  (for some  $n \geq 0$ ), the size of the corresponding BDD will be linear in  $n$ .

In other words, a BDD of size  $n$  can represent  $2^n$  (or even more) valuations.

However, the size of a BDD strongly depends on the ordering!

Example: Repeat the previous construction for the ordering

$$p_1 < p_3 < p_5 < p_2 < p_4 < p_6.$$

# Equivalence test

---

To implement CTL model checking, we need a test for equivalence between BDDs (e.g., to check the termination of a fixed-point computation).

**Problem:** Given BDDs  $B$  and  $C$  (w.r.t. the same ordering) do  $B$  and  $C$  represent equivalent formulae?

**Solution:** Test whether  $B$  and  $C$  are isomorphic.

**Special cases:**

Unsatisfiability test: Check if the BDD consists just of the  $0$  leaf.

Tautology test: Check if the BDD consists just of the  $1$  leaf.

# Implementing BDDs with hash tables

---

Suppose we want to write an application in which we need to manipulate multiple BDDs.

Efficient BDDs implementations exploit the uniqueness property by storing all BDD nodes in a hash table. (Recall that each node is in fact the root of some BDD.)

Each BDD is then simply represented by a pointer to its root.

Initially, the hash table has only two unique entries, the leaves **0** and **1**.



---

Every other node is uniquely identified by the triple  $(x, B_0, B_1)$ , where  $x$  is the atomic proposition labelling that node and  $B_0, B_1$  are the subtrees of that node, represented by pointers to their respective roots.

Usually, one implements a function  $mk(x, B_0, B_1)$  that checks whether the hash table already contains such a node; if yes, then the pointer to that node is returned, otherwise a new node is created.

A multitude of BDDs is then stored as a “forest” (a DAG with multiple roots).

Problem: garbage collection (by reference counting)

# Equivalence test II

---

Let us reconsider the equivalence-checking problem.

(Given two BDDs  $B$  and  $C$ , do  $B$  and  $C$  represent equivalent formulae?)

If  $B$  and  $C$  are stored in hash tables (as described previously), then  $B$  and  $C$  are representable by pointers to their roots.

Due to the uniqueness property, one then simply has to check whether the pointers are the same (a **constant-time** procedure).

# Logical operations I: Complement

---

Let  $F$  be a PL formula and  $B$  a BDD representing  $F$ .

**Problem:** Compute a BDD for  $\neg F$ .

**Solution:** Exchange the two leaves of  $B$ .

(Caution: This is not quite so simple with the hash-table implementation.)

## Logical operations II: Disjunction/union

---

Let  $F, G$  be PL formulae and  $B, C$  the corresponding BDDs (with the same ordering).

**Problem:** Compute a BDD for  $F \vee G$  from  $B$  and  $C$ .

We have the following equivalence:

$$F \vee G \equiv \text{ite}(x, (F \vee G)[x/1], (F \vee G)[x/0]) \equiv \text{ite}(x, F[x/1] \vee G[x/1], F[x/0] \vee G[x/0])$$

If  $x$  is the smallest variable occurring in either  $F$  or  $G$ , then

$F[x/1], F[x/0], G[x/1], G[x/0]$  are either the children of the roots of  $B$  and  $C$  (or the roots themselves).

---

We construct a BDD for disjunction according to the following, recursive strategy:

If  $B$  and  $C$  are equal, then return  $B$ .

If either  $B$  or  $C$  are the **1** leaf, then return **1**.

If either  $B$  or  $C$  are the **0** leaf, then return the other BDD.

Otherwise, compare the two variables labelling the roots of  $B$  and  $C$ , and let  $x$  be the smaller among the two (or the one labelling both).

If the root of  $B$  is labelled by  $x$ , then let  $B_1, B_0$  be the subtrees of  $B$ ; otherwise, let  $B_1, B_0 := B$ . We define  $C_1, C_0$  analogously.

Apply the strategy recursively to the pairs  $B_1, C_1$  and  $B_0, C_0$ , yielding BDDs  $E$  and  $F$ . If  $E = F$ , return  $E$ , otherwise  $mk(x, E, F)$ .

# Logical operations III: Intersection

---

Let  $F, G$  be PL formulae and  $B, C$  the corresponding BDDs (with the same ordering).

**Problem:** Compute a BDD for  $F \wedge G$  from  $B$  and  $C$ .

**Solution:** Analogous to union, with the rules for **1** and **0** leaves adapted accordingly.

**Complexity:** With dynamic programming:  $\mathcal{O}(|B| \cdot |C|)$  (every pair of nodes at most once).

# Computing predecessors

---

In the following, we derive a strategy for computing the set

$$pre(M) = \{s \mid \exists s' : (s, s') \in \rightarrow \wedge s' \in M\}.$$

Note that the relation  $\rightarrow$  is a subset of  $S \times S$  whereas  $M \subset S$ .

We represent  $M$  by a BDD with variables  $y_1, \dots, y_m$ .

$\rightarrow$  will be represented by a BDD with variables  $x_1, \dots, x_m$  and  $y_1, \dots, y_m$  (states “before” and “after”).

---

Remark: Every BDD for  $M$  is at the same time a BDD for  $S \times M$ !

Thus, we can rewrite  $pre(M)$  as follows:

$$\{s \mid \exists s' : (s, s') \in \rightarrow \cap (S \times M)\}$$

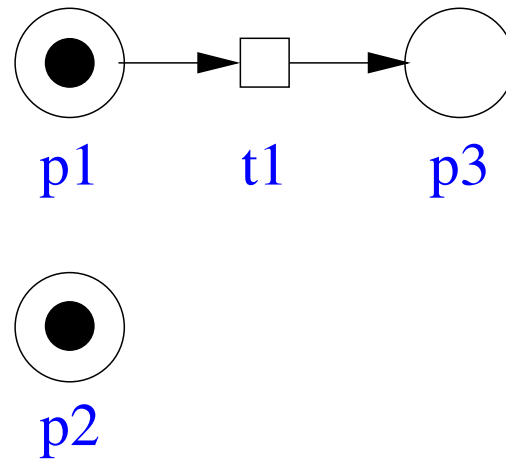
Then,  $pre$  reduces to the operations intersection and existential abstraction.



# Example

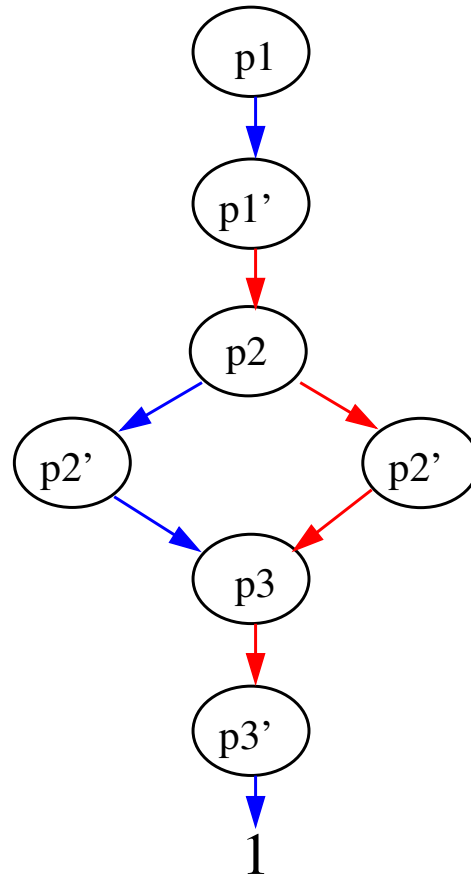
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Let us consider the following Petri net with just one transition:



---

The BDD  $F_{t_1}$  describes the effect of  $t_1$ , where  $p_1, p_2, p_3$  describe the state *before* and  $p'_1, p'_2, p'_3$  the state *after* firing  $t_1$ .



# Existential abstraction

---

Existential abstraction w.r.t. an atomic proposition  $x$  is defined as follows:

$$\exists x : F \equiv F[x/0] \vee F[x/1]$$

I.e.,  $\exists x : F$  is true for those valuations that can be extended with a value for  $x$  in such a way that they become models for  $F$ .

**Example:** Let  $F \equiv (x_1 \wedge x_2) \vee x_3$ . Then

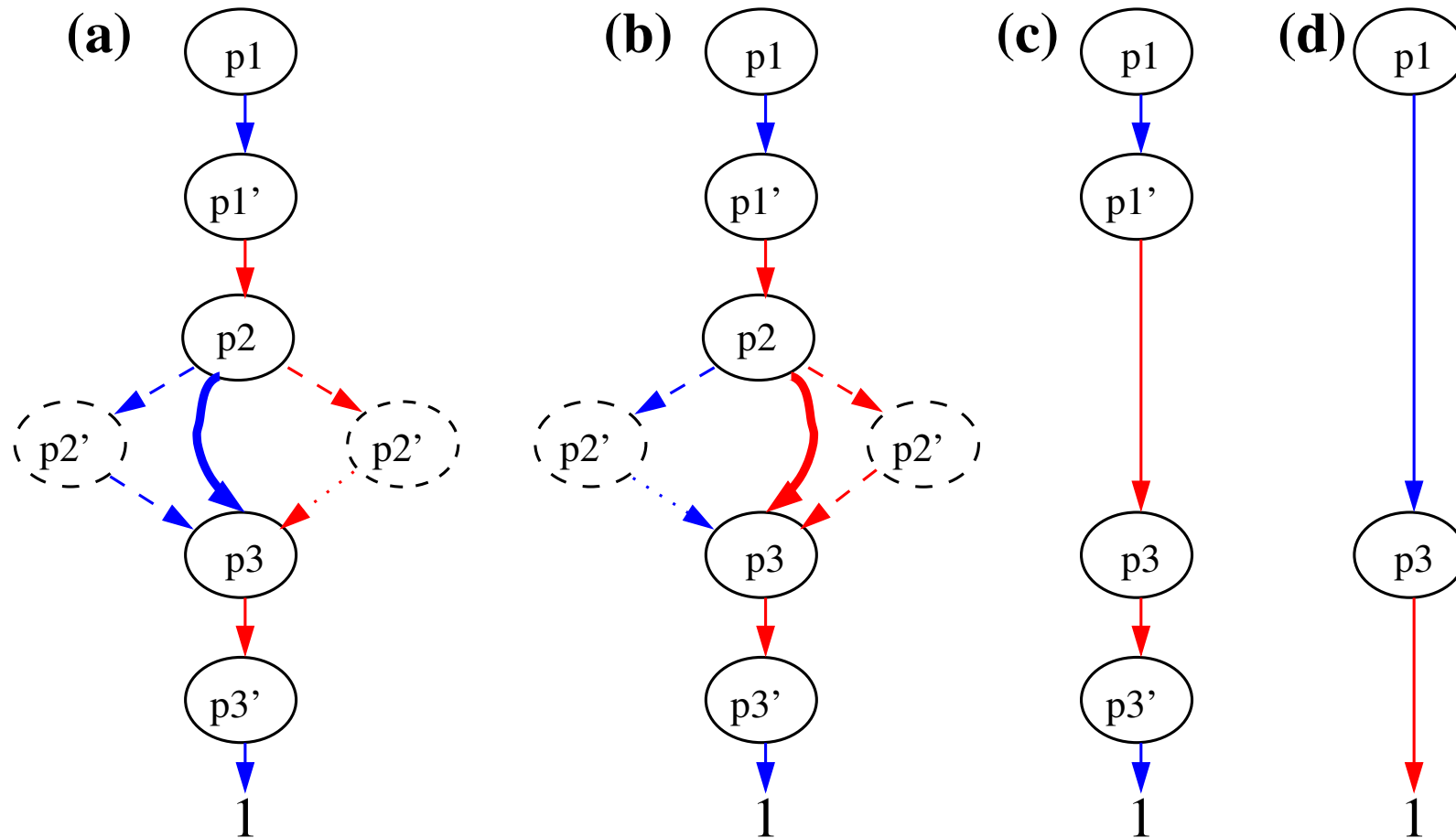
$$\exists x_1 : F \equiv F[x_1/0] \vee F[x_1/1] \equiv (x_3) \vee (x_2 \vee x_3) \equiv x_2 \vee x_3$$

By extension, we can consider existential abstraction over sets of atomic propositions (abstract from each of them in turn).

# Example: Existential abstraction

---

(a)  $F_{t_1}[p'_2/1]$ ; (b)  $F_{t_1}[p'_2/0]$ ; (c)  $\exists p'_2: F_{t_1}$ ; (d)  $\exists p'_1, p'_2, p'_3: F_{t_1}$



# BDDs with complement arcs

---

Implementation with hash tables makes negation a costly operation.

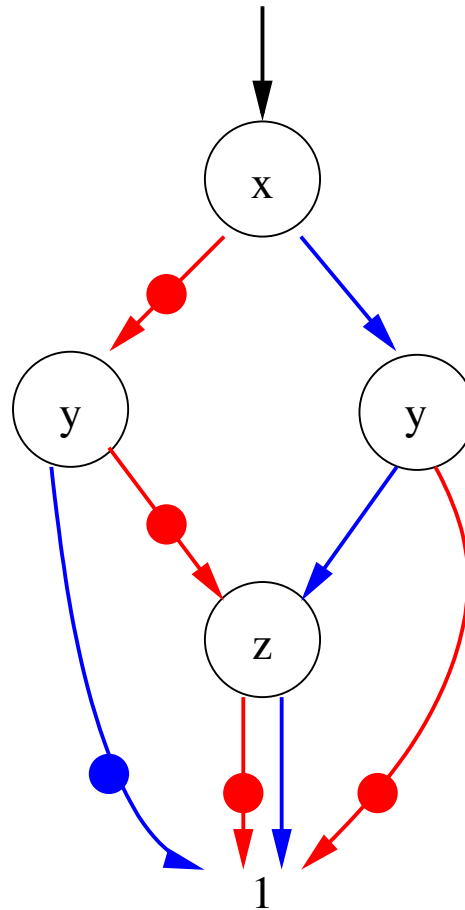
Therefore, BDD libraries often use a modification of BDDs, called **BDDs with complement arcs** (CBDDs).

In a CBDD, every edge is equipped with an additional bit. If the bit is true, then it means that the edge should really lead to the negation of its target.

Representation: if the bit is set, we put a filled circle onto the edge.

# CBDD: Example

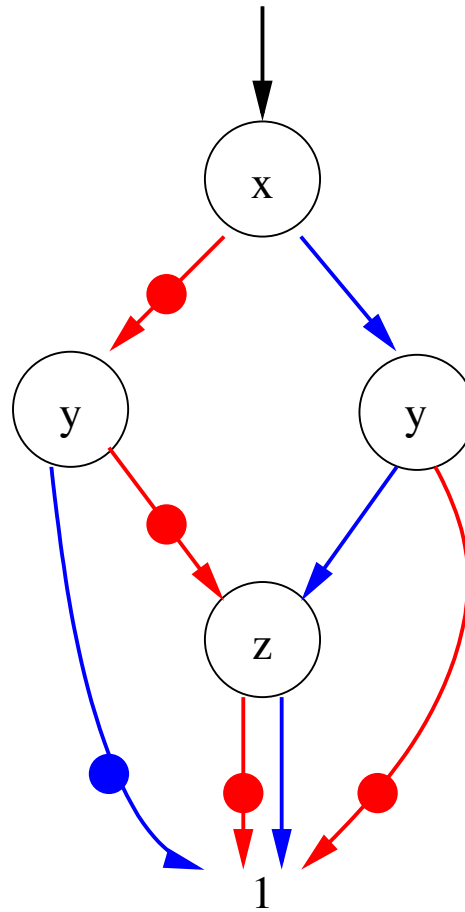
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The red arc leaving the **z**-labelled node has its negation bit set, it therefore effectively leads to **0**.

# CBDD: Example

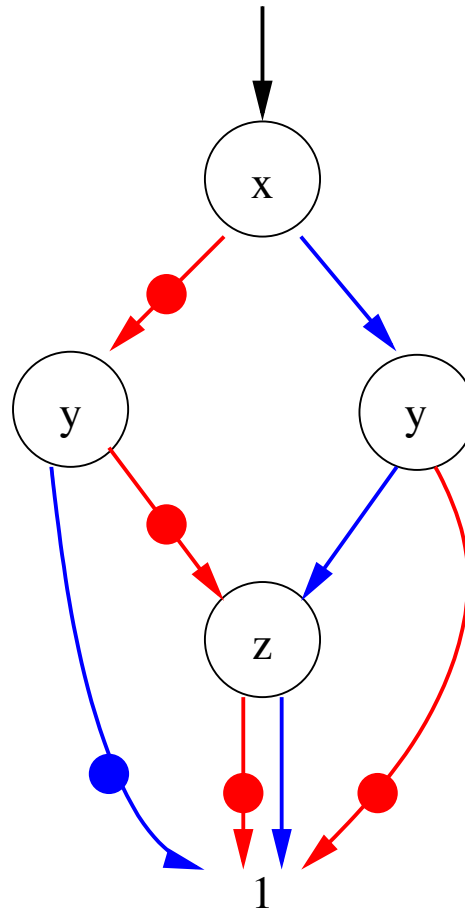
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For this reason, the **z**-labelled node is *not redundant*.  
The **0**-leaf can be omitted altogether.

# CBDD: Example

---

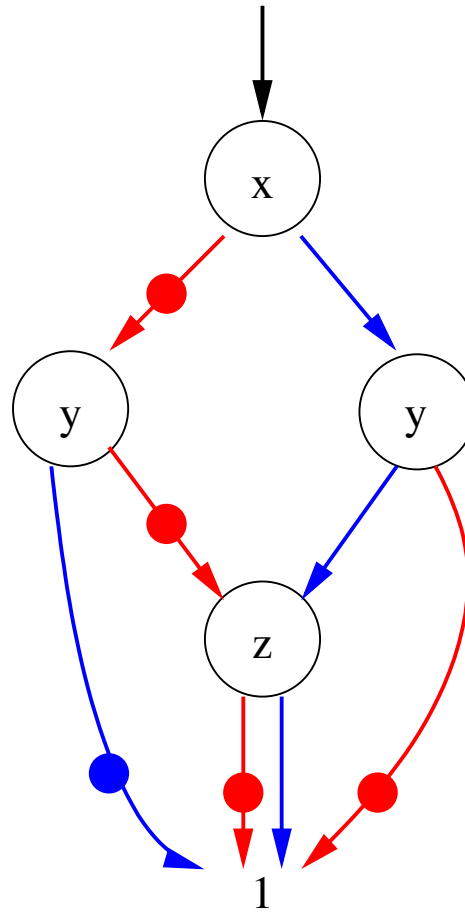


The left  $y$ -labelled node represents the formula  $\neg y \wedge \neg z$ .



# CBDD: Example

---



The pointer to the root is also equipped with a negation bit (false in this case).

# Remarks

---

A valuation  $\nu$  is a model of the formula represented by a CBDD iff the number of negations on the path corresponding to  $\nu$  is *even* (including the pointer to the root).

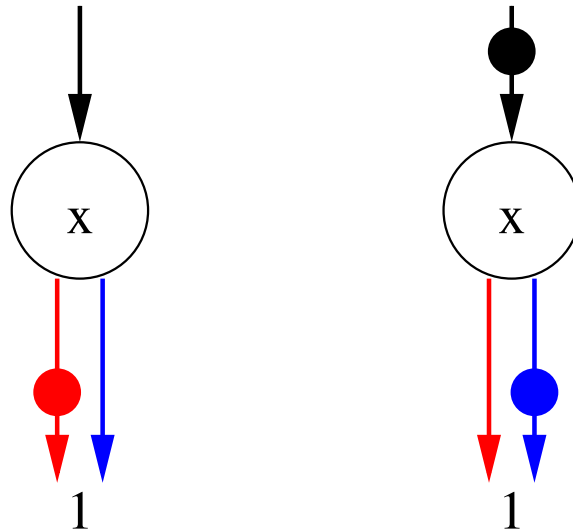
Negation with CBDDs: trivial, invert the negation bit of the pointer to the root (constant-time operation).

Implementation (e.g., in the CUDD library): coded into the least significant bit of the pointer

**Problem:** CBDDs (as presented until now) are not unique!

# CBDDs are not (yet) unique

---



Both of the CBDDs shown above represent the formula  $x$ .

# Canonical form

---

To ensure uniqueness, one can additionally prohibit negation bits on 0-labelled edges.

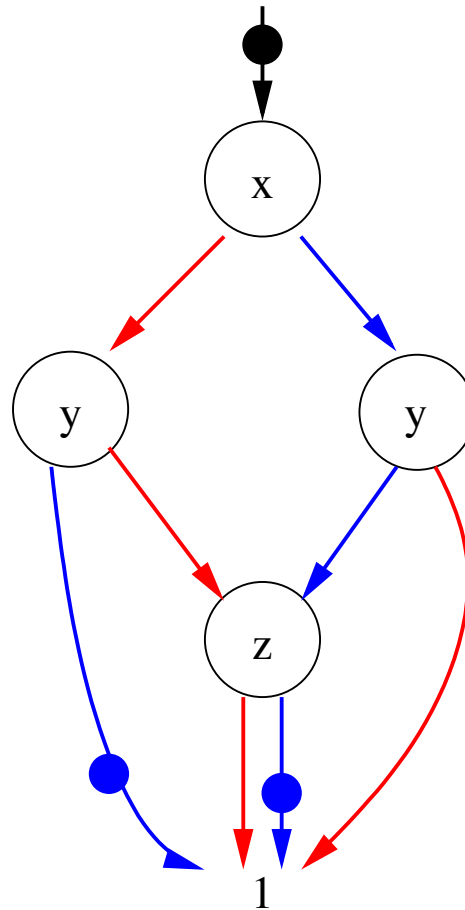
For this, we exploit the following equivalence:

$$ite(x, F, \neg G) \equiv \neg ite(x, \neg F, G)$$

Given any CBDD, one can eliminate negated 0-labelled edges by inverting all the negation bits on those edges that are incident with its source node (starting at the leaves, finishing with the root).

# Canonical form

---



The CBDD shown above represents the same formula as before and does not have any negated **0**-labelled edges.

# LTL with BDDs

---

**Question:** Can one implement also LTL model checking using BDDs?

**Answer:** Yes and no (worst-case: quadratical, but works ok in practice).

**Problems:** BDD not compatible with depth-first search, combination with partial-order reduction difficult.

# Symbolic algorithms for LTL

---

**Idea:** Find non-trivial SCCs with an accepting state, then search backwards for an initial state.

**Algorithms:** Emerson-Lei (EL), OWCTY

# Emerson-Lei (1986)

---

1. Assign to  $M$  the set of all states.
2. Let  $B := M \cap F$ .
3. Compute the set  $C$  of states that can reach elements of  $B$ .
4. Let  $M := M \cap \text{pre}(C)$ .
5. If  $M$  has changed, then go to step 2, otherwise stop.



# One-Way-Catch-Them-Young

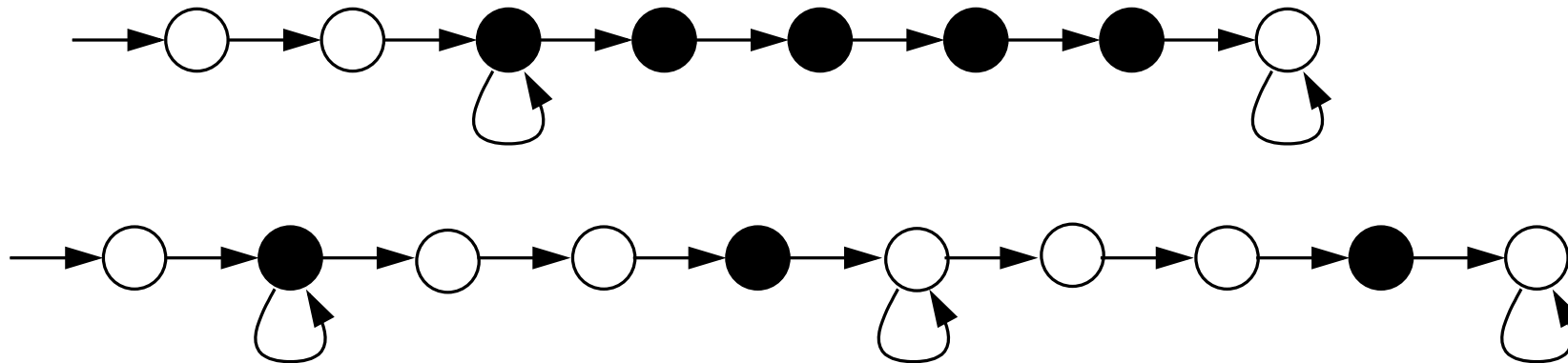
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(Hardin et al 1997, Fislser et al 2001)

Like EL, but step 4 is repeated until  $M$  does not change any more.

# EL and OWCTY

---



In the upper case, OWCTY is superior, in the lower case EL is.

In practice, OWCTY appears to work better.

# Part 13: Abstraction

## Example 1 (loop)

---

Consider the following program with three numeric variables  $x$ ,  $y$ ,  $z$ .

$l_1$ :  $y = x+1$ ;

$l_2$ :  $z = 0$ ;

$l_3$ : `while (z < 100) z = z+1;`

$l_4$ : `if (y < x) error;`

**Question:** Is the error location reachable?

## Example 2 (Sorting)

---

Another program with three numeric variables  $x, y, z$ .

$l_1$ : if  $x > y$  then swap  $x, y$  else skip;

$l_2$ : if  $y > z$  then swap  $y, z$  else skip;

$l_3$ : if  $x > y$  then swap  $x, y$  else skip;

$l_4$ : skip

**Assumption:** initially,  $x, y, z$  are all different

**Question:** Are  $x, y, z$  sorted in ascending order when reaching  $l_4$ ?

## Question 3 (Device driver)

---

C code for Windows device driver

Operations on a semaphore: lock, release

Lock and release must be used alternately

# Abstraction

---

**Idea:** throw away (abstract from) “unimportant” information

Handling *infinite* state spaces

Reduce (large) finite problems to smaller ones

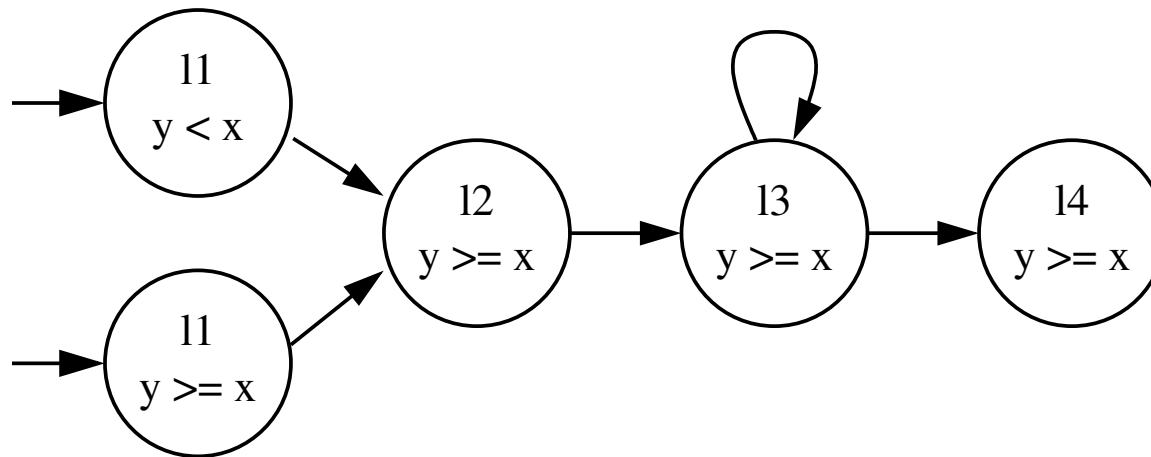
Alternative point of view: merge “equivalent” states

# Example 1

---

Omit concrete values of  $x, y, z$ ; retain only the following information: program counter, predicate  $y < x$

Resulting (abstract) Kripke structure:



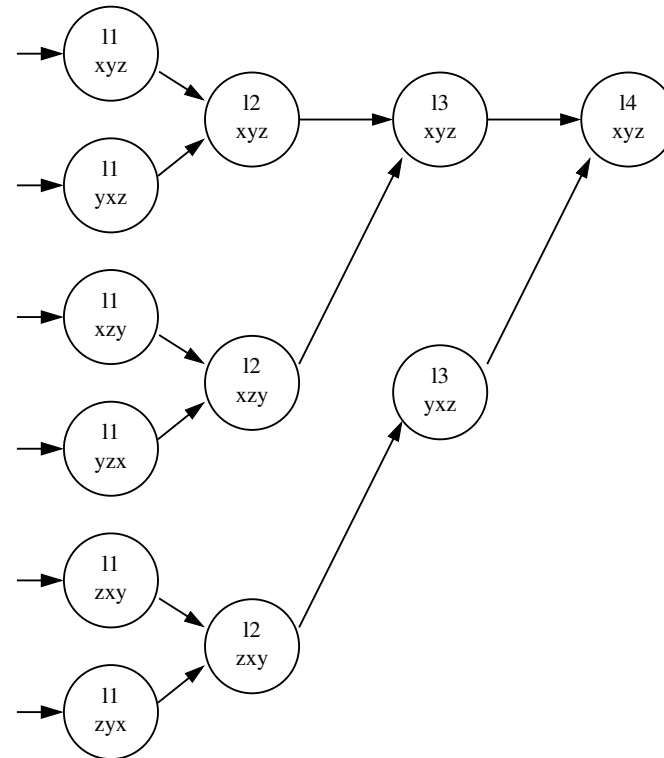
Result:  $\ell_4$  is reachable only with  $y \geq x$ ; the error will not happen.



## Example 2

---

Omit concrete values of  $x, y, z$ ; retain only program counter and permutation of  $x, y, z$



Result:  $\ell_4$  is reachable only with  $xyz$ ; no error.

---

**Questions:** What is the logical relation between the original programs and their abstract versions? What do the abstract versions really say about the original programs?

In Example 1, the error is unreachable in both the original and the abstract version.

However, in Example 1, the original structure terminates but the abstract version does not.

*Which conditions* must hold for the abstract structure in order to draw meaningful conclusions about the original structure?

# Simulation

---

Let  $\mathcal{K}_1 = (S, \rightarrow_1, s_0, AP, \nu)$  and  $\mathcal{K}_2 = (T, \rightarrow_2, t_0, AP, \mu)$  be two Kripke structures ( $S, T$  are possibly *infinite*), and let  $H \subseteq S \times T$  be a relation.

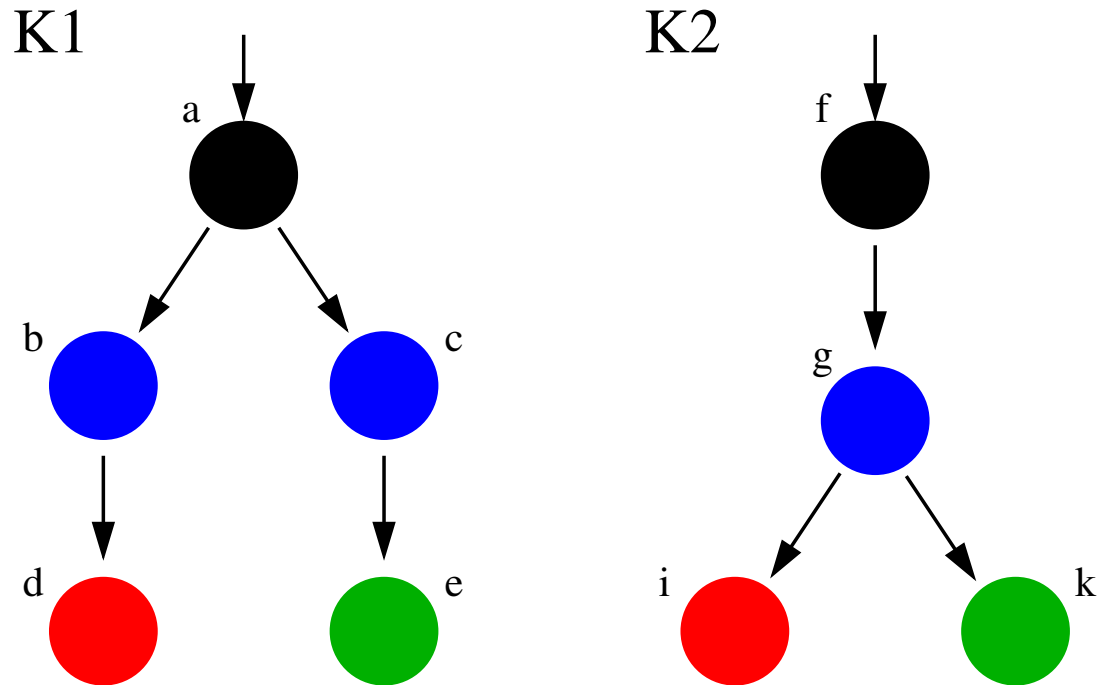
$H$  is called a **simulation from  $\mathcal{K}_1$  to  $\mathcal{K}_2$**  iff

- (i)  $(s_0, t_0) \in H$ ;
- (ii) for all  $(s, t) \in H$  we have:  $\nu(s) = \mu(t)$ ;
- (iii) if  $(s, t) \in H$  and  $s \rightarrow_1 s'$ , then there exists  $t'$  such that  $t \rightarrow_2 t'$  and  $(s', t') \in H$ .

We say:  $\mathcal{K}_2$  **simulates**  $\mathcal{K}_1$  (written  $\mathcal{K}_1 \leq \mathcal{K}_2$ ) if such a simulation  $H$  exists.

---

Intuition:  $\mathcal{K}_2$  can do anything that is possible in  $\mathcal{K}_1$ .



$\mathcal{K}_2$  simulates  $\mathcal{K}_1$  (with  $H = \{(a, f), (b, g), (c, g), (d, i), (e, k)\}$ ).

However,  $\mathcal{K}_1$  does *not* simulate  $\mathcal{K}_2$ !

# Bisimulation

---

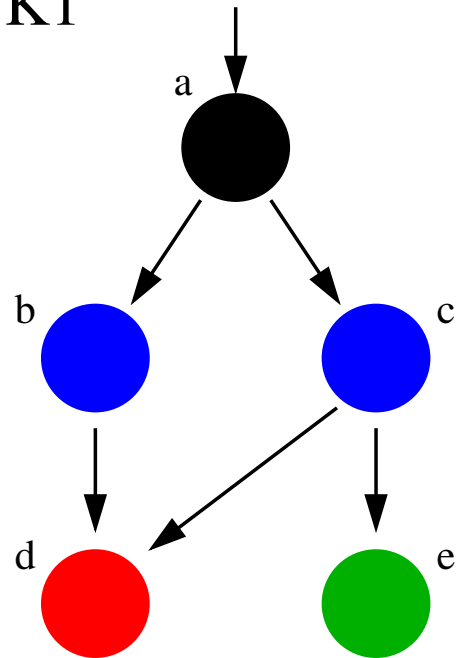
A relation  $H$  is called a **bisimulation between  $\mathcal{K}_1$  and  $\mathcal{K}_2$**  iff  $H$  is a simulation from  $\mathcal{K}_1$  to  $\mathcal{K}_2$  and  $\{(t, s) \mid (s, t) \in H\}$  is a simulation from  $\mathcal{K}_2$  to  $\mathcal{K}_1$ .

We say:  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are **bisimilar** (written  $\mathcal{K}_1 \equiv \mathcal{K}_2$ ) iff such a relation  $H$  exists.

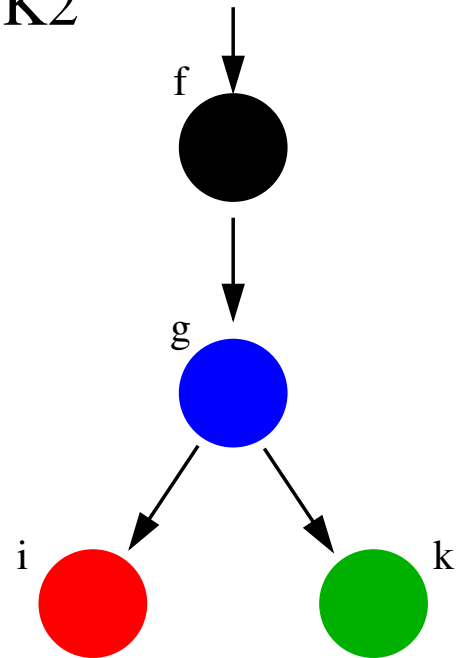
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Careful: In general,  $\mathcal{K}_1 \leq \mathcal{K}_2$  and  $\mathcal{K}_2 \leq \mathcal{K}_1$  do *not* imply  $\mathcal{K}_1 \equiv \mathcal{K}_2$ !

K1



K2



# (Bi-)Simulation and model checking

---

Let  $\mathcal{K}_1 \leq \mathcal{K}_2$  and  $\phi$  an LTL formula.

Then we have:  $\mathcal{K}_2 \models \phi$  implies  $\mathcal{K}_1 \models \phi$ .

Let  $\mathcal{K}_1 \equiv \mathcal{K}_2$  and  $\phi$  a CTL or LTL formula.

Then we have:  $\mathcal{K}_1 \models \phi$  iff  $\mathcal{K}_2 \models \phi$ .

# Existential abstraction

---

Let  $\mathcal{K} = (S, \rightarrow, r, AP, \nu)$  be a Kripke structure (**concrete structure**).

Let  $\approx$  be an equivalence relation on  $S$  such that for all  $s \approx t$  we have  $\nu(s) = \nu(t)$  (we say:  $\approx$  **respects**  $\nu$ ).

Let  $[s] := \{t \mid s \approx t\}$  denote the equivalence class of  $s$ ;  
 $[S]$  denotes the set of all equivalence classes.

The **abstraction of  $S$  w.r.t.  $\approx$**  denotes the structure  $\mathcal{K}' = ([S], \rightarrow', [r], AP, \nu')$ ,  
where

$[s] \rightarrow' [t]$  for all  $s \rightarrow t$ ;

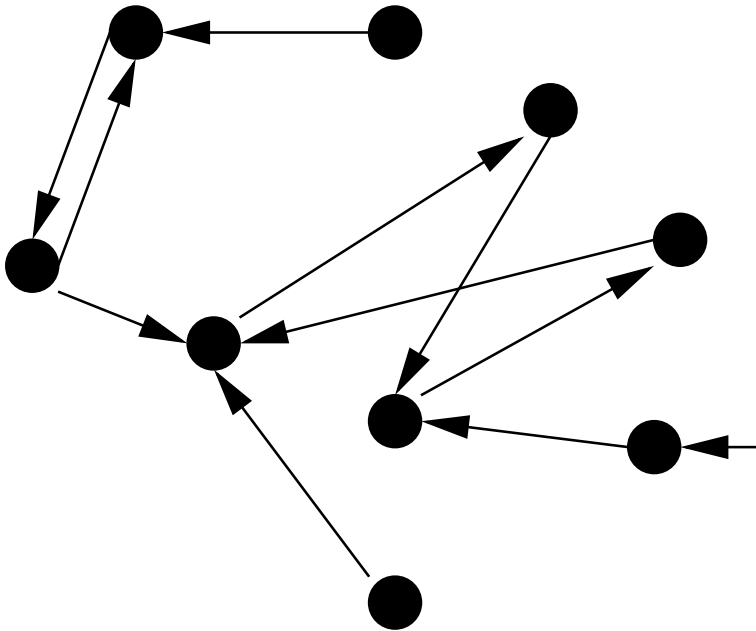
$\nu'([s]) = \nu(s)$  (this is well-defined!).



# Example

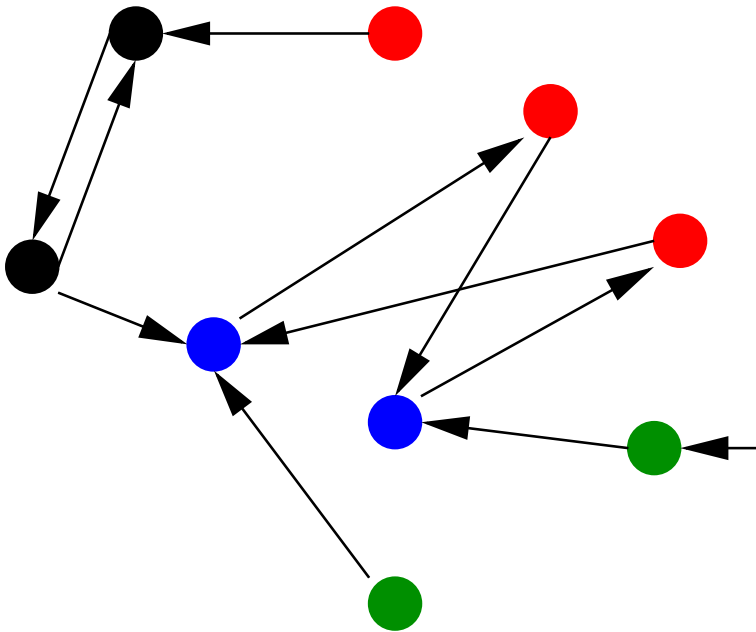
---

Consider the Kripke structure below:



---

States partitioned into equivalence classes:





---

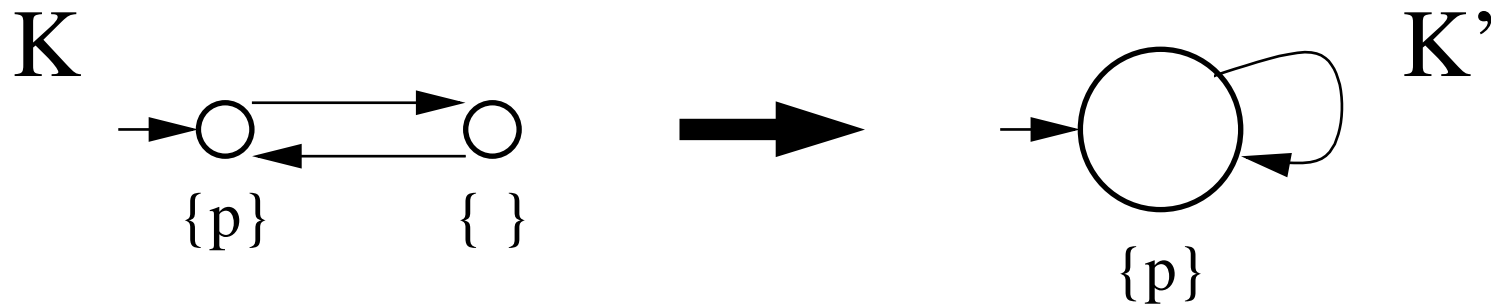
Let  $\mathcal{K}'$  be a structure obtained by abstraction from  $\mathcal{K}$ .

Then  $\mathcal{K} \leq \mathcal{K}'$  holds.

Thus, if  $\mathcal{K}'$  satisfies some LTL formula, so does  $\mathcal{K}$ .

---

What happens if  $\approx$  does *not* respect  $\nu$ ?



Then  $\mathcal{K} \not\approx \mathcal{K}'$  does not hold.

Example: The abstraction satisfies  $\mathbf{G}p$ , the concrete system does not.

---

Let  $\mathcal{K}'$  be a structure obtained by abstracting  $\mathcal{K}$ .

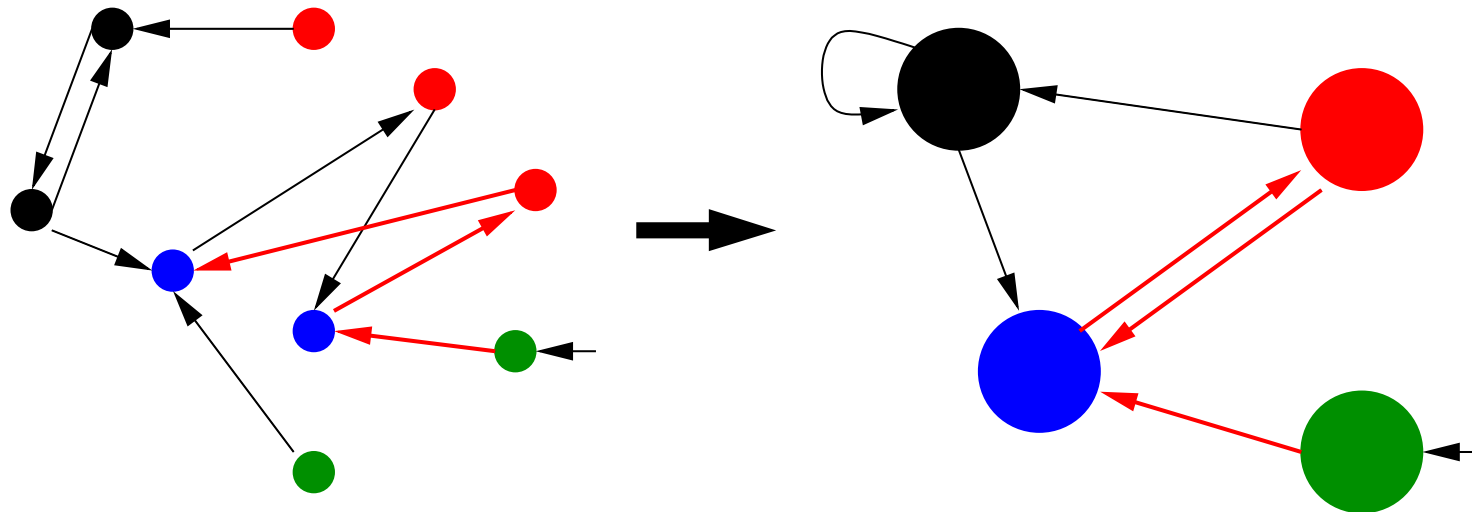
Then  $\mathcal{K} \leq \mathcal{K}'$  holds; thus, if  $\mathcal{K}'$  satisfies some LTL formula, then so does  $\mathcal{K}$ .

However, if  $\mathcal{K}' \not\models \phi$ , then  $\mathcal{K} \models \phi$  may or may not hold!

---

Abstraction gives rise to additional paths in the system:

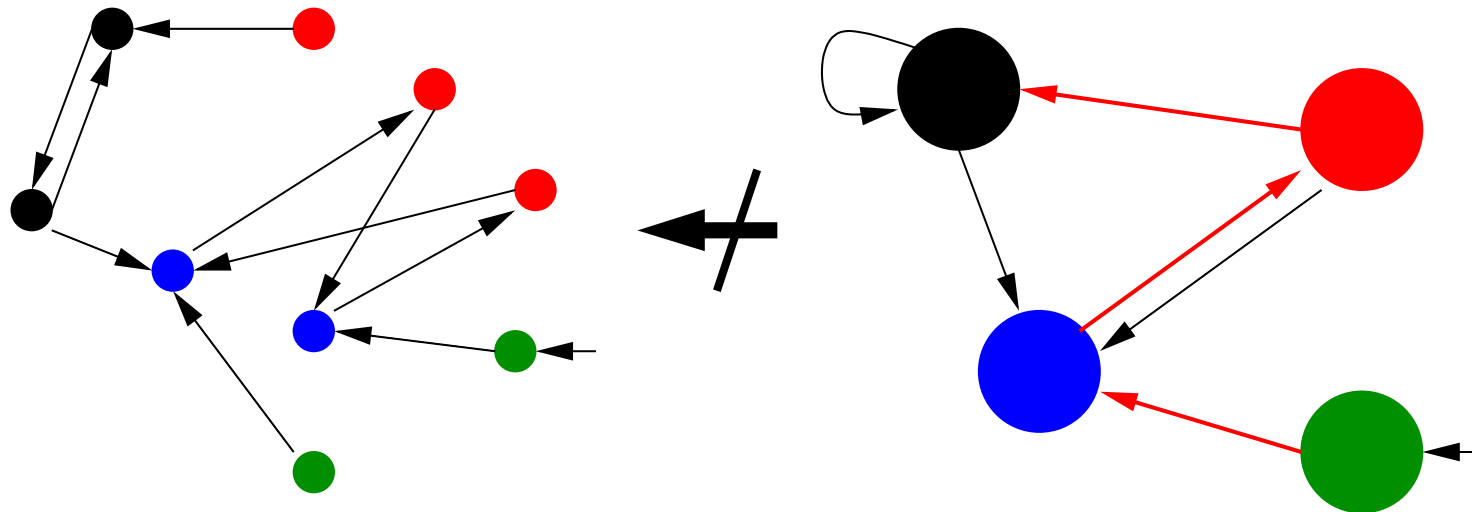
Every concrete run has got a corresponding run in the abstraction ...



---

Abstraction gives rise to additional paths in the system:

...but not every abstract run has got a corresponding run in the concrete system.





---

Suppose that  $\mathcal{K}' \not\models \phi$ , where  $\rho$  is a counterexample.

Check whether there is a run in  $\mathcal{K}$  that “corresponds” to  $\rho$ .

If yes, then  $\mathcal{K} \not\models \phi$ .

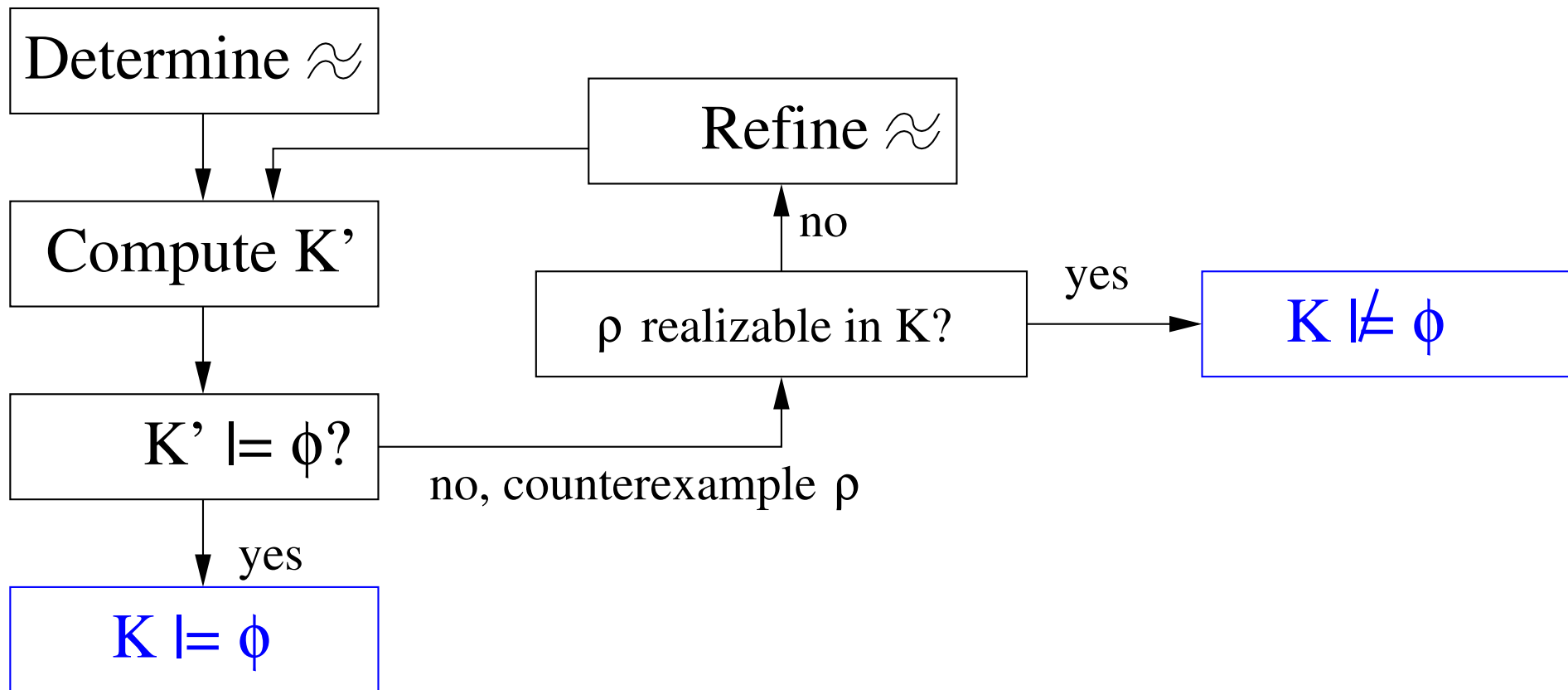
If no, then we can use  $\rho$  to **refine** the abstraction; i.e. we remove some equivalences from the relation  $H$ , introducing additional distinct states in  $\mathcal{K}'$  so that  $\rho$  disappears.

The refinement can be repeated until a definite answer for  $\mathcal{K} \models \phi$  (positive or negative) can be determined. This technique is called **counterexample-guided abstraction refinement** (CEGAR) [Clarke et al, 1994].

# The abstraction-refinement cycle

---

Input:  $\mathcal{K}, \phi$



# Simulation of $\rho$

---

**Problem:** Given a counterexample  $\rho$ , is there a run corresponding to  $\rho$  in  $\mathcal{K}$ ?

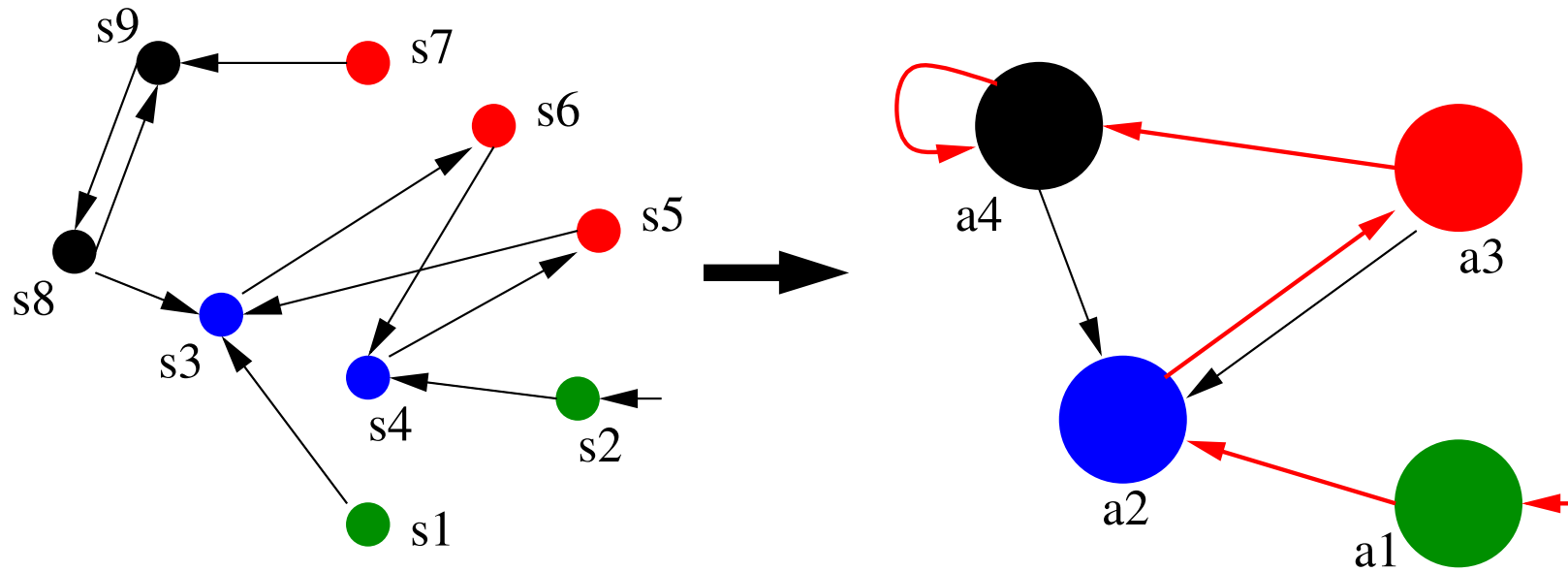
**Solution:** “Simulate”  $\rho$  on  $\mathcal{K}$ .

**Remark:** Any counterexample  $\rho$  can be partitioned into a finite **stem** and a finite **loop**, i.e.  $\rho = w_S w_L^\omega$  for suitable  $w_S, w_L$ .

**Case distinction:** The simulation may fail in the stem or in the loop.

# Example 1: $G \not\models \text{black}$

---



Abstraction yields a counterexample with stem  $a_1 a_2 a_3 a_4$  and loop  $a_4$ .

# Simulating the stem

---

Let  $w_S = b_0 \cdots b_k$ .

Start with  $S_0 = \{r\}$ . (We have  $b_0 = [r]$ .)

For  $i = 1, \dots, k$ , compute  $S_i = \{t \mid t \in b_i \wedge \exists s \in S_{i-1} : s \rightarrow t\}$ .

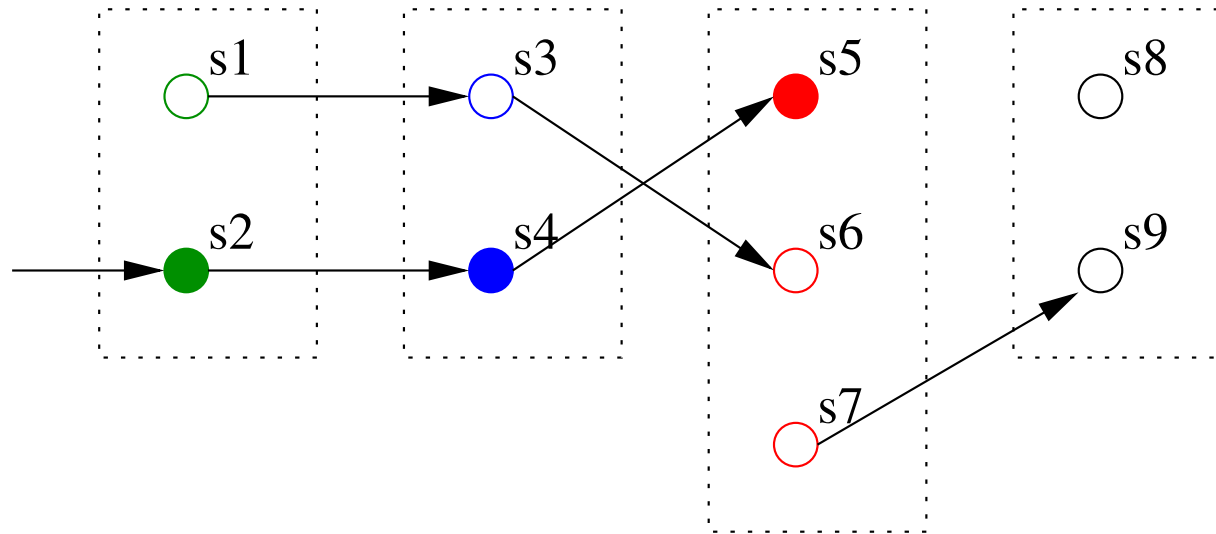
If  $S_k \neq \emptyset$ , then there is a concrete correspondence for  $w_S$ .

If  $S_k = \emptyset$ : Find the smallest index  $\ell$  with  $S_\ell = \emptyset$ : The refinement should distinguish the states in  $S_{\ell-1}$  and those  $b_{\ell-1}$ -states that have immediate successors in  $b_\ell$ .

# Example: $w_S = a_1 a_2 a_3 a_4$

---

$$S_0 = \{s_2\}, \quad S_1 = \{s_4\}, \quad S_2 = \{s_5\}, \quad S_3 = \emptyset.$$

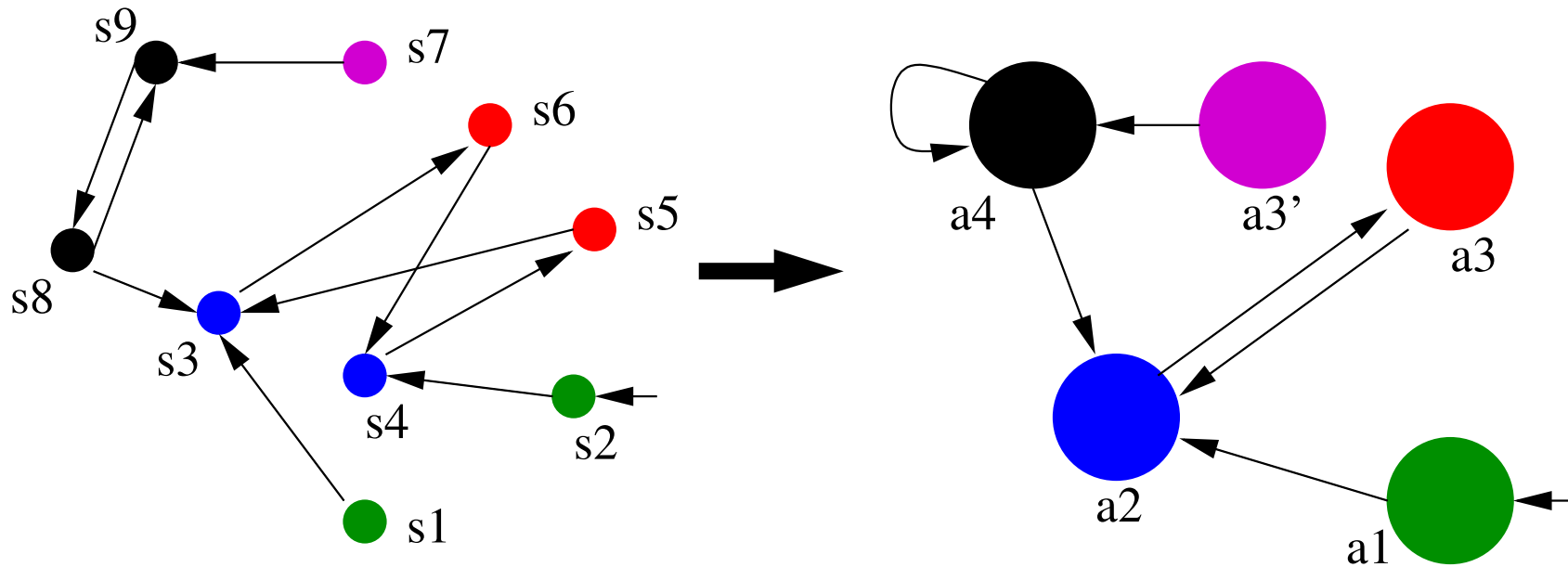


In the next refinement,  $s_5$  and  $s_7$  must be distinguished.

Possible new equivalence classes:  $\{s_5, s_6\}, \{s_7\}$  or  $\{s_5\}, \{s_6, s_7\}$ .

## Next try: $G \neg black$ with refinement

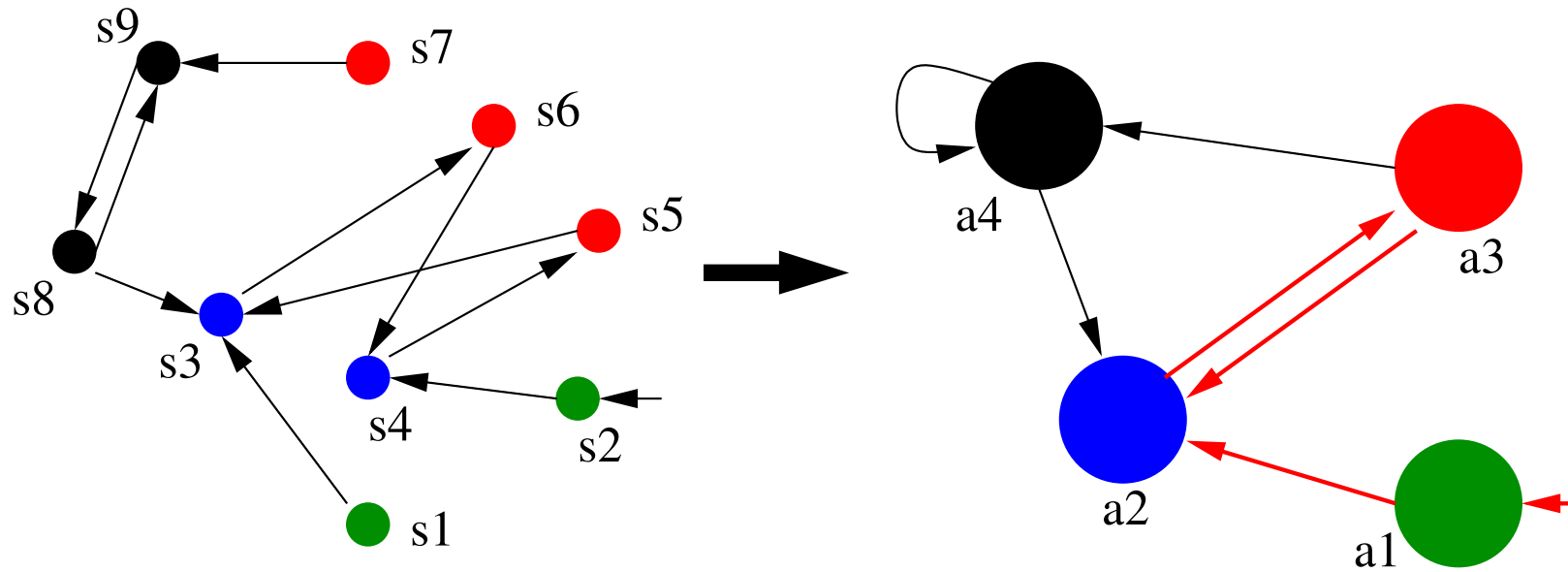
---



The new abstraction does not yield any counterexample; therefore,  $G \neg black$  also holds in the concrete system.

## Example 2: $\text{FG red}$

---



The abstraction yields a counterexample with stem  $a_1 a_2$  and loop  $a_3 a_2$ .



# Simulating a loop

---

Assume  $w_S = b_0 \cdots b_k$ ,  $w_L = c_1 \cdots c_\ell$

$w_S$  is simulated as before, however  $w_L$  may have to be simulated multiple times.

Let  $m$  be the size of the smallest equivalence class in  $w_L$ :

$$m = \min_{i=1, \dots, \ell} |c_i|$$

Then we simulate the path  $w_S w_L^{m+1}$ ; doing so, either the simulation will fail, or we will discover a real counterexample.

Refinement: same as before.



# Reasons for infinite state-spaces

---

So far, we have dealt with finite-state systems. However, many interesting real systems have infinitely many states.

**Data:** integers, reals, lists, trees, heap, ...

**Control:** procedures, dynamic thread creation, ...

**Communication:** unbounded message channels

**Parameters:** number of participants in a protocol, ...

**Time:** discrete or continuous clocks

Some (not all!) of these features (or combinations thereof) lead to **Turing-powerful** models of computation.

# Part 14: Pushdown systems

# Example 1

---

A small program (where  $n \geq 1$ ):

```
bool g=true;
void main() {
    level1();
    level1();
    assume(g);
}
void leveln() {
    g:=not g;
}

void leveli() {
    for j:=1 to 8 do skip;
    leveli+1();
    leveli+1();
}
```

Question: Will  $g$  be true when the program terminates?

---

Example 1 has got *finitely* many states.

(The call stack is bounded by  $n$ .)

Can be treated by “inlining” (replace procedure calls by a copy of the callee).

Inlining causes an exponential state-space explosion.

Inlining is inefficient: every copy of each procedure will be investigated separately.

Inlining not applicable for **recursiven** procedure calls.

## Example 2: Recursive program (plotter)

```

procedure p;
p0: if ? then
p1:     call s;
p2:     if ? then call p; end if;
           else
p3:     call p;
           end if
p4: return
  
```

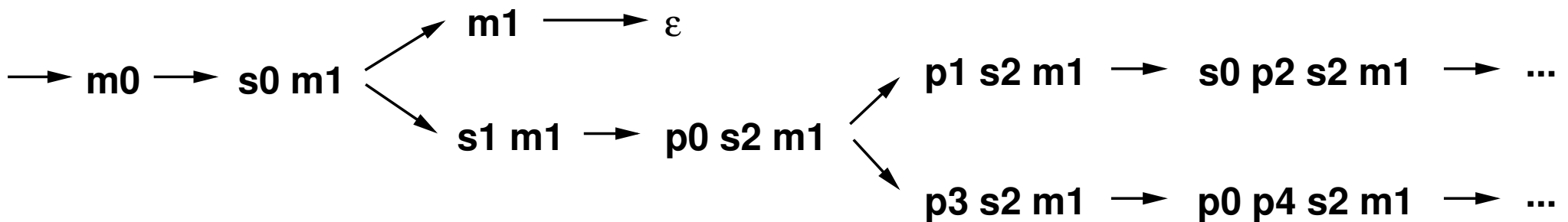
```

procedure s;
s0: if ? then return; end if;
s1: call p;
s2: return;

procedure main;
m0: call s;
m1: return;
  
```

$S = \{p_0, \dots, p_4, s_0, \dots, s_2, m_0, m_1\}^*$ ,

initial state  $m_0$



---

Example 2 has got infinitely many states.

Inlining not applicable!

Cannot be analyzed by naïvely searching all reachable states.

We shall require a *finite* representation of infinitely many states.



## Example 3: Quicksort

---

```
void quicksort (int left, int right) {
    int lo,hi,piv;
    if (left >= right) return;
    piv = a[right]; lo = left; hi = right;
    while (lo <= hi) {
        if (a[hi]>piv) {
            hi = hi - 1;
        } else {
            swap a[lo],a[hi];
            lo = lo + 1;
        }
    }
    quicksort(left,hi);
    quicksort(lo,right);
}
```

---

**Question:** Does Example 3 sort correctly? Is termination guaranteed?

The mere structure of Example 3 does not tell us whether there are infinitely many reachable states:

*finitely* many if the program terminates

*infinitely* many if it fails to terminate

Termination can only be checked by directly dealing with infinite state sets.

# A computation model for procedural programs

---

## Control flow:

sequential program (no multithreading)

procedures

mutual procedure calls (possibly recursive)

## Data:

global variables (restriction: only **finite memory**)

local variables in each procedure (one copy per call)

# Pushdown systems

---

A **pushdown system** (PDS) is a triple  $(P, \Gamma, \Delta)$ , where

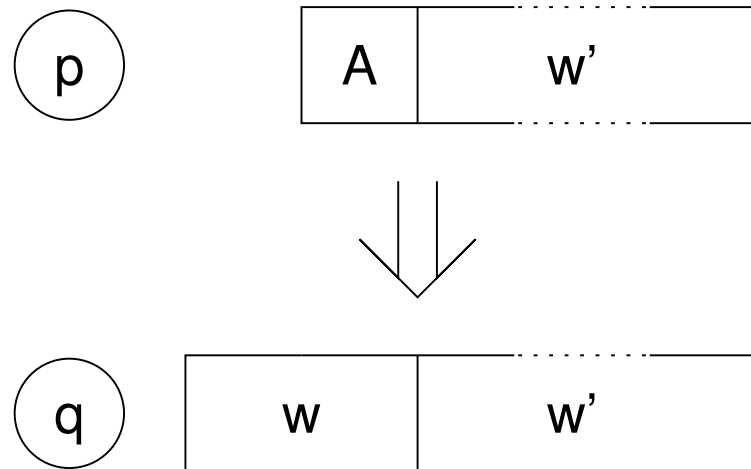
$P$  is a finite set of **control states**;

$\Gamma$  is a finite **stack alphabet**;

$\Delta$  is a finite set of **rules**.

---

Rules have the form  $pA \leftrightarrow qw$ , where  $p, q \in P$ ,  $A \in \Gamma$ ,  $w \in \Gamma^*$ .



Like acceptors for context-free language, but without any input!

# Behaviour of a PDS

---

Let  $\mathcal{P} = (P, \Gamma, \Delta)$  be a PDS and  $c_0 \in P \times \Gamma^*$ .

With  $\mathcal{P}$  we associate a transition system  $\mathcal{T}_{\mathcal{P}} = (S, \rightarrow, r)$  as follows:

$S = P \times \Gamma^*$  are the states (which we call **configurations**);

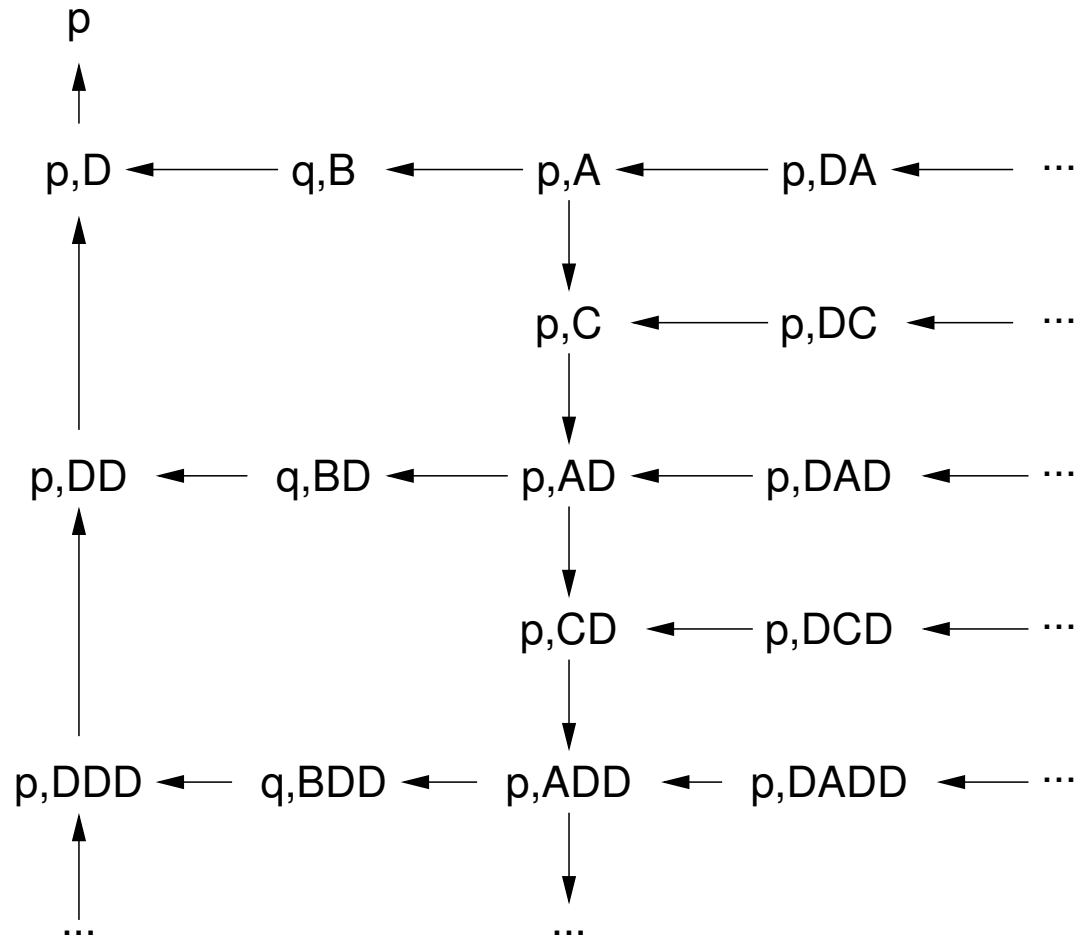
we have  $pAw' \rightarrow qww'$  for all  $w' \in \Gamma^*$  iff  $pA \hookrightarrow qw \in \Delta$ ;

$r = c_0$  is the initial configuration.

# Transition system of a PDS

---

$pA \hookrightarrow qB$   
 $pA \hookrightarrow pC$   
 $qB \hookrightarrow pD$   
 $pC \hookrightarrow pAD$   
 $pD \hookrightarrow p\varepsilon$



# Procedural programs and PDSs

---

$P$  may represent the valuations of global variables.

$\Gamma$  may contain tuples of the form (*program counter, local valuations*)

Interpretation of a configuration  $pAw$ :

global values in  $p$ , current procedure with local variables in  $A$

“suspended” procedures in  $w$

Rules:

$pA \hookrightarrow qB \hat{=} \text{statement within a procedure}$

$pA \hookrightarrow qBC \hat{=} \text{procedure call}$

$pA \hookrightarrow q\epsilon \hat{=} \text{return from a procedure}$



# Reachability in PDS

---

Let  $\mathcal{P}$  be a PDS and  $c, c'$  two of its configurations.

**Problem:** Does  $c \rightarrow^* c'$  hold in  $\mathcal{T}_{\mathcal{P}}$ ?

Note:  $\mathcal{T}_{\mathcal{P}}$  has got infinitely many (reachable) states.

Nonetheless, the problem is decidable!

# Finite automata

---

To represent (infinite) sets of configurations, we shall employ **finite automata**.

Let  $\mathcal{P} = (P, \Gamma, \Delta)$  be a PDS. We call  $\mathcal{A} = (\Gamma, Q, P, \delta, F)$  a  $\mathcal{P}$ -automaton.

The alphabet of  $\mathcal{A}$  is the stack alphabet  $\Gamma$ .

The initial states of  $\mathcal{A}$  are the control states  $P$ .

We say that  $\mathcal{A}$  **accepts** the configuration  $pw$  if  $\mathcal{A}$  has got a path labelled by input  $w$  starting at  $p$  and ending at some final state.

---

Let  $\mathcal{L}(\mathcal{A})$  be the set of configurations accepted by  $\mathcal{A}$ .

A set  $C$  of configurations is called **regular** iff there is some  $\mathcal{P}$ -automaton  $\mathcal{A}$  with  $\mathcal{L}(\mathcal{A}) = C$ .

An automaton is **normalized** if there are no transitions leading into initial states.

**Remark:** In the following, we shall use the following notation:

$pw \Rightarrow p'w'$  (in the PDS  $\mathcal{P}$ )    and     $p \xrightarrow{w} q$  (in  $\mathcal{P}$ -automata)

# Reachability in PDS

---

Let  $pre^*(C) = \{c' \mid \exists c \in C: c' \Rightarrow c\}$  denote the predecessors of  $C$ , and let  $post^*(C) = \{c' \mid \exists c \in C: c \Rightarrow c'\}$  the successors.

The following result is due to Büchi (1964):

Let  $C$  be a regular set and  $\mathcal{A}$  be a (normalized)  $\mathcal{P}$ -automaton accepting  $C$ .

If  $C$  is regular, then so are  $pre^*(C)$  and  $post^*(C)$ .

Moreover,  $\mathcal{A}$  can be transformed into an automaton accepting  $pre^*(C)$  resp.  $post^*(C)$ .

## The basic idea (for $pre$ )

---

**Saturation rule:** Add new transitions to  $\mathcal{A}$  as follows:

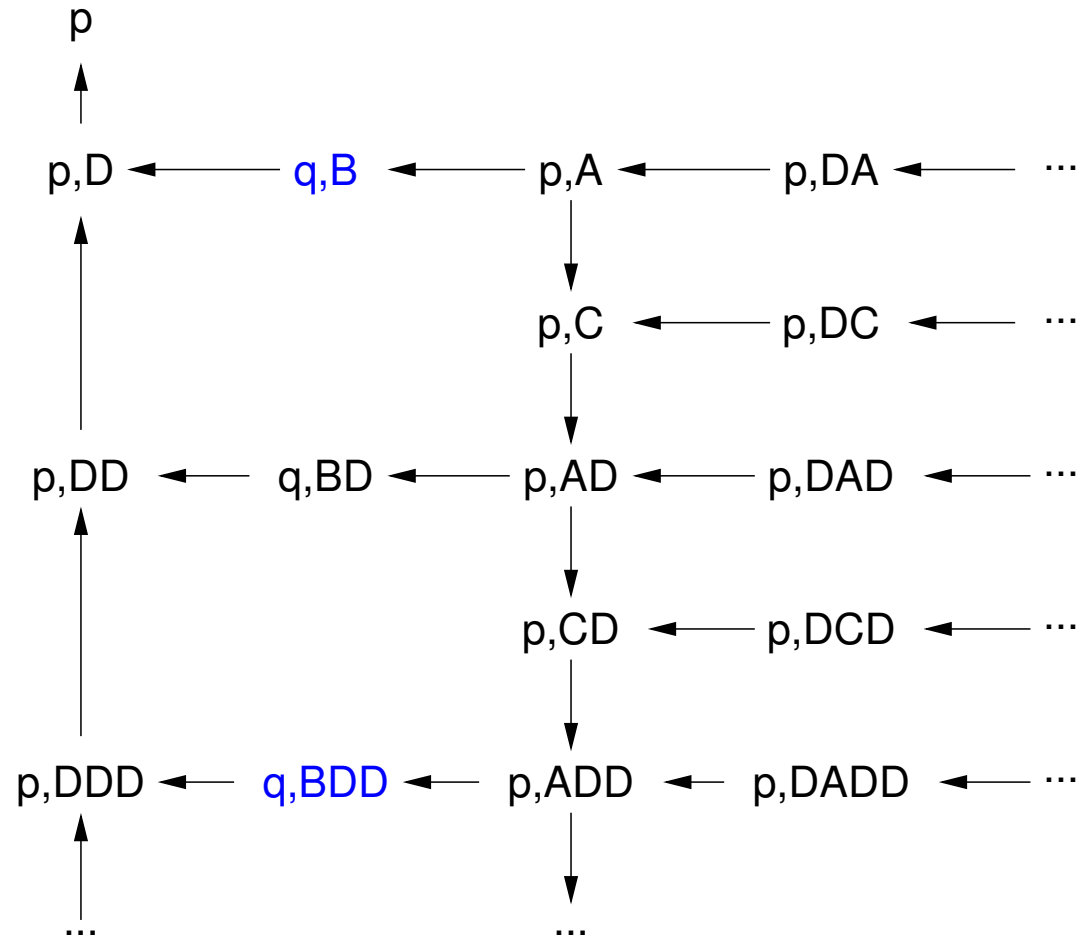
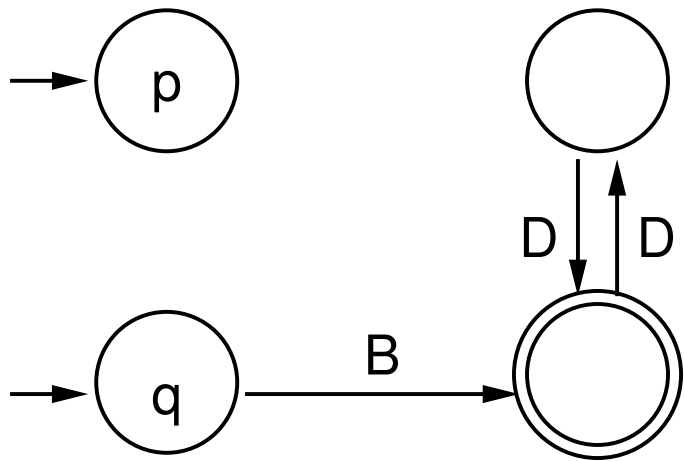
If  $q \xrightarrow{w} r$  currently holds in  $\mathcal{A}$  and  $pA \hookrightarrow qw$  is a rule, then add the transition  $(p, A, r)$  to  $\mathcal{A}$ .

Repeat this until no other transition can be added.

At the end, the resulting automaton accepts  $pre^*(C)$ .

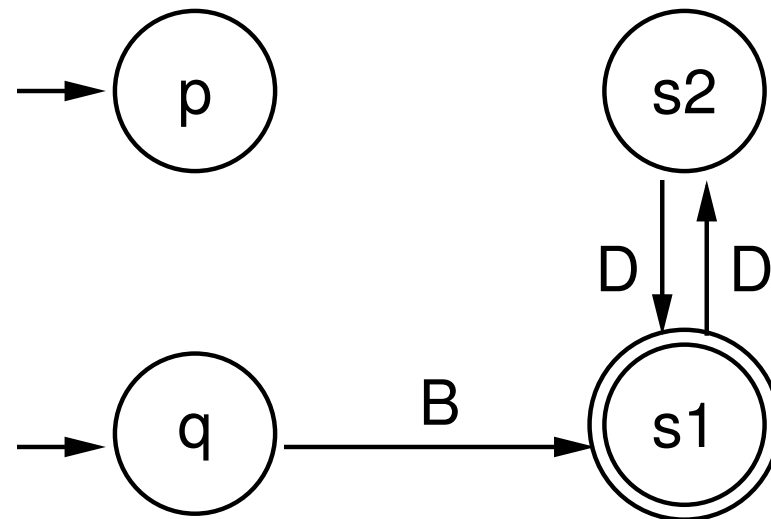
For  $post^*(C)$ : similar procedure.

# Automaton $\mathcal{A}$ for $C$



# Extending $\mathcal{A}$

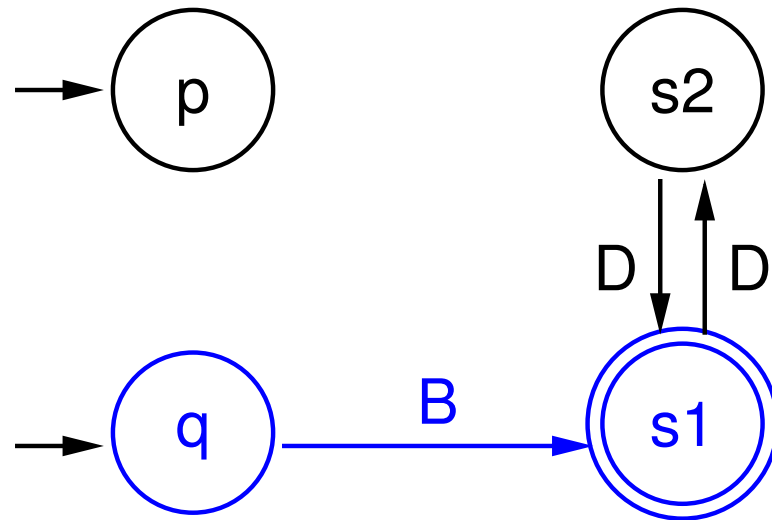
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# Extending $\mathcal{A}$

---

If the right-hand side of a rule can be read,



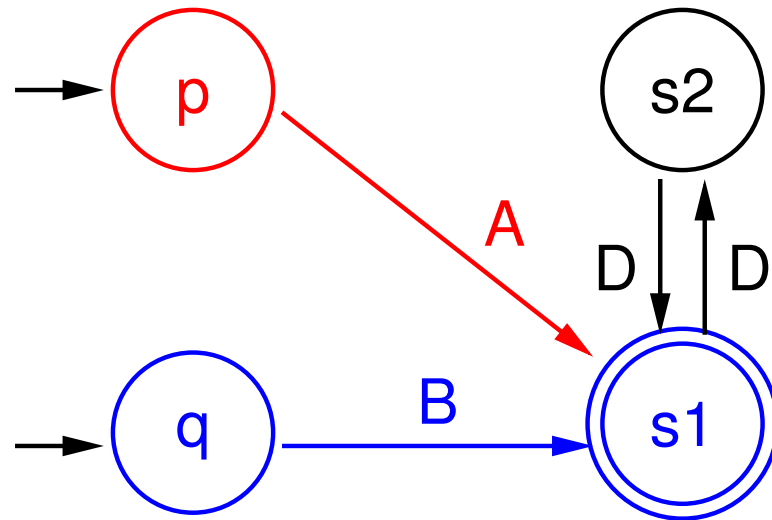
Rule:  $pA \hookrightarrow qB$       Path:  $q \xrightarrow{B} s_1$



# Extending $\mathcal{A}$

---

If the right-hand side of a rule can be read, add the left-hand side.



Rule:  $pA \hookrightarrow qB$

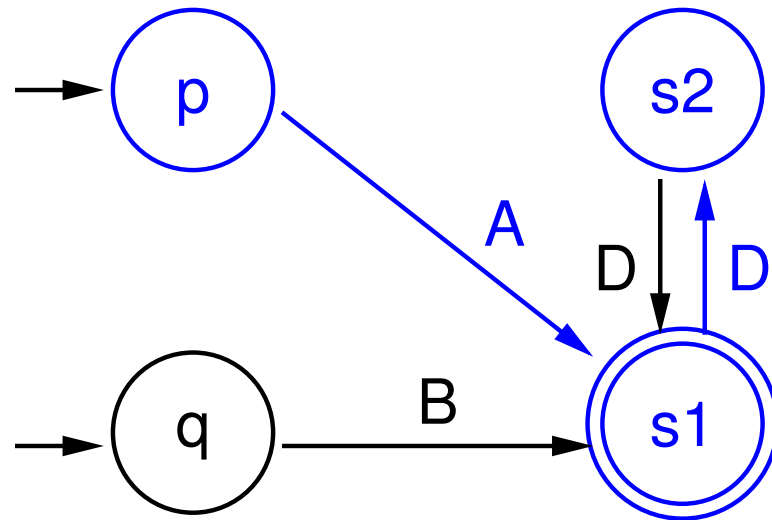
Path:  $q \xrightarrow{B} s_1$

New path:  $p \xrightarrow{A} s_1$

# Extending $\mathcal{A}$

---

If the right-hand side of a rule can be read,



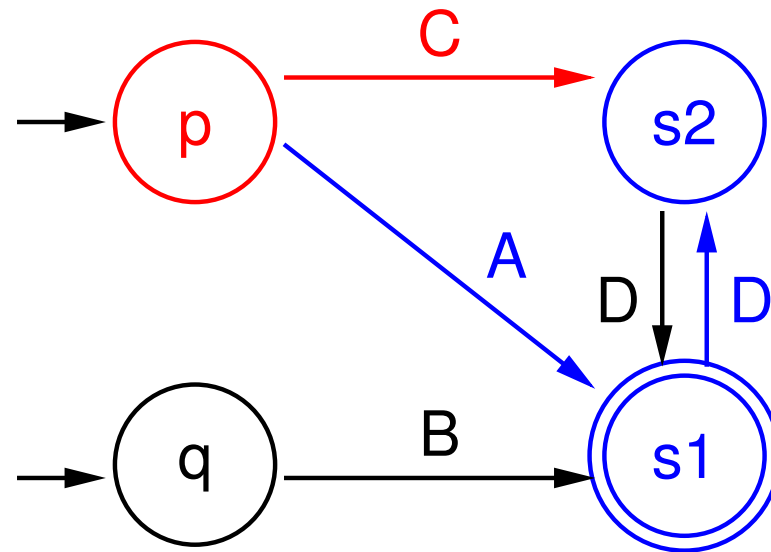
Rule:  $pC \hookrightarrow pAD$

Path:  $p \xrightarrow{A} s_1 \xrightarrow{D} s_2$

# Extending $\mathcal{A}$

---

If the right-hand side of a rule can be read, add the left-hand side.

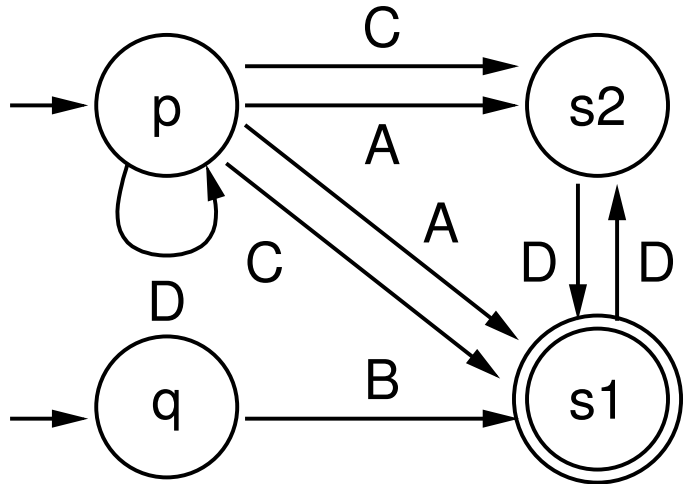


Rule:  $pC \hookrightarrow pAD$

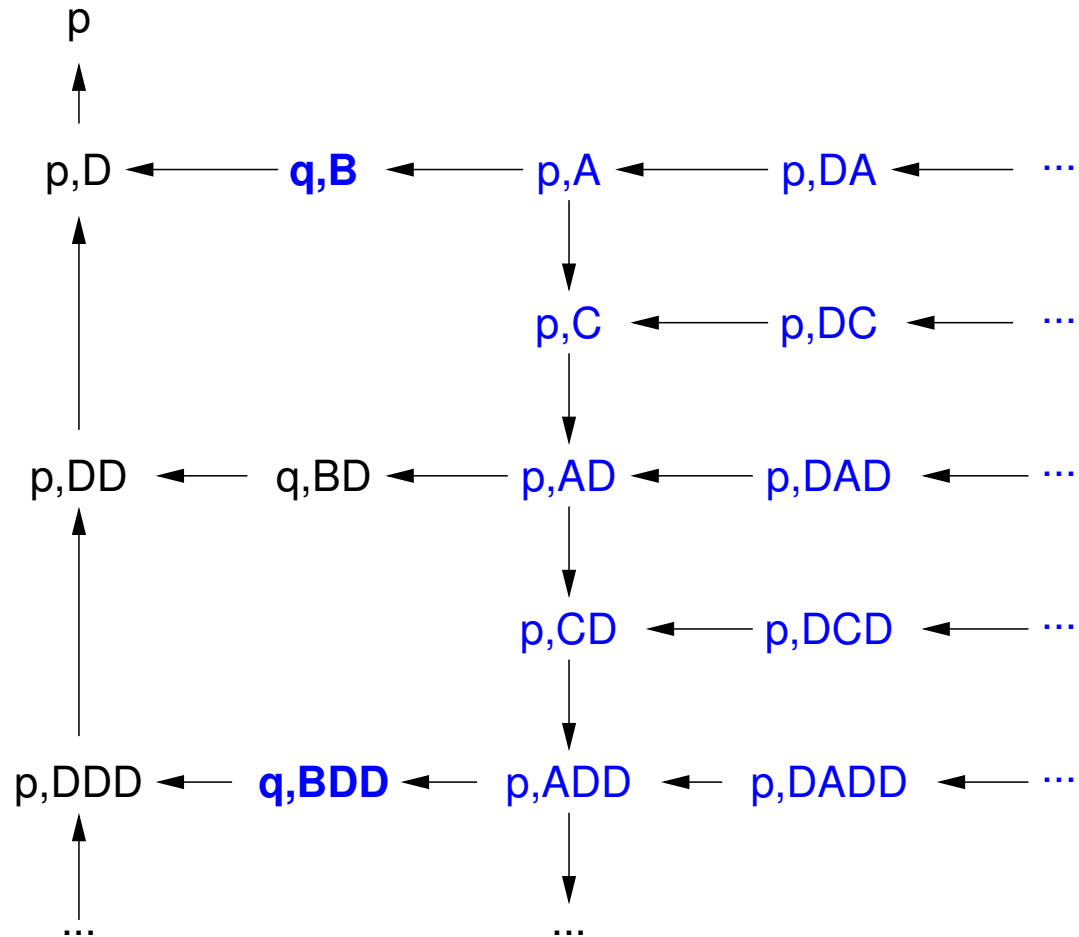
Path:  $p \xrightarrow{A} s_1 \xrightarrow{D} s_2$

New path:  $p \xrightarrow{C} s_2$

# Final result



Complexity:  
 $\mathcal{O}(|Q|^2 \cdot |\Delta|)$  time.



# Proof of correctness

---

We shall show:

Let  $\mathcal{B}$  be the  $\mathcal{P}$ -automaton arising from  $\mathcal{A}$  by applying the saturation rule.  
Then  $\mathcal{L}(\mathcal{B}) = pre^*(C)$ .

Part 1: Termination

The saturation rule can only be applied finitely many times because no states are added and there are only finitely many possible transitions.

Part 2:  $pre^*(C) \subseteq \mathcal{L}(\mathcal{B})$

Let  $c \in pre^*(C)$  and  $c' \in C$  such that  $c'$  is reachable from  $c$  in  $k$  steps. We proceed by induction on  $k$  (simple).

---

Part 3:  $\mathcal{L}(\mathcal{B}) \subseteq pre^*(\mathcal{C})$

Let  $\xrightarrow{i}$  denote the transition relation of the automaton after the saturation rule has been applied  $i$  times.

We show the following, more general property: If  $p \xrightarrow{i}^w q$ , then there exist  $p'w'$  with  $p' \xrightarrow{0}^{w'} q$  and  $pw \Rightarrow p'w'$ ; if  $q \in P$ , then additionally  $w' = \varepsilon$ .

Proof by induction over  $i$ : The base case  $i = 0$  is trivial.

Induction step: Let  $t = (p_1, A, q')$  be the transition added in the  $i$ -th application and  $j$  the number of times  $t$  occurs in the path  $p \xrightarrow{i}^w q$ .

Induction over  $j$ : Trivial for  $j = 0$ . So let  $j > 0$ .

---

There exist  $p_2, p', u, v, w', w_2$  with the following properties:

- (1)  $p \xrightarrow{i-1}^u p_1 \xrightarrow{i}^A q' \xrightarrow{i}^v q$  (splitting the path  $p \xrightarrow{i}^w q$ )
- (2)  $p_1 A \hookrightarrow p_2 w_2$  (pre-condition for saturation rule)
- (3)  $p_2 \xrightarrow{i-1}^{w_2} q'$  (pre-condition for saturation rule)
- (4)  $pu \Rightarrow p_1 \varepsilon$  (ind.hyp. on  $i$ )
- (5)  $p_2 w_2 v \Rightarrow p' w'$  (ind.hyp. on  $j$ )
- (6)  $p' \xrightarrow{0}^{w'} q$  (ind.hyp. on  $j$ )

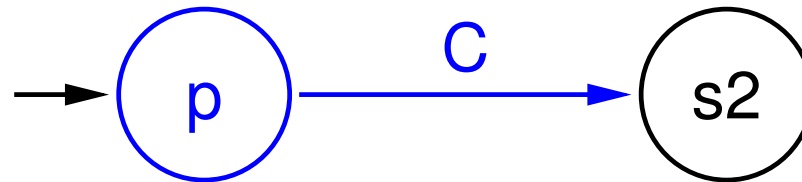
The desired proof follows from (1), (4), (2), and (5).

If  $q \in P$ , then the second part follows from (6) and the fact that  $A$  is normalized.

## Example: $post^*$ (without proof)

---

If the *left-hand side* of a rule can be read,



Rule:  $pC \hookrightarrow pAD$

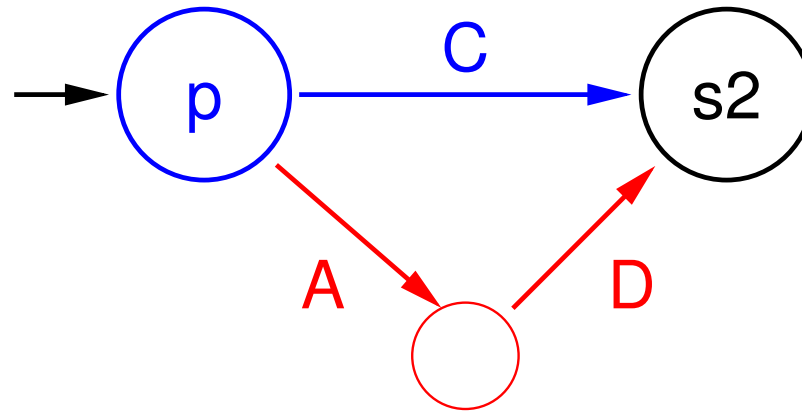
Path:  $p \xrightarrow{C} s_2$



# Example: $post^*$ (without proof)

---

If the *left-hand side* of a rule can be read, add the *right-hand side*.



Rule:  $pC \hookrightarrow pAD$

Path:  $p \xrightarrow{C} s_2$

New Path:  $p \xrightarrow{AD} s_2$

# LTL and Pushdown Systems

---

Let  $\mathcal{P} = (P, \Gamma, \Delta)$  be a PDS with initial configuration  $c_0$ , let  $\mathcal{T}_{\mathcal{P}}$  denote the corresponding transition system,  $AP$  a set of atomic propositions, and  $\nu: P \times \Gamma^* \rightarrow 2^{AP}$  a valuation function.

$\mathcal{T}_{\mathcal{P}}$ ,  $AP$ , and  $\nu$  form a Kripke structure  $\mathcal{K}$ ; let  $\phi$  be an LTL formula (over  $AP$ ).

**Problem:** Does  $\mathcal{K} \models \phi$ ?

Undecidable for arbitrary valuation functions!

(could encode undecidable decision problems in  $\nu \dots$ )

However, LTL model checking *is* decidable for certain “reasonable” restrictions of  $\nu$ .

---

In the following, we consider “simple” valuation functions satisfying the following restriction:

$$\nu(pAw) = \nu(pA), \text{ for all } p \in P, A \in \Gamma, \text{ and } w \in \Gamma^*.$$

In other words, the “head” of a configuration holds all information about atomic propositions.

LTL model checking is decidable for such “simple” valuations.

# Approach

---

Same principle as for finite Kripke structures:

Translate  $\neg\phi$  into a Büchi automaton  $\mathcal{B}$ .

Build the cross product of  $\mathcal{K}$  and  $\mathcal{B}$ .

Test the cross product for emptiness.

Note that the cross product is not a Büchi automaton in this case (e.g., it is not finite).

# Büchi PDS

---

The cross product is a new pushdown system  $\mathcal{Q}$ , as follows:

Let  $\mathcal{P} = (P, \Gamma, \Delta)$  be a PDS,  $p_0 w_0$  the initial configuration, and  $AP, \nu$  as usual.

Let  $\mathcal{B} = (2^{AP}, Q, q_0, \delta, F)$  be the Büchi automaton for  $\neg\phi$ .

Construction of  $\mathcal{Q}$ :

$\mathcal{Q} = (P \times Q, \Gamma, \Delta', P \times F)$ , where

$(p, q)A \hookrightarrow (p', q')w \in \Delta'$  iff

- $pA \hookrightarrow p'w \in \Delta$  and
- $(q, L, q') \in \delta$  such that  $\nu(pA) = L$ .

Initial configuration:  $(p_0, q_0)w_0$

---

Let  $\rho$  be a run of  $\mathcal{Q}$  with  $\rho(i) = (p_i, q_i)w_i$ .

We call  $\rho$  **accepting** if  $q_i \in F$  for infinitely many values of  $i$ .

The following is easy to see:

$\mathcal{P}$  does *not* satisfy  $\phi$  iff there exists an accepting run in  $\mathcal{Q}$ .

# Characterization of accepting runs

---

**Question:** If there an accepting run starting at  $(p_0, q_0)w_0$ ?

In the following, we shall consider the following, more general **global** model-checking problem:

Compute *all* configurations  $c$  such that there exists an accepting run starting at  $c$ .

**Lemma:** There is an accepting run starting at  $c$  iff there exists  $(p, q) \in P \times Q$ ,  $A \in \Gamma$  with the following properties:

(1)  $c \Rightarrow (p, q)Aw$  for some  $w \in \Gamma^*$

(2)  $(p, q)A \Rightarrow (p, q)Aw'$  for some  $w' \in \Gamma^*$ , where

the path from  $(p, q)A$  to  $(p, q)Aw'$  contains at least one step;

the path contains at least one accepting Büchi state.

# Repeating heads

---

We call  $(p, q)A$  a **repeating head** if  $(p, q)A$  satisfies properties (1) and (2).

Strategy:

1. Compute all repeating heads.

(naïvely: check for each pair  $(p, q)A$  whether

$(p, q)A \in pre^*(\{(p, q)Aw \mid w \in \Gamma^*\})$ . Visiting an accepting state can be encoded into the control state, see next slide.

2. Compute the set  $pre^*(\{(p, q)Aw \mid (p, q)A \text{ is a repeating head, } w \in \Gamma^*\})$



# Implementing step 1

---

First, we transform  $\mathcal{Q}$  into a modified PDS  $\mathcal{Q}'$ .

For each pair  $(p, q) \in P \times Q$ ,  $\mathcal{Q}'$  has got two control states:  $(p_0, q)$ ,  $(p_1, q)$ .

For each rule  $(p, q)A \hookrightarrow (p', q')w$  in  $\mathcal{Q}$ ,  $\mathcal{Q}'$  has got rules:

$(p_b, q)A] \hookrightarrow (p'_1, q)w$  for  $b = 0, 1$  if  $q \in F$ ;

$(p_b, q)A] \hookrightarrow (p'_b, q)w$  for  $b = 0, 1$  if  $q \notin F$ .

Then, for  $\mathcal{Q}'$ , we compute *once* the set  $pre^*(\{(p_1, q)\varepsilon \mid (p, q) \in P \times Q\})$

---

We then construct the following *finite* graph  $G = (V, \rightarrow)$ :

$$V = (P \times Q) \times \Gamma$$

$(p, q)A \rightarrow (p', q')B$  iff there exist  $(p, q)A \xleftrightarrow{v} (p'', q'')vBw$  with  $(p'', q'') \in P \times Q$ ,  $v, w \in \Gamma^*$ , and  $(p'', q'')v \Rightarrow (p', q')\varepsilon$

Label the edge  $(p, q)A \rightarrow (p', q')B$  by **1** iff either  $q \in F$  or  $(p''_0, q'')v \Rightarrow (p'_1, q')\varepsilon$  holds in  $\mathcal{Q}'$ .

Notice that an edge is labelled by **1** iff there is a stack-increasing computation leading from one head to another visiting an accepting Büchi state.

Find those SCCs in  $G$  that contain a **1**-labelled edge.

$(p, q)A$  is a repeating head iff it is contained in such an SCC.

# CTL\* and PDS

---

The model-checking approach for PDS can even be lifted to CTL\*.

The approach is analogous to finite-state systems (sketch only):

Solve the innermost **E**-subformulae using the global LTL model-checking algorithm we have developed.

The result is a finite-state automaton  $\mathcal{A}$  giving all the configurations satisfying that subformula.

Modify the PDS: Encode the states of  $\mathcal{A}$  into the stack alphabet and synchronize the push/pop operations with the actions of  $\mathcal{A}$  such that the top-of-stack symbol shows the final state of  $\mathcal{A}$  iff the current stack content is accepted by  $\mathcal{A}$ .

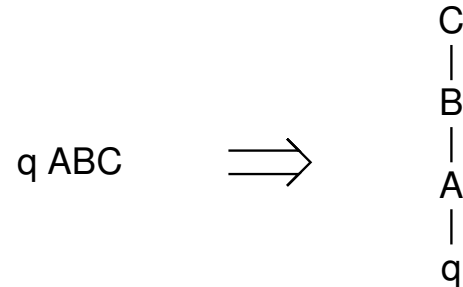
Replace the subformula by a fresh atomic proposition and continue as in the finite-state case.

# Part 15: Tree-Rewriting Systems

# From sequences to trees

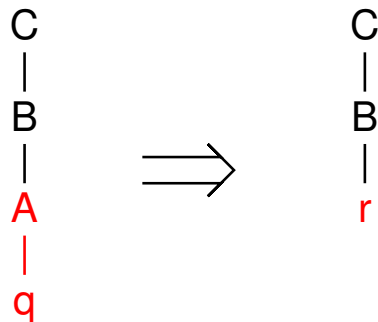
---

Pushdown configurations are *words* (i.e., sequences of symbols).



Alternative view: a sequence is a (degenerated) tree.

PDS rules replace one (degenerated) subtree by another, e.g. for  $qA \hookrightarrow r\varepsilon$ :



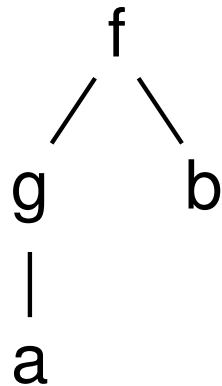
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Let us consider the case where configurations are general trees, and where rules replace one subtree by another.

Motivation:

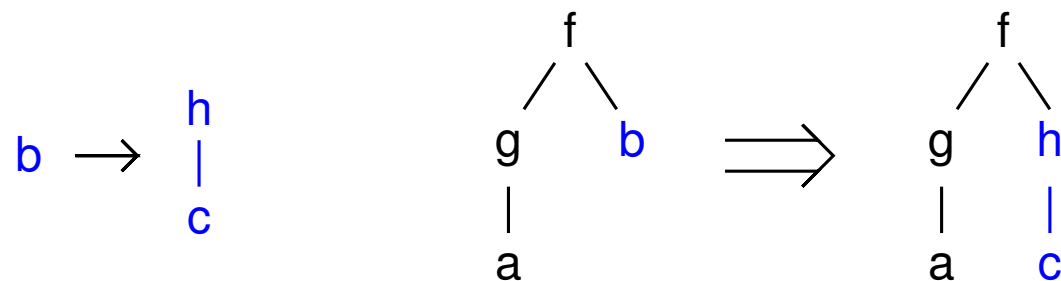
Systems with procedures and threads

Functional languages, e.g., the term  $f(g(a), b)$  is a tree:



---

An equality  $b = h(c)$  corresponds to the rewrite rule:



Notice that if a node has more than one child, their order matters (i.e., there is a designated *first*, *second* etc. child); we speak of **ordered** trees.

In the trees we are interested in, each node is labelled by some symbol. If the symbols of a tree are all taken from an alphabet  $\Sigma$ , we speak of  $\Sigma$ -labelled trees.

# Ground Tree Rewrite Systems

---

A **Ground Tree Rewrite System** (GTRS) is a tuple  $(\Sigma, \Delta)$ , where:

$\Sigma$  is a finite **alphabet**;

$\Delta$  is a finite set of **rewrite rules**;

each rewrite rule is a pair of ordered  $\Sigma$ -labelled trees.

A configuration of a GTRS  $\mathcal{G} = (\Sigma, \Delta)$  is a  $\Sigma$ -labelled tree.

If  $t$  is a configuration of  $\mathcal{G}$ , and  $t'$  a subtree of  $t$  such that  $(t', u') \in \Delta$ , then  $t$  can be **rewritten** to  $u$ , where  $u$  is the tree obtained from  $t$  by replacing  $t'$  with  $u'$  (see previous example).



# GTRS and PDS

---

It is easy to see that GTRS are a generalization of PDS; i.e. each PDS can be expressed as a GTRS (but not necessarily vice versa).

Using the rewriting relation of GTRS, we can define reachability between configurations, i.e.,  $t \Rightarrow u$  if  $u$  can be obtained from  $t$  by zero or more rewritings;  $pre^*$  and  $post^*$  are then defined as in PDS.

We shall see the following:

Like for PDS, there is a representation for (“regular”) sets of trees.

Like for PDS, regularity is closed under reachability.

Unlike for PDS, LTL model checking becomes undecidable!

# Representing sets of trees

---

Like PDS, GTRS are infinite-state systems (i.e., any GTRS may have infinitely many different configurations).

Therefore, like for PDS, we need a representation for infinite sets of configurations.

For PDS, we used *finite automata*. We shall generalize these to *tree automata*.

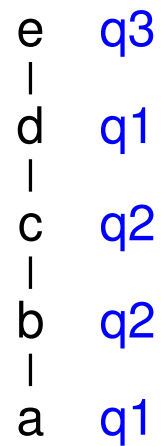
A run of a finite automaton, given input *abcde*, may have the following form (corresponding automaton not shown):

$$\begin{array}{cccccc} & a & b & c & d & e \\ q_0 & \longrightarrow & q_1 & \longrightarrow & q_2 & \longrightarrow & q_2 & \longrightarrow & q_1 & \longrightarrow & q_3 \end{array}$$

A run is considered *accepting* if at the end of the input, we reach a final state.

---

Alternative view: A word is a (degenerated) tree, a run labels each node with a state:



In this view, a run is *accepting* if the root is labelled with a final state.

---

Consider a different way of writing the rules of a finite automaton:

instead of  $q \xrightarrow{a} q'$ , we write  $a(q) \rightarrow q'$ ;

additionally, if  $q$  is the initial state, we also write  $a \rightarrow q'$ .

In a tree automaton, the input is a tree, and a run labels each node with a state. A run is considered accepting if the root is labelled by a final state.

Rules of the form  $a \rightarrow q$  specify how to label *leaves*;

rules of the form  $a(q') \rightarrow q$  specify how to label nodes with one child;

rules of the form  $a(q', q'') \rightarrow q$  are for nodes with two children etc.

The next slides contain the formal definitions.

# Tree automata

---

A **tree automaton** is a tuple  $\mathcal{T} = (\Sigma, Q, F, \delta)$ , where:

$\Sigma$  is a finite **input alphabet**;

$Q$  is a finite set of states;

$F \subset Q$  are the **final states**;

$\delta \subseteq \Sigma \times Q^* \times Q$  is a finite set of **transitions**.

$\mathcal{T}$  is called **deterministic** if for each pair  $(a, \vec{q}) \in \Sigma \times Q^*$  there is at most one  $r \in Q$  such that  $(a, \vec{q}, r) \in \delta$ .

Remark: We denote transition  $(a, q_1 \cdots q_n, q)$  as  $a(q_1, \dots, q_n) \rightarrow q$ .

# Regular tree languages

---

Let  $\mathcal{T} = (\Sigma, Q, F, \delta)$  be a tree automaton and  $t$  a  $\Sigma$ -labelled tree.

Suppose that  $n$  is a node of  $t$  labelled by  $a \in \Sigma$ .

We say that  $n$  can be labelled by  $q \in Q$  if

either  $n$  is a leaf and  $\delta$  contains  $a \rightarrow q$ ,

or  $n$  has children that can be labelled by  $q_1, \dots, q_n$  (in that order), and  $\delta$  contains  $a(q_1, \dots, q_n) \rightarrow q$ .

We say that  $t$  is **accepted** by  $\mathcal{T}$  if its root can be labelled by some final state of  $\mathcal{T}$ .

The set of trees accepted by  $\mathcal{T}$  is called the **language** of  $\mathcal{T}$ , denoted  $\mathcal{L}(\mathcal{T})$ .

A set of trees is called **regular** if there is a tree automaton accepting it.

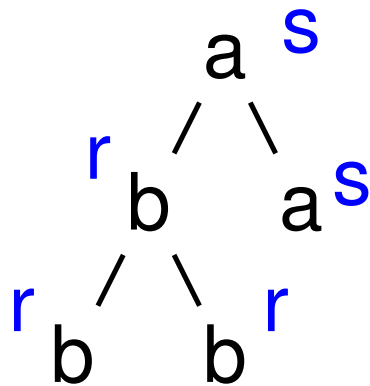
# Example

---

Let  $\mathcal{T} = (\Sigma, Q, F, \delta)$ , where  $\Sigma = \{a, b\}$ ,  $Q = \{r, s\}$ ,  $F = \{s\}$ , and  $\delta$  contains the following rules:

$$a \rightarrow s, \quad a(r, s) \rightarrow s, \quad b \rightarrow r, \quad b(r, r) \rightarrow r.$$

In the tree shown below, the possible labels are shown in blue.



Since the root is labelled by final state  $s$ , the tree is accepted.

# Tree automata with empty moves

---

Like in finite automata, it is sometimes convenient to consider tree automata with “empty” moves.

An “empty” move is denoted by a rule  $q \rightarrow q'$ , meaning that any state that can be labelled with  $q$  can also be labelled with  $q'$ .

Like for finite automata, empty moves can only occur in nondeterministic automata; they can be eliminated by adding, for each pair of rules  $a(\vec{q}) \rightarrow q$  and  $q \rightarrow q'$ , another rule  $a(\vec{q}) \rightarrow q'$ .

Thus, like for finite automata, empty moves do not increase the expressive power of tree automata.



# Notes on tree automata

---

The tree automata considered here are also called **bottom-up tree automata**.

Like regular word languages, regular tree languages are closed under union, intersection, and negation.

Like regular word languages, deterministic tree automata are equally powerful as non-deterministic ones.

The corresponding operations on tree automata can be adapted from those on finite automata.

# Reachability on GTRS

---

Let  $\mathcal{G} = (\Sigma, \Delta)$  be a GTRS and  $\mathcal{L}$  be a set of  $\Sigma$ -labelled trees.

We make the following claims (analogous to PDS):

If  $\mathcal{L}$  is regular, then so are  $pre^*(\mathcal{L})$  and  $post^*(\mathcal{L})$ .

If  $\mathcal{T}$  is a tree automaton accepts  $\mathcal{L}$ , then  $\mathcal{T}$  can be transformed into an automaton accepting  $pre^*(\mathcal{L})$  (or  $post^*(\mathcal{L})$ , respectively).

---

In the following, we shall transform  $\mathcal{L}$  into a tree automaton  $\mathcal{T}'$  such that  $\mathcal{L}(\mathcal{T}') = pre^*(\mathcal{L})$ .

1. Set  $\mathcal{T}' := \mathcal{T}$ .
2. For each pair  $(t, u) \in \Delta$ , build a tree automaton  $\mathcal{T}_t$  that accepts just  $t$ . Assume that  $\mathcal{T}_t$  has one single final state  $q_t$ . Add the states and transitions of  $\mathcal{T}_t$  to  $\mathcal{T}'$ , however  $q_t$  will not be accepting in  $\mathcal{T}'$ .
3. If  $(t, u) \in \Delta$  and  $u$  can be labelled by some state  $q$  of  $\mathcal{T}'$ , then add an empty move  $q_t \rightarrow q$  to  $\mathcal{T}'$ .
4. Repeat step 3 until no more additions are possible.

Note: For  $post^*$ , just switch the left/right-hand sides of the rules.

# LTL on GTRS

---

We now show that the following problem is **undecidable**.

Let  $\mathcal{G}$  be a GTRS and  $\phi$  an LTL formula, does  $\mathcal{G} \models \phi$ ?

Beweisidee:

Reduction to the halting problem of Turing machines on empty tape.

Given a TM  $\mathcal{M}$  we construct  $\mathcal{G}$  and  $\phi$ , such that  $\mathcal{G} \models \phi$  iff  $\mathcal{M}$  stops.

# Recap: Turing machines

---

Let  $(\Sigma, Q, q_0, q_f, \#, \delta)$  be a Turing machine with

tape alphabet  $\Sigma$ , control states  $Q$ , initial state  $q_0$ ,

accepting state  $q_f$ , empty tape symbol  $\#$ , transitions  $\delta$ .

W.l.o.g., we assume that  $\mathcal{M}$  is deterministic.

Moreover, let  $\# \notin \Sigma$ . We define  $\Sigma' := \Sigma \cup \{\#\}$ .

The transitions in  $\delta$  are of the form  $(p, a, X, q, b)$  with  $p, q \in Q$ ,  $a, b \in \Sigma$ ,  $X \in \{L, N, R\}$ .

# Construction of the GTRS

---

Our GTRS has the following alphabet:

$$\{S\} \cup \{(Q, q) \mid q \in Q\} \cup \{(X, a) \mid X \in \{L, R\}, a \in \Sigma\}$$

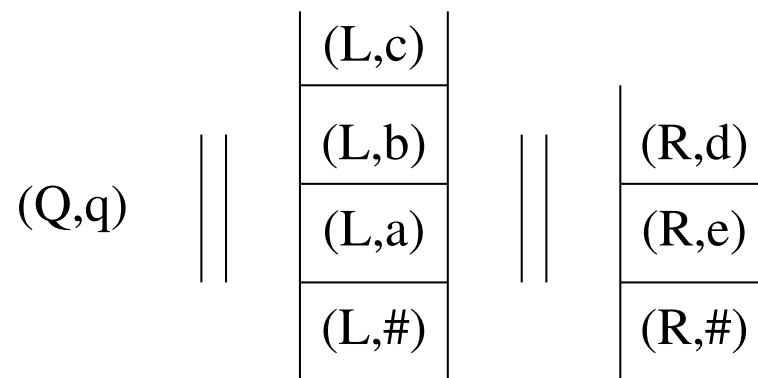
The initial configuration is the tree with a root and three children labelled  $(Q, q_0)$ ,  $(L, \#)$ , and  $(R, \#)$ , respectively. (In the following, these are called the Q/L/R-components.)

All other rules modify one of these three components.

# Reachable trees of the GTRS

---

Each reachable tree will have three components below the root, one consisting of a single node, and two stacks:



Each such configuration can be interpreted as a configuration of  $\mathcal{M}$  (for instance, the tape content  $\#abcde\#$  with control state  $q$ , where the reading head is on  $c$ ).

# Rules of the GTRS

---

The other rules simulate the usual operations on the stack, where the actions “record” what happens.

Operations on the Q-component:

$$(Q, q) \xrightarrow{(Q, q, q')} (Q, q') \text{ for all } q, q' \in Q \text{ and } q \neq q_f \text{ and } (Q, q_f) \xrightarrow{\text{Halt}} \varepsilon$$

Operations on the L/R-components:

$$(X, a) \xrightarrow{(X, a, b)} (L, b) \text{ for all } X \in \{L, R\}, a, b \in \Sigma'$$

$$(X, a) \xrightarrow{(X, a, \varepsilon)} \varepsilon \text{ for all } X \in \{L, R\}, a \in \Sigma$$

$$(X, \#) \xrightarrow{(X, \#, \varepsilon)} (X, \#) \text{ for all } X \in \{L, R\}$$

$$(X, a) \xrightarrow{(X, a, bc)} (X, c) \cdot (X, b) \text{ for all } X \in \{L, R\}, a, b, c \in \Sigma'$$



# Behaviour of the GTRS

---

Note:  $\mathcal{G}$  can emulate the (single) behaviour of  $\mathcal{M}$ , but additionally it can do completely different things.

Our LTL formula  $\phi$  will have the following meaning: if  $\mathcal{G}$  behaves like  $\mathcal{M}$ , then it reachabes the accepting state, i.e.,  $\phi$  has the form

$$\phi_{\mathcal{M}} \rightarrow \mathbf{F} \textit{Halt}$$

$\phi_{\mathcal{M}}$  specifies how  $\mathcal{G}$  should behave in order to emulate  $\mathcal{M}$ .

# Construction of $\phi_{\mathcal{M}}$

---

Each step of  $\mathcal{M}$  modifies the three components; we demand that  $\mathcal{G}$  works on them in turns.

$$\phi_1 := Q \wedge \mathbf{G}(Q \rightarrow (\mathbf{X}L \wedge \mathbf{X}\mathbf{X}R \wedge \mathbf{X}\mathbf{X}\mathbf{X}(Q \vee \text{Halt})))$$

Here,  $Q, L, R$  are abbreviations for the disjunction of all actions starting with  $Q, L, R$ , respectively.

For each transition  $t = (p, a, X, q, b) \in \delta$ , we define a formula that is true iff a sequence of three steps corresponds to an execution of  $t$ .

$$\text{falls } X = N: \phi_t := (Q, p, q) \wedge \mathbf{X}(L, a, b) \wedge \mathbf{X}\mathbf{X} \bigvee_{c \in \Sigma'} (R, c, c)$$

$$\text{falls } X = L: \phi_t := (Q, p, q) \wedge \mathbf{X}(L, a, \varepsilon) \wedge \mathbf{X}\mathbf{X} \bigvee_{c \in \Sigma'} (R, c, cb)$$

$$\text{falls } X = R: \phi_t := (Q, p, q) \wedge \bigvee_{c \in \Sigma'} (\mathbf{X}(L, a, bc) \wedge \mathbf{X}\mathbf{X}(R, c, \varepsilon))$$

# The end

---

We now define

$$\phi_{\mathcal{M}} := \phi_1 \wedge \mathbf{G}(Q \rightarrow \bigvee_{t \in \delta} \phi_t)$$

Thus,  $\phi$  says that each correct run (there is only one!) reaches the accepting state. This is the case iff  $\mathcal{M}$ , starting on the empty tape, holds, which is undecidable.