Resolution for predicate logic

Gilmore's algorithm is correct, but useless in practice.

We upgrade resolution to make it work for predicate logic.

Recall: resolution in propositional logic

Resolution step:

$$\frac{\{L_1, \ldots, L_n, A\} \qquad \{L'_1, \ldots, L'_m, \neg A\}}{\{L_1, \ldots, L_n, L'_1, \ldots, L'_m\}}$$

Mini-example:

$$\frac{\dfrac{\{\neg A, B\} \qquad \{A\}}{\{B\}} \qquad \{\neg B\}}{\square}$$

A set of clauses is unsatisfiable iff the empty clause can be derived.

Adapting Gilmore's Algorithm

Gilmore's Algorithm:

Let F be a closed formula in Skolem form and let $\{F_1, F_2, F_3, \ldots,\}$ be an enumeration of $E(F)$.

$n := 0$;
repeat $n := n + 1$;
until $(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$ is unsatisfiable;
(this can be checked with any calculus
for propositional logic)
report "unsatisfiable" and **halt**

"Any calculus" $\rightsquigarrow$ use resolution for the unsatisfiability test

Recall: Definition of Res

Definition: Let F be a set of clauses. The set of clauses $Res(F)$ is defined by

$$Res(F) = F \cup \{R \mid R \text{ is a resolvent of two clauses } F\}.$$

We set:

$$\begin{aligned} Res^0(F) &= F \\ Res^{n+1}(F) &= Res(Res^n(F)) \qquad \text{for } n \geq 0 \end{aligned}$$

and define

$$Res^*(F) = \bigcup_{n \geq 0} Res^n(F).$$

Ground clauses

A ground term is a term without occurrences of variables.

A ground formula is a formula in which only ground terms occur.

A predicate clause is a disjunction of atomic formulas.

A ground clause is a disjunction of ground atomic formulas.

A ground instance of a predicate clause K is the result of substituting ground terms for the variables of K.

Clause Herbrand expansion

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n F^*$ be a closed formula in Skolem form with matrix F^* in clause form, and let $C_1, \ldots, C_m$ be the set of predicate clauses of F^*.

The clause Herbrand expansion of F is the set of ground clauses

$$CE(F) = \bigcup_{i=1}^{m} \{C_i[y_1/t_1][y_2/t_2] \ldots [y_n/t_n] \mid t_1, t_2, \ldots, t_n \in D(F)\}$$

Lemma: $CE(F)$ is unsatisfiable iff $E(F)$ is unsatisfiable.

Proof: Follows immediately from the definition of satisfiability for sets of formulas.

Ground resolution algorithm

Let $C_1, C_2, C_3, \ldots$ be an enumeration of $CE(F)$.

$n := 0$;
$S := \emptyset$;
repeat
  $n := n + 1$;
  $S := S \cup \{C_n\}$;
  $S := \mathit{Res}^*(S)$
until $\square \in S$

report "unsatisfiable" and **halt**

Ground resolution theorem

Ground Resolution Theorem: A formula $F = \forall y_1 \ldots \forall y_n \ F^*$ with matrix F^* in clause form is unsatisfiable iff there is a set of ground clauses $C_1, \ldots, C_m$ such that:

- C_m is the empty clause, and
- for every $i = 1, \ldots, m$
  - either C_i is a ground instance of a clause $K \in F^*$, i.e., $C_i = K[y_1/t_1] \ldots [y_n/t_n]$ where $t_j \in D(F)$,
  - or C_i is a resolvent of two clauses C_a, C_b with $a < i$ and $b < i$

Proof sketch: If F is unsatisfiable, then $C_1, \ldots, C_m$ can be easily extracted from S by leaving clauses out.

Substitutions

A substitution sub is a (partial) mapping of variables to terms.

An atomic substitution is a substitution which maps one single variable to a term.

$F sub$ denotes the result of applying the substitution sub to the formula F.

$t\ sub$ denotes the result of applying the substitution sub to the term t

Substitutions

The concatenation $sub_1 sub_2$ of two substitutions sub_1 and sub_2 is the substitution that maps every variable x to $sub_2(sub_1(x))$.

(First apply sub_1 and then sub_2.)

Substitutions

Two substitutions sub_1, sub_2 are equivalent if $t\, sub_1 = t\, sub_2$ for every term t.

Every substitution is equivalent to a concatenation of atomic substitutions. For instance, the substitution

$$x \mapsto f(h(w)) \quad y \mapsto g(a, h(w)) \quad z \mapsto h(w)$$

is equal to the concatenation

$$[x/f(z)]\,[y/g(a, z)]\,[z/h(w)].$$

Swapping Lemma for substitutions

Lemma: If $x \notin \mathrm{dom}(sub)$ and x appears in none of the terms $ysub$ with $y \in \mathrm{dom}(sub)$, then

$$[x/t]sub = sub[x/tsub].$$

Examples:

- $[x/f(y)]\underbrace{[y/g(z)]}_{sub} = [y/g(z)][x/f(g(z))]$
- but $[x/f(y)]\underbrace{[x/g(z)]}_{sub} \neq [x/g(z)][x/f(y)]$
- and $[x/z]\underbrace{[y/x]}_{sub} \neq [y/x][x/z]$

Unifier and most general unifier

Let $\mathbf{L} = \{L_1, \ldots, L_k\}$ be a set of literals of predicate clauses (terms).
A substitution sub is a unifier of $\mathbf{L}$ if

$$L_1 sub = L_2 sub = \ldots = L_k sub$$

i.e., if $|\mathbf{L}sub| = 1$, where $\mathbf{L}sub = \{L_1 sub, \ldots, L_k sub\}$.

A unifier sub of $\mathbf{L}$ is a most general unifier of $\mathbf{L}$ if for every unifier sub' of $\mathbf{L}$ there is a substitution s such that $sub' = sub\, s$.

Exercise

Unifiable?			Yes	No
	$P(f(x))$	$P(g(y))$		
	$P(x)$	$P(f(y))$		
	$P(x, f(y))$	$P(f(u), z)$		
	$P(x, f(y))$	$P(f(u), f(z))$		
	$P(x, f(x))$	$P(f(y), y)$		
	$P(x, g(x), g^2(x))$	$P(f(z), w, g(w))$		
$P(x, f(y))$	$P(g(y), f(a))$	$P(g(a), z)$		

Unification algorithm

Input: a set $\mathbf{L} \neq \emptyset$ of literals

$sub := []$ (the empty substitution)

while $|\mathbf{L}sub| > 1$ **do**

  Find the first position at which two literals $L_1, L_2 \in \mathbf{L}sub$ differ

  if none of the two characters at that position is a variable then

  then report "non-unifiable" and **halt**

  else let x be the variable and t the term starting at that position

    (possibly another variable)

    if x occurs in t

    then report "non-unifiable" and **halt**

    else $sub := sub\,[x/t]$

report "unifiable" and **return** sub

Correctness of the unification algorithm

Lemma: The unification algorithm terminates.

Proof: Every execution of the **while**-loop (but the last) substitutes a variable x by a term t not containing x, and so the number of variables occurring in $\mathbf{L}sub$ decreases by one.

Lemma: If $\mathbf{L}$ is non-unifiable then the algorithm reports "non-unifiable".

Proof: If $\mathbf{L}$ is non-unifiable then the algorithm can never exit the loop.

Lemma: If $\mathbf{L}$ is unifiable then the algorithm reports "unifiable" and returns the most general unifier of $\mathbf{L}$ (and so in particular every unifiable set $\mathbf{L}$ has a most general unifier).

Resolution for predicate logic

A clause R is a resolvent of two predicate clauses C_1, C_2 if the following holds:

- There are renamings of variables s_1, s_2 (particular cases of substitutions) such that no variable occurs in both $C_1\, s_1$ and $C_2\, s_2$.
- There are literals $L_1, \ldots, L_m$ (with $m \geq 1$) in $C_1\, s_1$ and literals $L'_1, \ldots, L'_n$ (with $n \geq 1$) in $C_2\, s_2$ such that the set

$$\mathbf{L} = \{\overline{L_1}, \ldots, \overline{L_m}, L'_1, \ldots, L'_n\}$$

  is unifiable. Let sub be a most general unifier of $\mathbf{L}$.
- $R = ((C_1\, s_1 - \{L_1, \ldots, L_m\}) \cup (C_2\, s_2 - \{L'_1, \ldots, L'_n\}))sub.$

Correctness and completeness

Questions:

- If using predicate resolution $\Box$ can be derived from F then F is unsatisfiable (correctness)
- If F is unsatisfiable then predicate resolution can derive the empty clause $\Box$ from F (completeness)

Exercise

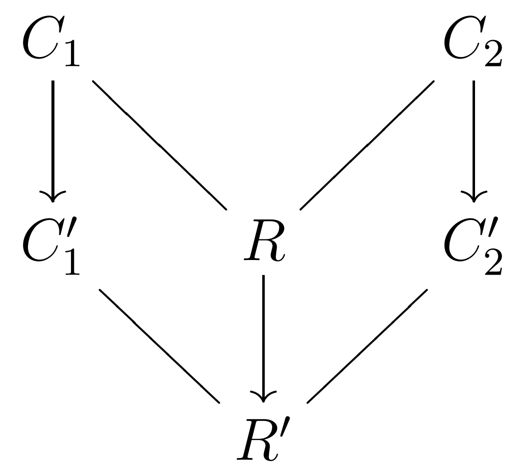
Do the following pairs of predicate clauses have a resolvent?
How many resolvents are there?

C_1	C_2	Resolvents
$\{P(x), Q(x,y)\}$	$\{\neg P(f(x))\}$	
$\{Q(g(x)), R(f(x))\}$	$\{\neg Q(f(x))\}$	
$\{P(x), P(f(x))\}$	$\{\neg P(y), Q(y,z)\}$	

Lifting-Lemma

Let C_1, C_2 be predicate clauses and let C_1', C_2' be two ground instances of them that can be resolved into the resolvent R'.

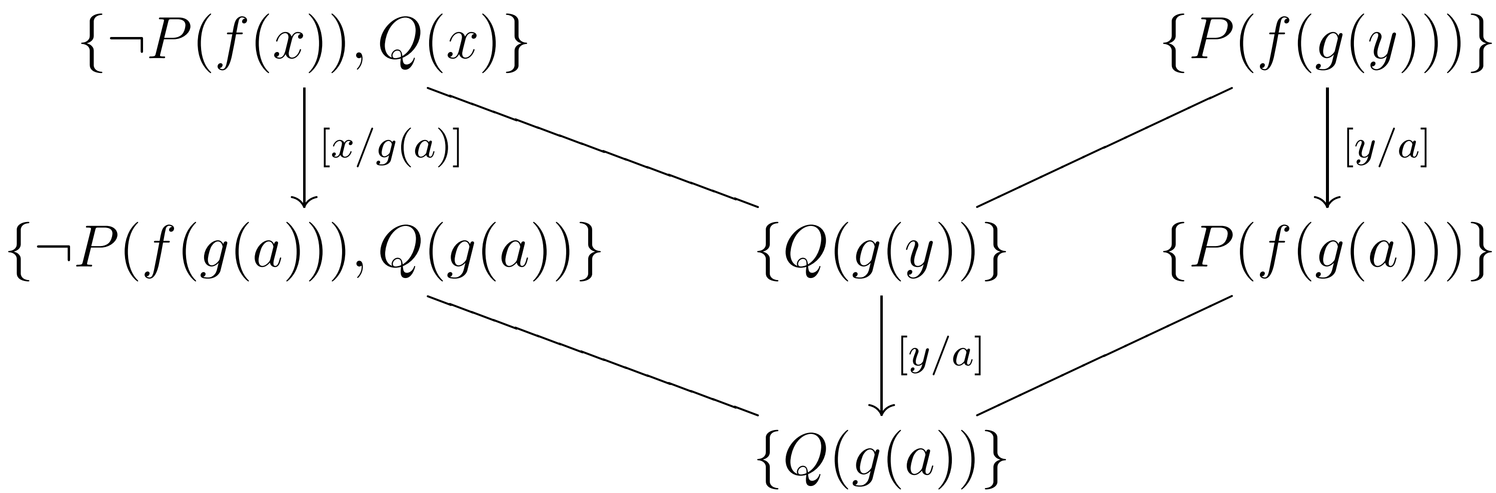
Then there is predicate resolvent R of C_1, C_2 such that R' is a ground instance of R.



—: Resolution

$\rightarrow$: Substitution

Lifting-Lemma: example



Universal closure

The universal closure of a formula H with free variables $x_1, \ldots, x_n$ is the formula

$$\forall H = \forall x_1 \forall x_2 \ldots \forall x_n H$$

Let F be a closed formula in Skolem form with matrix F^*. Then

$$F \equiv \forall F^* \equiv \bigwedge_{K \in F^*} \forall K$$

Example:

$$\begin{aligned} F^* &= P(x,y) \wedge \neg Q(y,x) \\ F &\equiv \forall x \forall y (P(x,y) \wedge \neg Q(y,x)) \equiv \forall x \forall y P(x,y) \wedge \forall x \forall y (\neg Q(y,x)) \end{aligned}$$

Predicate Resolution Theorem

Resolution Theorem of Predicate Logic:

Let F be a closed formula in Skolem form with matrix F^* in predicate clause form. F is unsatisfiable iff $\Box \in Res^*(F^*)$.

Exercise

Is the set of clauses

$$\{\{P(f(x))\}, \{\neg P(f(x)), Q(f(x), x)\}, \{\neg Q(f(a), f(f(a)))\},$$
$$\{\neg P(x), Q(x, f(x))\}\}$$

unsatisfiable?

Demo

We consider the following set of predicate clauses (Schöning):

$$F = \{\{\neg P(x), Q(x), R(x, f(x))\}, \{\neg P(x), Q(x), S(f(x))\}, \{T(a)\}, \{P(a)\}, \{\neg R(a, x), T(x)\}, \{\neg T(x), \neg Q(x)\}, \{\neg T(x), \neg S(x)\}\}$$

and prove it is unsatisfiable with `otter`.

Refinements of resolution

Problems of predicate resolution:

- Branching degree of the search space too large
- Too many dead ends
- Combinatorial explosion of the search space

Solution:

Strategies and heuristics: forbid certain resolution steps, which narrows the search space.

But: Completeness must be preserved!