Resolution

For every formula F:

$$(F \vee A) \wedge (F' \vee \neg A) \equiv (F \vee A) \wedge (F' \vee \neg A) \wedge (F \vee F')$$

Or in clause form

$$\{\ \{\clubsuit, A\}\ ,\ \{\spadesuit, \neg A\}\ \}\ \equiv\ \{\ \{\clubsuit, A\}\ ,\ \{\spadesuit, \neg A\}\ ,\ \{\clubsuit, \spadesuit\}\ \}$$

If $F \vee F'$ is the "empty disjunction" ($=$ empty clause) then the formula is unsatisfiable.

- Is it always possible to derive the empty clause from any unsatisfiable formula?
(completeness)
- Can we represent derivations in more compact form?
(without carrying always all clauses around)

Clause representation of CNF formulas

- Clause: set of literals (disjunction).

$$\{A, B\} \text{ stands for } A \vee B.$$

- Formula: set of clauses (conjunction).

$$\{\{A, B\}, \{\neg A, B\}\} \text{ stands for } (A \vee B) \wedge (\neg A \vee B).$$

- Block: set of formulas (disjunction).

$$\{F, G\} \text{ stands for } F \vee G.$$

The empty clause stands for $\mathbf{false}$ or 0.

The empty formula stands for $\mathbf{true}$ or 1.

The empty block stands for $\mathbf{false}$.

Advantages of the clause form

We get “for free”:

- Commutativity:
  $A \vee B \equiv B \vee A$, both represented by $\{A, B\}$
- Associativity:
  $(A \vee B) \vee C \equiv A \vee (B \vee C)$, both represented by $\{A, B, C\}$
- Idempotence:
  $(A \vee A) \equiv A$, both represented by $\{A\}$

Resolvent (I)

Definition: Let C_1, C_2 and R be clauses. R is a resolvent of C_1 and C_2 if there is a literal L such that $L \in C_1$, $\overline{L} \in C_2$ and

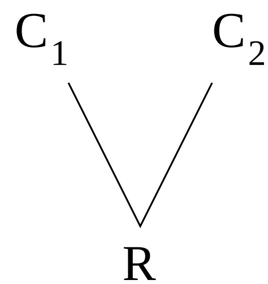
$$R = (C_1 - \{L\}) \cup (C_2 - \{\overline{L}\})$$

where $\overline{L}$ is defined by

$$\overline{L} = \begin{cases} \neg A_i & \text{if } L = A_i \\ A_i & \text{if } L = \neg A_i \end{cases}$$

Resolvent (II)

Graphical representation:



If $C_1 = \{L\}$ and $C_2 = \{\overline{L}\}$ then the empty clause is a resolvent of C_1 and C_2. We represent it with the special symbol $\square$.

Recall: $\square \equiv \mathbf{false}$.

Resolution Lemma

Resolution Lemma: Let F be a formula in **CNF**, represented as a set of clauses, and let R be a resolvent of two clauses C_1 and C_2 in F. Then the formulas F and $F \cup \{R\}$ are equivalent.

Proof: Follows immediately from

$$\underbrace{(F_1 \vee A)}_{C_1} \wedge \underbrace{(F_2 \vee \neg A)}_{C_2} \equiv \underbrace{(F_1 \vee A)}_{C_1} \wedge \underbrace{(C_2 \vee \neg A)}_{C_2} \wedge \underbrace{(F_1 \vee F_2)}_{R}$$

Resolution calculus

A calculus is a set of syntactic transformation rules allowing to decide semantic properties.

- Syntactic rules: resolution, halt when the empty clause is derived.
- Semantic property: unsatisfiabilty.

Example

We wish to prove that

$$((Ab \vee Bb) \wedge (Ab \rightarrow Bb) \wedge (Bb \wedge Ro \rightarrow \neg Ab) \wedge Ro) \rightarrow (\neg Ab \wedge Bb)$$

is valid. This is the case iff

$$(Ab \vee Bb) \wedge (\neg Ab \vee Bb) \wedge (\neg Bb \vee \neg Ro \vee \neg Ab) \wedge Ro \wedge (Ab \vee \neg Bb)$$

is unsatisfiable. (Recall: $F \rightarrow G$ valid iff $F \wedge \neg G$ unsatisfiable.)

Desirable properties of a calculus

- Correctness (or consistency): If the application of the syntactic rules say that the semantic property holds, then this is indeed the case.
  If the empty clause can be derived from F then F is unsatisfiable.
- Completeness: If the semantic property holds, then this can be shown with the help of the syntactic rules.
  If F is unsatisfiable then the empty clause can be derived from F.

Definition of $Res(F)$

Definition: Let F be a set of clauses. The formula $Res(F)$ is defined as follows:

$$Res(F) = F \cup \{R \mid R \text{ ist a resolvent of two clauses in } F\}.$$

Furthermore, define

$$\begin{array}{rcll} Res^0(F) & = & F & \\ Res^{n+1}(F) & = & Res(Res^n(F)) & \text{für } n \geq 0 \end{array}$$

and finally let

$$Res^*(F) = \bigcup_{n \geq 0} Res^n(F).$$

Exercise

Assume n atomic formulas occur in F. Then:

A	$\lvert Res^*(F)\rvert \leq 2^n$	**B**	$\lvert Res^*(F)\rvert \leq 4^n$
C	$\lvert Res^*(F)\rvert$ can be arbitrarily large		

Resolution Theorem

We prove that resolution is correct and complete:

Resolution Theorem (of propositional logic):
A set of clauses F is unsatisfiable iff $\square \in Res^*(F)$.

Correctness: $\square \in Res^*(F) \Rightarrow F$ is unsatisfiable follows immediately from the resolution lemma.

Completeness proof (I)

Completeness: F is unsatisfiable $\Rightarrow \Box \in Res^*(F)$
By induction on the number of atomic formulas in F.

Here: Induction step with $n+1=4$

$$F = \{\{A_1\}, \{\neg A_2, A_4\}, \{\neg A_1, A_2, A_4\}, \{A_3, \neg A_4\}, \{\neg A_1, \neg A_3, \neg A_4\}\}$$

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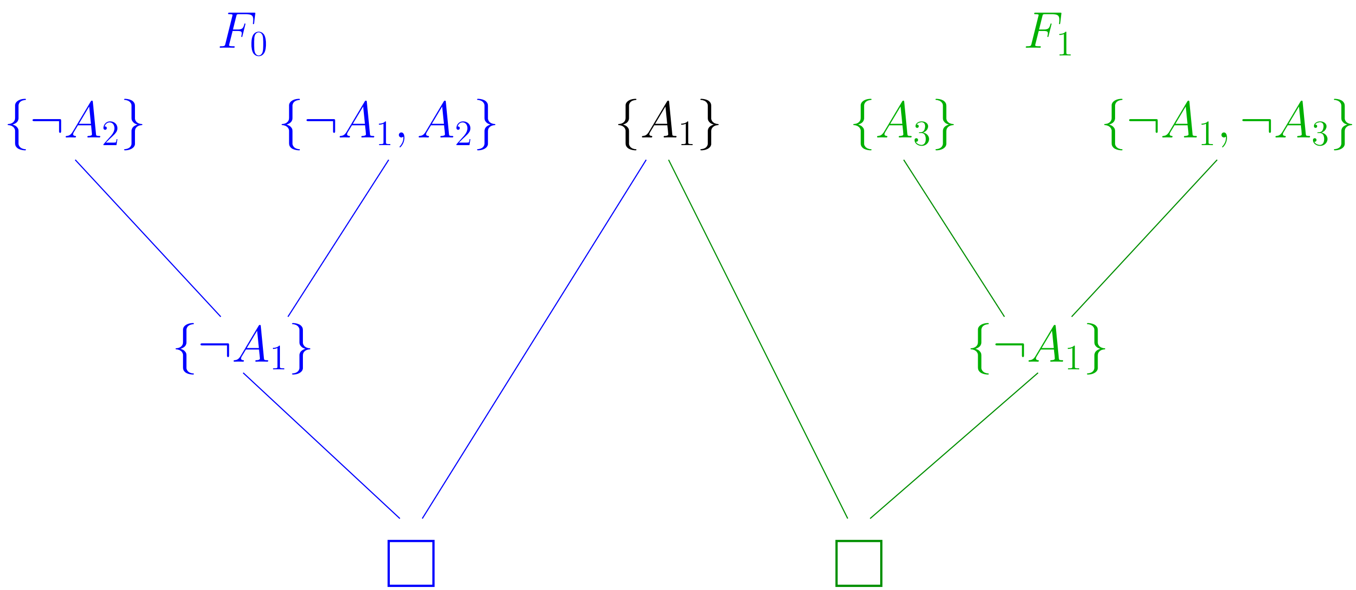
Completeness proof (II)



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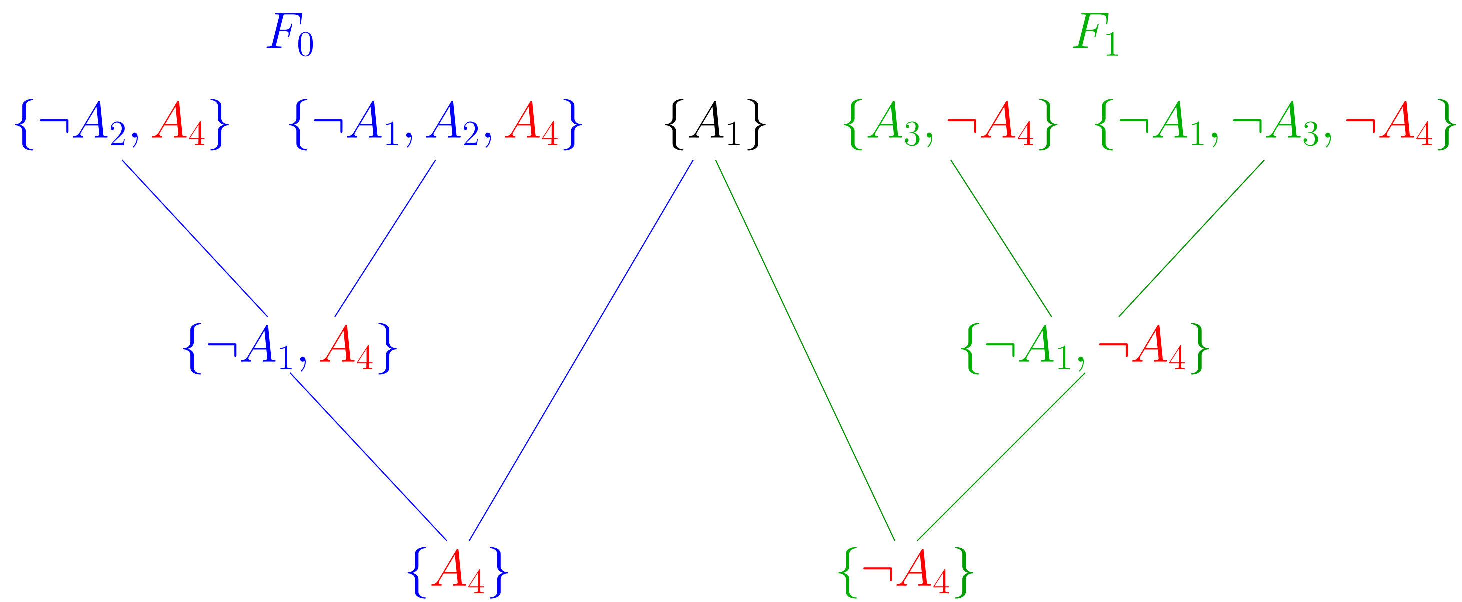

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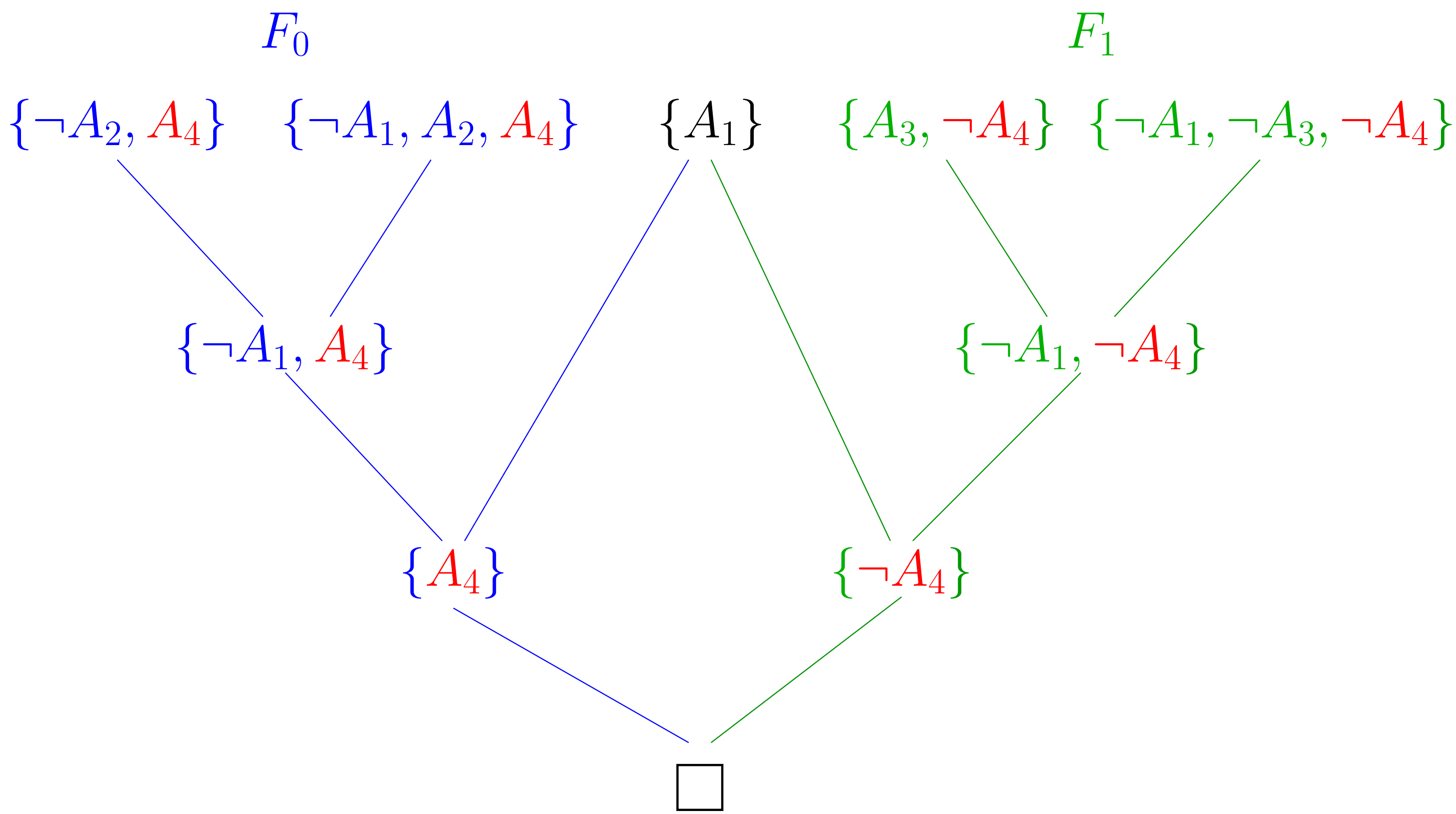

Definition

A derivation (or proof) of the empty clause from a set F of clauses is a sequence $C_1, C_2, \ldots, C_m$ of clauses such that:

> C_m is the empty clause and for every $i = 1, \ldots, m$ it holds that C_i is either a clause in F or a resolvent of two clauses C_a,C_b with $a, b < i$.

F is unsatisfiable iff a derivation of the empty clause from F exists.