

Predicate logic with equality

Predicate logic

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distinguished predicate symbol “=” of arity 2.

Semantics : a structure \mathcal{A} of predicate logic with equality always maps the predicate symbol = to the identity relation:

$$\mathcal{A}(=) = \{(d, d) \mid d \in U_{\mathcal{A}}\} .$$

Fact: A structure is model of $\exists x \forall y \ x=y$ iff its universe is a singleton.

Modelling equality

Let F be a formula of predicate logic with equality, and let Eq be a predicate symbol that does not occur in F .

Let G_F be the conjunction of the following formulas:

$$\forall x Eq(x, x)$$

$$\forall x \forall y (Eq(x, y) \rightarrow Eq(y, x))$$

$$\forall x \forall y \forall z ((Eq(x, y) \wedge Eq(y, z)) \rightarrow Eq(x, z))$$

$$\forall x_1 \dots \forall x_n \forall y (Eq(x_i, y) \rightarrow Eq(f(x_1, \dots, x_i, \dots, x_n), f(x_1, \dots, y, \dots, x_n)))$$

for every function symbol f of F and every $1 \leq i \leq n$

$$\forall x_1 \dots \forall x_n \forall y (Eq(x_i, y) \rightarrow (P(x_1, \dots, x_i, \dots, x_n) \leftrightarrow P(x_1, \dots, y, \dots, x_n)))$$

for every predicate symbol P of F and every $1 \leq i \leq n$

Let H_F be the formula obtained from F by substituting every occurrence of “=” by “ Eq ”.

Theorem: The formulas F and $G_F \wedge H_F$ are sat-equivalent.

Proof: We show that if $G_F \wedge H_F$ is satisfiable then F is satisfiable. Let \mathcal{A} be a model of $G_F \wedge H_F$. Then $Eq^{\mathcal{A}}$ is an equivalence relation. For every $d \in U_{\mathcal{A}}$ let $[d]$ be the equivalence class of d . Define the structure \mathcal{B} as follows:

- $U_{\mathcal{B}} = \{[d] \mid d \in U_{\mathcal{A}}\}$.
- For every function symbol f of F :
 $f^{\mathcal{B}}([d_1], \dots, [d_n]) = [f^{\mathcal{A}}(d_1, \dots, d_n)]$
- For every predicate symbol P of F :
 $([d_1], \dots, [d_n]) \in P^{\mathcal{B}}$ iff $(d_1, \dots, d_n) \in P^{\mathcal{A}}$

\mathcal{B} is well defined because $\mathcal{A} \models G_F$.

Since $\mathcal{A} \models H_F$ we get $\mathcal{B} \models F$.

An application

Theorem: Every formula without function symbols of the form $\forall x \exists u \forall y F^*$ is sat-equivalent to a formula without function symbols of the form $\forall x \forall y \forall z \exists v G^*$.

Proof: Let P be a predicate symbol not occurring in F . Let

$$H = \forall x \forall y \forall z \exists v (P(x, v) \wedge (P(x, y) \rightarrow y=v) \wedge (P(x, z) \rightarrow F^*[u/z]))$$

It is easy to see that the original formula and H are sat-equivalent

Replace $y = v$ by $Eq(y, v)$ in H , add the formulas expressing that Eq is a congruence, convert the resulting formula into prenex form, and let G be the result.

We have that H and G are sat-equivalent, and so the original formula and G are sat-equivalent.