Equivalences

Theorem. Let F and G be arbitrary formulas.

(1) $\neg\forall x F \equiv \exists x \neg F$
$\neg\exists x F \equiv \forall x \neg F$

(2) If x does not occur free in G then:
$(\forall x F \wedge G) \equiv \forall x(F \wedge G)$
$(\forall x F \vee G) \equiv \forall x(F \vee G)$
$(\exists x F \wedge G) \equiv \exists x(F \wedge G)$
$(\exists x F \vee G) \equiv \exists x(F \vee G)$

(3) $(\forall x F \wedge \forall x G) \equiv \forall x(F \wedge G)$
$(\exists x F \vee \exists x G) \equiv \exists x(F \vee G)$

(4) $\forall x \forall y F \equiv \forall y \forall x F$
$\exists x \exists y F \equiv \exists y \exists x F$

Conversion Lemma

Let F be a formula, let x be a variable and let t be a term. By $F[x/t]$ we denote the formula one obtains from F by replacing every free occurrence of x by t.

Similarly by $\mathcal{A}[x/t]$ we denote the mapping that agrees with $\mathcal{A}$ on everything but on x, we set $\mathcal{A}_{[x/t]}(x) = \mathcal{A}(t)$.

Lemma (Conversion Lemma). Let F be a formula, let x be a variable and let t be a term that does not contain any occurrence of a variable that is bound by F. Then $\mathcal{A}(F[x/t]) = \mathcal{A}_{[x/\mathcal{A}(t)]}(F)$.

Rectified Formulas

A formula is rectified if no variable occurs both bound and free and if all quantifiers in the formula refer to different variables.

Lemma. Let $F = QxG$ be a formula where $Q \in \{\forall, \exists\}$. Let y be a variable that does not occur free in G. Then $F \equiv QyG[x/y]$.

Lemma. Every formula F is equivalent to a rectified formula G.

Prenex form

A formula is in prenex form if it has the form

$$Q_1 y_1 Q_2 y_2 \dots Q_n y_n F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, all the y_i are variables, and F contains no quantifiers.

Theorem. Every formula is F equivalent to a rectified formula in prenex form (a formula in $\mathbf{RPF}$ G).

Skolem form

The Skolem form of a formula F in $\mathbf{RPF}$ is the result of applying the following algorithm to F:

$\mathbf{while}$ F contains an existential quantifier $\mathbf{do}$

Let G be the formula in $\mathbf{RPF}$
such that $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\ G$
(the block of universal quantifiers may be empty).

Let f be a fresh function symbol of arity n
that does not occur in F.

$F := \forall y_1 \forall y_2 \ldots \forall y_k\ G[z/f(y_1, y_2, \ldots, y_n)]$

i.e., cancel the first existential quantifier in F and
substitute every occurrence of z in G by $f(y_1, y_2, \ldots, y_n)$

We say that two formulas are sat-equivalent if they are both satisfiable or unsatisfiable.

Theorem. A formula in $\mathbf{RPF}$ and its Skolem form are sat-equivalent.

Clause form

A closed formula is in clause form if it is of the form

$$\forall y_1 \forall y_2 \ldots \forall y_n \, F$$

where F contains no quantifiers and is in $\mathbf{CNF}$.

A closed formula in clause form can be represented as a set of clauses.

Example: the clause form of $\forall x \forall y \; ((P(x,y) \wedge Q(x)) \wedge P(f(y),a)$ is

$$\{ \; \{P(x,y), Q(x)\} \; , \; \{P(f(y),a)\} \; \}$$

Converting into clause form up to sat-equivalence

Given: a formula F of predicate logic (with possible occurrences of free variables).

1. Rectify F by systematic renaming of bound variables.
   The result is a formula F_1 equivalent to F.
2. Let $y_1, y_2, \ldots, y_n$ be the variables occurring free in F_1.
   Produce the formula $F_2 = \exists y_1 \exists y_2 \ldots \exists y_n F_1$.
   F_2 is sat-equivalent to F_1 and closed.
3. Produce a formula F_3 in prenex form equivalent to F_2.

4. Eliminate the existential quantifiers in F_3 by transforming F_3 into a Skolem formula F_4.
   The formula F_4 is sat-equivalent to F_3.
5. Convert the matrix of F_4 into $\mathbf{CNF}$ (and write the resulting formula F_5 as set of clauses).

Exercise

Which formulas are rectified, in prenex, Skolem, or clause form?

	R	P	S	C
$\forall x(\mathit{Tet}(x) \lor \mathit{Cube}(x) \lor \mathit{Dodec}(x))$				
$\exists x \exists y(\mathit{Cube}(y) \lor \mathit{BackOf}(x,y))$				
$\forall x(\neg \mathit{FrontOf}(x,x) \land \neg \mathit{BackOf}(x,x))$				
$\neg \exists x \mathit{Cube}(x) \leftrightarrow \forall x \neg \mathit{Cube}(x)$				
$\forall x(\mathit{Cube}(x) \to \mathit{Small}(x)) \to \forall y(\neg \mathit{Cube}(y) \to \neg \mathit{Small}(y))$				
$(\mathit{Cube}(a) \land \forall x \mathit{Small}(x)) \to \mathit{Small}(a)$				
$\exists x(\mathit{Larger}(a,x) \land \mathit{Larger}(x,b)) \to \mathit{Larger}(a,b)$				