Hilbert Calculus

Two kinds of calculi:

- Calculi as basis for automatic techniques
  Examples: Resolution, DPLL, BDDs
- Calculi formalizing mathematical reasoning
  (axiom, hypothesis, lemma . . . , derivation)
  Examples: Hilbert Calculus, Natural Deduction

Resolution Calculus vs. Hilbert Calculus

Resolution calculus	Hilbert calculus
Proves unsatisfiability	Proves consequence $(F_1, \ldots, F_n \models G)$
Formulas in CNF	Formulas with $\neg$ and $\rightarrow$
Syntactic derivation of the empty clause from F	Syntactic derivation of $F_1, \ldots, F_n \vdash G$ from axioms and hypothesis
Goal: automatic proofs	Goal: model mathematical reasoning
Completeness proof comparatively simple	Completeness proof comparatively involved

Recall: Consequence

A formula G is a consequence or follows from the formulas $F_1, \ldots, F_k$ if every model $\mathcal{A}$ of $F_1, \ldots, F_k$ that is suitable for G is also a model of G

If G is a consequence of $F_1, \ldots, F_k$ then we write $F_1, \ldots, F_k \models G$.

Preliminaries

In the following slides, formulas contain only the operators $\neg$ and $\rightarrow$.
Recall: $F \vee G \equiv \neg F \rightarrow G$ and $F \wedge G \equiv \neg(F \rightarrow \neg G)$.

The calculus defines a syntactic consequence relation $\vdash$ (notation: $F_1, \ldots, F_n \vdash G$), intended to "mirror" semantic consequence.

We will have:

$$F_1, \ldots, F_n \vdash G \quad \text{iff} \quad F_1, \ldots, F_n \models G$$

(syntactic consequence and semantic consequence will coincide).

Axiom schemes

We take five axiom schemes or axioms, with F, G as place-holders for formulas:

(1) $F \rightarrow (G \rightarrow F)$

(2) $(F \rightarrow (G \rightarrow H)) \rightarrow ((F \rightarrow G) \rightarrow (F \rightarrow H))$

(3) $(\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$

(4) $F \rightarrow (\neg F \rightarrow G)$

(5) $(\neg F \rightarrow F) \rightarrow F$

An instance of an axiom is the result of substituting the place-holders of the axiom by formulas.
Easy to see: all instances are valid formulas.

Example: Instance of (4) with $\neg A \rightarrow B$ and $\neg C$ for F and G:

$$(\neg A \rightarrow B) \rightarrow (\neg(\neg A \rightarrow B) \rightarrow \neg C)$$

Derivations in Hilbert calculus

Let S be a set of formulas - also called hypothesis - and let F be a formula. We write $S \vdash F$ and say that F is a syntactic consequence of S in Hilbert Calculus if one of these conditions holds:

Axiom:	F is an instance of an axiom
Hypothesis:	$F \in S$
Modus Ponens:	$S \vdash G \rightarrow F$ and $S \vdash G$, i.e. both $G \rightarrow F$ and G are syntactic consequences of S.

Modus Ponens

Derivation rule of the calculus, allowing to generate new syntactic consequences from old ones:

$$\begin{array}{rcl} S & \vdash & G \rightarrow F \\ S & \vdash & G \\ \hline S & \vdash & F \end{array}$$

Example of derivation

1.	$\vdash A \rightarrow ((B \rightarrow A) \rightarrow A)$	Instance of Axiom (1)
2.	$\vdash (A \rightarrow ((B \rightarrow A) \rightarrow A))$ $\rightarrow$ $((A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A))$	Instance of Axiom (2)
3.	$\vdash (A \rightarrow (B \rightarrow A)) \rightarrow (A \rightarrow A)$	Modus Ponens with 1. & 2.
4.	$\vdash A \rightarrow (B \rightarrow A)$	Instance of Axiom (1)
5.	$\vdash A \rightarrow A$	Modus Ponens with 3. & 4.

Remark: The same derivation works for arbitrary formulas F, G instead of A, B, and so we can derive $\vdash F \rightarrow F$ for any formula F.

We can therefore see a derivation as a way of producing new axioms (the axiom $F \rightarrow F$ in this case).

Correctness and completeness

Correctness: If F is a syntactic consequence from S, then F is a consequence of S.

Completeness: If F is a consequence of S, then F is a syntactic consequence from S.

Correctness proof of the Hilbert calculus

Correctness Theorem: Let F be an arbitrary formula, and let S be a set of formulas such that $S \vdash F$. Then $S \models F$.

Proof: Easy induction on the length of the derivation of $S \vdash F$.

Completeness proof: preliminaries

Wie wish to prove: if $S \models F$, then $S \vdash F$. How could this work?

- Induction on the derivation?
  $\leadsto$ there is no derivation!
- Induction on the structure of the formula F?
  For the induction basis we would have to prove for an atomic formula A:

  if $S \models A$ then $S \vdash A$.

  But how do we construct a derivation of $S \vdash A$ if all we know is $S \models A$?

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Proof sketch: Assume $S \models F$.
Then $S \cup \{\neg F\}$ is unsatisfiable by (1).
Then $S \cup \{\neg F\}$ is inconsistent by (4).
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We prove (3) and (4).

(In)consistency

Definition: A set S of formulas is inconsistent if there is a formula F such that $S \vdash F$ and $S \vdash \neg F$, otherwise it is consistent.

Observe: inconsistency is a purely syntactic notion!!

Examples of inconsistent sets

- $\{A, \neg A\}$
- $\{\neg(A \to (B \to A))\}$
- $\{\neg B, \neg B \to B\}$
- $\{C, \neg(\neg C \to D)\}$

Important tool: the Deduction Theorem

Theorem: $S \cup \{F\} \vdash G$ iff $S \vdash F \to G$.

Proof: Assume $S \vdash F \to G$. Then $S \cup \{F\} \vdash F \to G$.
Using $S \cup \{F\} \vdash F$ and Modus Ponens we get $S \cup \{F\} \vdash G$.

Assume $S \cup \{F\} \vdash G$. Proof by induction on the derivation (length):

Axiom/Hypothesis: G is instance of an axiom or $G \in S \cup \{F\}$.
If $F = G$ use example of derivation to prove $S \vdash F \to F$.
Otherwise $S \vdash G$ and by Axiom (1) $S \vdash G \to (F \to G)$.
By Modus Ponens we get $S \vdash F \to G$.

Modus Ponens: Then $S \cup \{F\} \vdash G$ is derived by Modus Ponens
from some $S \cup \{F\} \vdash H \to G$ and $S \cup \{F\} \vdash H$.
By ind. hyp we have $S \vdash F \to (H \to G)$ and $S \vdash F \to H$.
From Axiom (2) we get
$S \vdash (F \to (H \to G)) \to ((F \to H) \to (F \to G))$.
Two applications of Modus Ponens yield $S \vdash F \to G$.

Consequences of the Deduction Theorem

Lemma I: $S \cup \{\neg F\} \vdash F$ iff $S \vdash F$

Proof: Assume $S \cup \{\neg F\} \vdash F$ holds.
By the Deduction Theorem $S \vdash \neg F \rightarrow F$.
Using Axiom (5) we get $S \vdash (\neg F \rightarrow F) \rightarrow F$.
By Modus Ponens we get $S \vdash F$.

The other direction is trivial.

Completeness - Proof of (3)

Assertion (3): $S \vdash F$ iff $S \cup \{\neg F\}$ is inconsistent.

Proof: Assume $S \vdash F$.
Then $S \cup \{\neg F\} \vdash F$.
Since $S \cup \{\neg F\} \vdash \neg F$, the set $S \cup \{\neg F\}$ is inconsistent.

Assume $S \cup \{\neg F\}$ is inconsistent.
Then there is a formula G s.t. $S \cup \{\neg F\} \vdash G$ and $S \cup \{\neg F\} \vdash \neg G$.
By Axiom (4) we get $S \cup \{\neg F\} \vdash G \rightarrow (\neg G \rightarrow F)$.
Two applications of Modus Ponens yield $S \cup \{\neg F\} \vdash F$.
Lemma I yields $S \vdash F$.

Completeness - Proof of (4)

Recall assertion (4):

> Unsatisfiable sets are inconsistent.

We prove the equivalent assertion:

> Consistent sets are satisfiable.

How do we prove an assertion like this?

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How do we prove an assertion like this?

Answer: Construct a satisfying truth assignment $\mathcal{A}$ as follows:

If $A \in S$ then set $\mathcal{A}(A) := 1.$

If $\neg A \in S$ then set $\mathcal{A}(A) := 0.$

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Problem: What do we do if neither $A \in S$ nor $\neg A \in S$?

Perhaps we can avoid the problem?

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Definition: A set S of formulas is maximally consistent if it is consistent and for every formula F either $F \in S$ or $\neg F \in S$.

We extend S to a maximally consistent set $\overline{S} \supseteq S$.

Completeness - Proof sketch for (4)

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(4.1) Every consistent set can be extended to a maximally consistent set.

(4.2) Let S be maximally consistent and let $\mathcal{A}$ be the assignment given by $\mathcal{A}(A) = 1$ if $A \in S$ and $\mathcal{A}(A) = 0$ if $A \notin S$.
Then $\mathcal{A}$ satisfies S.

Proof of (4.1) - Preliminaries

Lemma II: Let S be a consistent set and let F be an arbitrary formula. Then: $S \cup \{F\}$ or $S \cup \{\neg F\}$ (or both) are consistent.

Proof: Assume S is consistent but both $S \cup \{F\}$ and $S \cup \{\neg F\}$ are inconsistent.
Since $S \cup \{\neg F\}$ is inconsistent we have $S \vdash F$ by Assertion (3).
Since $S \cup \{F\}$ is inconsistent there is a formula G s.t. $S \cup \{F\} \vdash G$ and $S \cup \{F\} \vdash \neg G$, and the Deduction Theorem yields $S \vdash F \rightarrow G$ and $S \vdash F \rightarrow \neg G$.
Modus Ponens yields $S \vdash G$ and $S \vdash \neg G$.
This contradicts the assumption that S is consistent.

Proof of (4.1)

Assertion (4.1): Every consistent set can be extended to a maximally consistent set.

Proof: Let $F_0, F_1, F_2 \ldots$ be an enumeration of all formulas. Let $S_0 = S$ and

$$S_{i+1} = \begin{cases} S_i \cup \{F_i\} & \text{if } S_i \cup \{F_i\} \text{ consistent} \\ S_i \cup \{\neg F_i\} & \text{if } S_i \cup \{\neg F_i\} \text{ consistent} \end{cases}$$

(this is well defined by Lemma II)

By definition, every S_i is consistent.
Let $\overline{S} = \bigcup_{i=1}^{\infty} S_i$. If $\overline{S}$ were inconsistent, some finite subset would also be inconsistent. So $\overline{S}$ is consistent.
By definition, $\overline{S}$ is maximally consistent.

Proof of (4.2) - Preliminaries

Lemma III: Let S be a maximally consistent set:

(1) For every formula F: $F \in S$ iff $S \vdash F$.

(2) For every formula F: $\neg F \in S$ iff $F \notin S$.

(3) For every two formulas F, G: $F \to G \in S$ iff $F \notin S$ or $G \in S$.

Proof: We prove only: if $F \notin S$ then $F \to G \in S$ (others similar).
From $\neg F \in S$ we get:

1.	S	$\vdash$	$\neg F$	because $\neg F \in S$
2.	S	$\vdash$	$\neg F \to (\neg G \to \neg F)$	Axiom (1)
3.	S	$\vdash$	$\neg G \to \neg F$	Modus Ponens to 1. & 2.
4.	S	$\vdash$	$(\neg G \to \neg F) \to (F \to G)$	Axiom (3)
5.	S	$\vdash$	$F \to G$	Modus Ponens to 3. & 4.

Proof of (4.2)

Assertion (4.2): Let S by maximally consistent, and let $\mathcal{A}$ be the assignment given by: $\mathcal{A}(A) = 1$ iff $A \in \overline{S}$. Then $\mathcal{A}$ satisfies S.

Proof: Let F be a formula. We prove: $\mathcal{A}(F) = 1$ iff $F \in \overline{S}$.
By induction on the structure of F (and using Lemma III):

Atomic formulas: $F = A$. Easy.

Negation: $F = \neg G$. We have: $\mathcal{A}(F) = 1$ iff $\mathcal{A}(G) = 0$ iff
$G \notin \overline{S}$ iff $\neg G \in \overline{S}$ iff $F \in \overline{S}$.

Implication: $F = F_1 \to F_2$. We have: $\mathcal{A}(F) = 1$ iff
$\mathcal{A}(F_1 \to F_2) = 1$ iff $(\mathcal{A}(F_1) = 0$ or $\mathcal{A}(F_2) = 1)$ iff
$(F_1 \notin \overline{S}$ or $F_2 \in \overline{S})$ iff $F_1 \to F_2 \in \overline{S}$ iff $F \in \overline{S}$.

A Hilbert Calculus for predicate logic

We extend formulas by allowing universal quantification.

Three new axiom schemes:

(6) $(\forall x\ F) \rightarrow F[x/t]$ for every term t.

(7) $(\forall x\ (F \rightarrow G)) \rightarrow (\forall x\ F \rightarrow \forall x\ G)$.

(8) $F \rightarrow \forall x\ F$ if x does not occur free in F.

Theorem: The extension of the Hilbert Calculus is correct and complete for predicate logic.