Expressiveness of predicate logic: Some motivation

In computer science the analysis of the expressiveness of predicate logic (a.k.a. first-order logic) is of particular importance, for instance

- In database theory (where finite models are being studied): can every desirable property be expressed in predicate logic/ SQL?
- Hardware/software verification: Which properties of systems can be expressed in predicate logic?
- Formal language theory: Which language class corresponds to the languages expressible in predicate logic (over words, trees,...)?

Expressiveness: Some examples

Let R be a binary relational symbol and let f be a unary function symbol and let $\mathcal{A}$ be a suitable structure for $\{R, f\}$.

Expressing "$R^{\mathcal{A}}$ is an equivalence relation":

$$\forall x \forall y \forall z\, (R(x,x) \wedge (R(x,y) \rightarrow R(y,x)) \wedge (R(x,y) \wedge R(y,z) \rightarrow R(x,z)))$$

Expressing "$f^{\mathcal{A}}$ is injective":

$$\forall x \forall y\, (f(x) = f(y) \rightarrow x = y)$$

Expressing "$R^{\mathcal{A}}$ contains no directed triangle":

$$\forall x \forall y \forall z\, (x \neq y \wedge y \neq z \wedge x \neq z \wedge R(x,y) \wedge R(y,z)) \rightarrow \neg R(z,x))$$

Isomorphic structures

Let S be a signature and let $\mathcal{A}$ and $\mathcal{B}$ be suitable for S.

Then we say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic (w.r.t. S) if there is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, i.e. a bijection $\pi : U_{\mathcal{A}} \to U_{\mathcal{B}}$ such that

- for each n-ary functional symbol $f \in S$ and for each $u_1, \ldots, u_n \in U_{\mathcal{A}}$ we have $\pi(f^{\mathcal{A}}(u_1, \ldots, u_n)) = f^{\mathcal{B}}(\pi(u_1), \ldots, \pi(u_n))$ and
- for each n-ary relational symbol and for each $u_1, \ldots, u_n \in U_{\mathcal{A}}$ we have $(u_1, \ldots, u_n) \in R^{\mathcal{A}}$ if and only if $(\pi(u_1), \ldots, \pi(u_n)) \in R^{\mathcal{B}}$.

We also write $\mathcal{A} \simeq \mathcal{B}$ in case $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Properties of predicate logic

Let S be a signature. An S-property is a class of structures suitable for S that is closed under isomorphism.

A property P is expressible in predicate logic if there exists a sentence F over the signature S such that for each suitable structure $\mathcal{A}$ we have $\mathcal{A} \in P$ if and only $\mathcal{A} \models F$.

- $P_1 = \{\mathcal{A} \mid R^{\mathcal{A}}$ is an equivalence relation$\}$
- $P_2 = \{\mathcal{A} \mid f^{\mathcal{A}}$ is injective$\}$
- $P_3 = \{\mathcal{A} \mid R^{\mathcal{A}}$ contains no directed triangle$\}$
- ...

Expressiveness: What do we know?

Löwenheim-Skolem's theorem and related results delivered the following inexpressibility results (of interest in mathematics):

- Finiteness of structures
- Countability/uncountability of structures.

Exercise.

However, in computer science other properties are often of interest.

- Proving expressibility of a property is often easier than showing inexpressibility.
- $\rightarrow$ New techniques are required for showing inexpressibility results (in particular on finite structures).

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé games provide an elegant proof technique for proving inexpressibility results of predicate logic.

Good news for computer science: Ehrenfeucht-Fraïssé games can be applied both on infinite and on finite structures.

The latter is not true for other results that we have proven (such as compactness, recursive enumerability of tautologies).

Ehrenfeucht-Fraïssé games

- Two players: Spoiler and Duplicator.
- The game board consists of two structures $\mathcal{A}$ and $\mathcal{B}$ over some signature S.
- The players move alternatingly and Spoiler begins.
- The number of rounds (denoted by $k \in \mathbb{N}$) is fixed a priori.
- In each round Spoiler first chooses a structure ($\mathcal{A}$ or $\mathcal{B}$), and then an element of the universe of that structure. Duplicator answers with an element of the universe of the other structure.
- Intuition: Spoiler wants to show that $\mathcal{A}$ and $\mathcal{B}$ are "different", whereas Duplicator wants to show that $\mathcal{A}$ and $\mathcal{B}$ are "similar".
- Winning condition to be defined on next slide.

Restricted structures and partial isomorphisms

For simplicity, we only treat signatures without functional symbols.

Let $\mathcal{A}$ be a structure and let $V \subseteq U_{\mathcal{A}}$. Then $\mathcal{A}\restriction_V$ denotes the restriction of $\mathcal{A}$ on V:

- Universe $U_{\mathcal{A}\restriction_V} = V$
- for all n-ary relational symbols R:

$$R^{\mathcal{A}\restriction_V} = \{(a_1, \ldots, a_n) \in R^{\mathcal{A}} \mid a_1, \ldots, a_n \in V\}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be structures suitable for some signature S and let $\delta : U_{\mathcal{A}} \to U_{\mathcal{B}}$ be a partial function with domain $\mathrm{dom}(\delta)$ and range $\mathrm{ran}(\delta)$. Then δ is called partial isomorphism if δ is an isomorphism from $\mathcal{A}\restriction_{\mathrm{dom}(\delta)}$ to $\mathcal{B}\restriction_{\mathrm{ran}(\delta)}$.

Winning condition of EF games

Who wins an EF game?

- Assume all k rounds have been played and in round i elements $a_i \in U_\mathcal{A}$ and $b_i \in U_\mathcal{B}$ have been selected.
- If the set
$$\{(a_1, b_1), \ldots, (a_k, b_k)\}$$
is a partial isomorphism, then Duplicator wins.
- Otherwise Spoiler wins.

We are less interested in the winner in a simple game but rather in the player that has a winning strategy.

Winning strategies in EF games

- Let us denote the k round game on $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$.
- A player has a winning strategy if she/he can win the game for every possible moves of the other player.
- Winning strategies can be depicted by "game trees" of depth k.
- For each game $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$ Spoiler or Duplicator has a winning strategy (a.k.a. determinacy: this is the case for every two-player game of finite duration that admits no draws).

Winning strategies in EF games

Note that

- alternating moves correspond to quantifier alternations and
- Winning strategies of Spoiler and Duplicator are dual.

Winning strategy for Spoiler:

- $\exists$ Spoiler move $\forall$ Duplicator moves $\exists$ Spoiler move $\cdots$
  $\cdots$ $\forall$ Duplicator moves: the game yields no partial isomorphism.

Winning strategy for Duplicator:

- $\forall$ Spoiler moves $\exists$ Duplicator move $\forall$ Spoiler moves $\cdots$
  $\cdots$ $\exists$ Duplicator move: the game yields a partial isomorphism.

Quantifier rank

The connection of EF games and predicate logic will be established by taking the quantifier rank of formulas into account.

The quantifier rank $\mathsf{qr}(F)$ of a formula F is the nesting depth of quantifiers, more formally

- $\mathsf{qr}(F) = 0$ if F is quantifier-free,
- $\mathsf{qr}(\neg F) = \mathsf{qr}(F)$,
- $\mathsf{qr}(F \wedge G) = \mathsf{qr}(F \vee G) = \max\{\mathsf{qr}(F), \mathsf{qr}(G)\}$, and
- $\mathsf{qr}(\exists x F) = \mathsf{qr}(\forall x F) = \mathsf{qr}(F) + 1$.

Example:

$$\mathsf{qr}(\exists x(\forall y P(x, y) \vee \exists z \forall y Q(x, y, z))) = 3$$

Quantifier rank

Lemma: Let S be any finite signature, n some arity and k some quantifier rank. Then there are only finitely many pairwise inequivalent formulas F over the signature S having m free variables and quantifier rank k.

Example. Assume $S = \{P\}$, where P has arity 1.

For $k = 0$ there are four equivalence classes:

$$P(x), \qquad \neg P(x), \qquad P(x) \wedge \neg P(x), \qquad P(x) \vee \neg P(x)$$

For instance $P(x) \vee P(x) \equiv P(x)$.

For $k = 1$ and $n = 1$ there are already 14 equivalence classes!

Ehrenfeucht-Fraïssé Theorem

Theorem Let $\mathcal{A}$ and $\mathcal{B}$ be structures over S. For each $k \geq 0$ the following two statements are equivalent:

(1) $\mathcal{A} \models F$ if and only $\mathcal{B} \models F$ for all sentences F over S satisfying $\mathsf{qr}(F) \leq k$.

(2) Duplicator has a winning strategy in $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$.

To prove the theorem by induction we have to consider games in which a certain number of rounds have already been played:

- Assume after i moves position $\{(a_1, b_1), \ldots, (a_i, b_i)\}$ has been reached.
- The remaining game with ℓ moves is denoted by $\mathcal{G}_\ell(\mathcal{A}, a_1, \ldots, a_i, \mathcal{B}, b_1, \ldots b_i)$.
- Winning strategies for subgames are defined analogously as for the whole game.

Ehrenfeucht-Fraïssé Theorem

We will prove the following more general statement by induction on k.

Theorem. Assume $\mathcal{A}$ and $\mathcal{B}$ are structures over S and let $\overline{a} = a_1, \ldots, a_r \in U_{\mathcal{A}}$ and let $\overline{b} = b_1, \ldots, b_r \in U_{\mathcal{B}}$. Then for each $k \geq 0$ the following statements are equivalent:

(1) $\mathcal{A}_{[\overline{x}/\overline{a}]} \models F$ and $\mathcal{B}_{[\overline{x}/\overline{b}]} \not\models F$ for a formula F over S with qr$(F) \leq k$ and free variables $\overline{x}$.

(2) Spoiler has a winning strategy in $\mathcal{G}_k(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

Please note:

- We consider games that already have some history.
- Winning strategies for Spoiler and distinguishability instead of winning strategies for Duplicator and indistinguishability.

The upper theorem obviously implies the Ehrenfeucht-Fraïssé Theorem.

Methodology Theorem

The following theorem is a basis for proving non-expressibility via EF games.

Theorem. Let P be a property. Assume for each $k \geq 0$ there are structures $\mathcal{A}_k$ and $\mathcal{B}_k$ satisfying

(1) $\mathcal{A}_k \in P$ and $\mathcal{B}_k \notin P$ and

(2) Duplicator has a winning strategy for $\mathcal{G}_k(\mathcal{A}_k, \mathcal{B}_k)$.

Then P is not expressible in predicate logic.

This proof prinicple works for any class $\mathcal{C}$ of structures (for instance all finite structures) as long as the $\mathcal{A}_k$ and $\mathcal{B}_k$ are from $\mathcal{C}$.

Parity

Recall: Predicate logic can count up to any constant $c \in \mathbb{N}$:

$$\forall x_0 \cdots \forall x_c \left(\bigvee_{0 \le i < j \le c} x_i = x_j \right)$$

Consider the following properties:

- FINITE $= \{\mathcal{A} : |U_{\mathcal{A}}| \text{ is finite}\}$.
- EVEN $= \{\mathcal{A} : |U_{\mathcal{A}}| \text{ is finite and even}\}$ and
  ODD $= \{\mathcal{A} : |U_{\mathcal{A}}| \text{ is finite and odd}\}$.

Theorem. For any subset X of infinite structures neither EVEN $\cup X$ nor ODD $\cup X$ are expressible in predicate logic, neither in the class of all structures nor in the class of all finite structures.

Connectivity

An undirected graph $G = (V, E)$ is connected if for any two $v, v' \in V$ there exists a sequence $v_0, \ldots, v_n \in V (n \geq 0)$ such that $v_0 = v$, $v_n = v'$ and (v_{i-1}, v_i) for all $i \in \{1, \ldots, n\}$.

We show that connectivity is a property inexpressible in predicate logic.

We choose undirected graphs $\mathcal{A}_k$ and $\mathcal{B}_k$ such that:

- $\mathcal{A}_k$ has a cycle of length 2^k (and hence is connected)
- $\mathcal{B}_k$ is the disjoint union of two cycles of length 2^k each (and hence not connected).

We have to prove: Duplicator has a winning strategy for $\mathcal{G}_k(\mathcal{A}_k, \mathcal{B}_k)$.

Connectivity

For two nodes u and v of a graph $G = (V, E)$ let $d(u, v)$ denote

- the length of a shortest path from u to v if such a path exists and
- $d(u, v) = \infty$ if such a path does not exist.

For $\ell \geq 0$, let $N_\ell(u) = \{v \in V \mid d(u, v) \leq \ell\}$ denote the neighborhood of radius ℓ around u.

Lemma. Duplicator has a strategy in $\mathcal{G}_k(\mathcal{A}_k, \mathcal{B}_k)$ such that after i moves a configuration $\{(a_1, b_1), \ldots, (a_i, b_i)\}$ is reached such that for all $1 \leq j < \ell \leq i$ we have

$$d(a_j, a_\ell) = d(b_j, b_\ell) \quad \text{or} \quad d(a_j, a_\ell), d(b_j, b_\ell) > 2^{k-i} \qquad (\bigstar)$$

Corollary. Connectivity is not expressible in predicate logic.

Transitive Closure

For many applications it is helpful to access the transitive closure of a binary relation.

Example in databases: Given a database of direct flight connections of an airline. The transitive closure comprises all connections between airport (by possibly taking transfer flights).

Lemma. Assume $S = \{E\}$ for a binary relational symbol E. There is no formula $F(x, y)$ in predicate logic such that for all structures $\mathcal{A}$ suitable for S we have for all $a, b \in U_{\mathcal{A}}$:

$$\mathcal{A}_{[x/a,y/b]} \models F \quad \Leftrightarrow \quad \text{there is a path from } a \text{ to } b \text{ in } \mathcal{A}$$

Further inexpressibility results

The following properties are also not expressible in predicate logic:

- Acyclicity
- Being a tree
- Planarity
- k-colorability for each $k \geq 2$
- $\cdots$